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ON THE CAUCHY-PROBLEM FOR GENERALIZED KADOMTSEV-PETVIASHVILI-II EQUATIONS

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Abstract. The Cauchy-problem for the generalized Kadomtsev-Petviashvili-II equation

$$u_t + u_{xxx} + \partial_x^{-1} u_{yy} = (u^l)_x, \quad l \geq 3,$$

is shown to be locally well-posed in almost critical anisotropic Sobolev spaces. The proof combines local smoothing and maximal function estimates as well as bilinear refinements of Strichartz type inequalities via multilinear interpolation in $X_{s,b}$-spaces.

1. Introduction

Inspired by the work of Kenig and Ziesler [13, 14], we consider the Cauchy problem

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \mathbb{R}^2 \tag{1.1}$$

for the generalized Kadomtsev-Petviashvili-II equation (for short: gKP-II)

$$u_t + u_{xxx} + \partial_x^{-1} u_{yy} = (u^l)_x, \tag{1.2}$$

where $l \geq 3$ is an integer. Concerning earlier results on related problems for this equation we refer to the works of Saut [15], Iório and Nunes [8], and Hayashi, Naumkin, and Saut [7].

For the Cauchy data we shall assume $u_0 \in H^{(s)}$, where for $s = (s_1, s_2, \varepsilon)$ the Sobolev type space $H^{(s)}$ is defined by its norm in the following way: Let $\xi := (k, \eta) \in \mathbb{R}^2$ denote the Fourier variables corresponding to $(x,y) \in \mathbb{R}^2$ and $\langle D_x \rangle^{\sigma_1} = \mathcal{F}^{-1} \langle k \rangle^{\sigma_1} \mathcal{F}$, $\langle D_y \rangle^{\sigma_2} = \mathcal{F}^{-1} \langle \eta \rangle^{\sigma_2} \mathcal{F}$, as well as $\langle D_x^{-1} D_y \rangle^{\sigma_3} = \mathcal{F}^{-1} \langle k^{-1} \eta \rangle^{\sigma_3} \mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform and $\langle x \rangle^{\sigma} = (1+x^2)^{\frac{\sigma}{2}}$. Setting

$$\|u_0\|_{\sigma_1,\sigma_2,\sigma_3} := \|\langle D_x \rangle^{\sigma_1} \langle D_y \rangle^{\sigma_2} \langle D_x^{-1} D_y \rangle^{\sigma_3} u_0\|_{L^2_{xy}}$$

we define

$$\|u_0\|_{H^{(s)}} := \|u_0\|_{s_1+2s_2+\varepsilon,0,0} + \|u_0\|_{s_1,s_2,\varepsilon}.$$

Almost the same data spaces are considered in [13], [14], the only new element here is the additional parameter s_2, which will play a major role only for powers $l \geq 4$.

Using the contraction mapping principle we will prove a local well-posedness result for (1.1), (1.2) with regularity assumptions on the data as weak as possible.

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Two types of estimates will be be involved in the proof: On the one hand there is the combination of local smoothing effect ([13], see also [10]) and maximal function estimate (proven in [13, 14]), which has been used in [13], a strategy that had been developed in [11] in the context of generalized KdV equations. On the other hand we rely on Strichartz type estimates (cf. [15]) and especially on the bilinear refinement thereof taken from [5, 6], see also [9, 16, 18, 17]. A similar bilinear estimate involving y-derivatives (to be proven below) will serve to deal with the $\langle D_y\rangle$ containing part of the norm. To make these two elements meet, we use Bourgain's Fourier restriction norm method [1], especially the following function spaces of $X_{s,b}$-type. The basic space $X_{0,b}$ is given as usual by the norm

$$\|u\|_{X_{0,b}} := \|\langle \tau - \phi(\xi)\rangle^b \mathcal{F}u\|_{L^2_{\xi\tau}},$$

where $\phi(\xi) = k^3 - \frac{\eta^2}{k}$ is the phase function of the linearized KP-II equation. According to the data spaces chosen above we shall also use

$$\|u\|_{X_{\sigma_1,\sigma_2,\sigma_3;b}} := \|\langle D_x\rangle^{\sigma_1}\langle D_y\rangle^{\sigma_2}\langle D_x^{-1}D_y\rangle^{\sigma_3} u\|_{X_{0,b}}$$

as well as

$$\|u\|_{X_{(s),b}} := \|u\|_{X_{s_1+2s_2+\varepsilon,0,0;b}} + \|u\|_{X_{s_1,s_2,\varepsilon;b}}.$$

Finally the time restriction norm spaces $X_{(s),b}(\delta)$ constructed in the usual manner will become our solution spaces. Now we can state the main result of this note.

Theorem 1.1. *Let $s_1 > 1/2$, $s_2 \geq \frac{l-3}{2(l-1)}$ and $0 < \varepsilon \leq \min(s_1, 1)$. Then for $s = (s_1, s_2, \varepsilon)$ and $u_0 \in H^{(s)}$ there exist $\delta = \delta(\|u_0\|_{H^{(s)}}) > 0$ and $b > \frac{1}{2}$ such that there is a unique solution $u \in X_{(s),b}(\delta)$ of (1.1), (1.2). This solution is persistent and the flow map $S : u_0 \mapsto u$, $H^{(s)} \to X_{(s),b}(\delta)$ is locally Lipschitz.*

The lower bounds on s_1, s_2, and ε are optimal (except for endpoints) in the sense that scaling considerations strongly suggest the necessity of the condition $s_1 + 2s_2 + \varepsilon \geq \frac{1}{2} + \frac{l-3}{l-1}$. Moreover, for $l = 3$ C^2-illposedness is known for $s_1 < \frac{1}{2}$ or $\varepsilon < 0$, see [13, Theorem 4.1]. The affirmative result in [13] concerning the cubic gKP-II equation, that was local well-posedness for $s_1 > \frac{3}{4}$, $s_2 = 0$, and $\varepsilon > \frac{1}{2}$ is improved here by $\frac{3}{4}$ derivatives. Some effort was made to keep the number of y-derivatives as small as possible[1], but we shall not attempt to give evidence for the necessity of the individual lower bounds on s_1, s_2, and ε, respectively.

By the general arguments concerning the Fourier restriction norm method introduced in [1] and further developed in [2, 12], matters reduce to proving the following multilinear estimate.

Theorem 1.2. *Let $s_1 > \frac{1}{2}$, $s_2 \geq \frac{l-3}{2(l-1)}$ and $0 < \varepsilon \leq \min(s_1, 1)$. Then there exists $b' > -\frac{1}{2}$, such that for all $b > \frac{1}{2}$ and all $u_1, \dots, u_l \in X_{(s),b}$ supported in $\{|t| \leq 1\}$ the estimate*

$$\|\partial_x \prod_{j=1}^{l} u_j\|_{X_{(s),b'}} \lesssim \prod_{j=1}^{l} \|u_j\|_{X_{(s),b}}$$

holds.

[1]For $l \geq 5$ the use of local smoothing effect and maximal function estimate *alone* yields LWP for $s_1 > \frac{l-3}{l-1}$, $s_2 = 0$, and $\varepsilon > \frac{1}{2}$, which is optimal from the scaling point of view, too. This result may be seen as essentially contained in [13, Theorem 2.1 and Lemma 3.2] plus [11, proofs of Theorem 2.10 and Theorem 2.17]. This variant always requires $\frac{1}{2}+$ derivatives in y.

To prepare for the proof of Theorem 1.2 let us first recall those estimates for free solutions $W(t)u_0$ of the linearized KP-II equation, which we take over from the literature. First we have the local smoothing estimate from [13, Lemma 3.2]: For $0 \leq \lambda \leq 1$,

$$\|D_x^{(1-\lambda)}(D_x^{-1}D_y)^{\lambda}W(t)u_0\|_{L_x^\infty L_{yt}^2} \lesssim \|u_0\|_{L_{xy}^2} \tag{1.3}$$

and hence by the transfer principle [2, Lemma 2.3] for $b > \frac{1}{2}$,

$$\|D_x^{(1-\lambda)}(D_x^{-1}D_y)^{\lambda}u\|_{L_x^\infty L_{yt}^2} \lesssim \|u\|_{X_{0,b}}. \tag{1.4}$$

Interpolation with the trivial case $L_{xyt}^2 = X_{0,0}$ and duality give for $0 \leq \theta \leq 1$, $\frac{1}{p_\theta} = \frac{1-\theta}{2}$,

$$\|[D_x^{(1-\lambda)}(D_x^{-1}D_y)^{\lambda}]^{\theta}u\|_{L_x^{p_\theta}L_{yt}^2} \lesssim \|u\|_{X_{0,\theta b}} \tag{1.5}$$

as well as

$$\|[D_x^{(1-\lambda)}(D_x^{-1}D_y)^{\lambda}]^{\theta}u\|_{X_{0,-\theta b}} \lesssim \|u\|_{L_x^{p'_\theta}L_{yt}^2}. \tag{1.6}$$

(For Hölder exponents p we will always have $\frac{1}{p} + \frac{1}{p'} = 1$.) To complement the local smoothing effect, we shall use the maximal function estimate

$$\|W(t)u_0\|_{L_x^4 L_{yT}^\infty} \lesssim \|\langle D_x\rangle^{\frac{3}{4}+}\langle D_x^{-1}D_y\rangle^{\frac{1}{2}+}u_0\|_{L_{xy}^2} \tag{1.7}$$

due to Kenig and Ziesler [13, Theorem 2.1], which is probably the hardest part of the whole story. The capital T here indicates, that this estimate is only valid local in time, the $+$-signs at the exponents on the right denote positive numbers, which can be made arbitrarily small at the cost of the implicit constant but independent of other parameters. (This notation will be used repeatedly below.) The transfer principle implies for u supported in $\{|t| \leq 1\}$

$$\|u\|_{L_x^4 L_{yt}^\infty} \lesssim \|\langle D_x\rangle^{\frac{3}{4}+}\langle D_x^{-1}D_y\rangle^{\frac{1}{2}+}u\|_{X_{0,b}}, \tag{1.8}$$

where $b > \frac{1}{2}$. The Strichartz type estimate

$$\|W(t)u_0\|_{L_{xyt}^4} \lesssim \|u_0\|_{L_{xy}^2}, \tag{1.9}$$

taken from [15, Proposition 2.3], becomes

$$\|u\|_{L_{xyt}^4} \lesssim \|u\|_{X_{0,b}}, \quad b > \frac{1}{2}. \tag{1.10}$$

For its bilinear refinement

$$\|uv\|_{L_{xyt}^2} \lesssim \|D_x^{1/2}u\|_{X_{0,b}}\|D_x^{-\frac{1}{2}}v\|_{X_{0,b}}, \quad b > \frac{1}{2}, \tag{1.11}$$

and the dualized version thereof

$$\|D_x^{1/2}(uv)\|_{X_{0,-b}} \lesssim \|D_x^{1/2}u\|_{X_{0,b}}\|v\|_{L_{xyt}^2}, \quad b > \frac{1}{2}, \tag{1.12}$$

we refer to [6, Theorem 3.3 and Proposition 3.5]. In order to estimate the $X_{s_1,s_2,\varepsilon;b'}$-norm of the nonlinearity the following bilinear estimate involving y-derivatives will be useful. We introduce the bilinear pseudodifferential operator $M(u,v)$ in terms of its Fourier transform

$$\widehat{M(u,v)}(\xi) := \int_{\xi=\xi_1+\xi_2} |k_1\eta - k\eta_1|^{1/2}\widehat{u}(\xi_1)\widehat{v}(\xi_2)d\xi_1$$

and define the auxiliary space $\widehat{L}^p_x L^q_{yt}$ by $\|f\|_{\widehat{L}^p_x L^q_{yt}} := \|\mathcal{F}_x f\|_{L^{p'}_x L^q_{yt}}$, where $\mathcal{F}_x$ denotes the partial Fourier transform with respect to the first space variable x only. Then we have:

Lemma 1.3.
$$\|M(W(t)u_0, W(t)v_0)\|_{\widehat{L}^1_x L^2_{yt}} \lesssim \|D_x^{1/2} u_0\|_{L^2_{xy}} \|D_x^{1/2} v_0\|_{L^2_{xy}}. \tag{1.13}$$

Proof. [2] We have
$$\begin{aligned} &\widehat{M(W(t)u_0, W(t)v_0)}(\xi,\tau)\\ &= \int_{\xi=\xi_1+\xi_2} |k_1\eta - k\eta_1|^{1/2} \delta(\tau - k_1^3 - k_2^3 + \frac{\eta_1^2}{k_1} + \frac{\eta_2^2}{k_2}) \widehat{u_0}(\xi_1)\widehat{v_0}(\xi_2) d\xi_1. \end{aligned}$$
Because of $\frac{\eta_1^2}{k_1} + \frac{\eta_2^2}{k_2} = \frac{\eta^2}{k} + \frac{k}{k_1k_2}(\eta_1 - \frac{k_1}{k}\eta)^2$ and with $a = \tau - k_1^3 - k_2^3 + \frac{\eta^2}{k}$ as well as $g(\eta_1) = \frac{k}{k_1k_2}(\eta_1 - \frac{k_1}{k}\eta)^2 - a$ this equals
$$\int_{\xi=\xi_1+\xi_2} |k_1\eta - k\eta_1|^{1/2} \delta(g(\eta_1)) \widehat{u_0}(\xi_1)\widehat{v_0}(\xi_2) d\eta_1 dk_1.$$
The zero's of g are $\eta_1^\pm = \frac{k_1}{k}\eta \pm \sqrt{\frac{k_1k_2a}{k}}$ and for the derivative we have $|g'(\eta_1)| = \frac{2}{|k_1k_2|}|k_1\eta - k\eta_1|$. So we get the two contributions
$$I^\pm(\xi,\tau) = \frac{1}{2}\int_{k_1+k_2=k} \frac{|k_1k_2|}{|k_1\eta - k\eta_1^\pm|^{1/2}} \widehat{u_0}(k_1,\eta_1^\pm)\widehat{v_0}(k_2,\eta-\eta_1^\pm) dk_1.$$
By Minkowski's integral inequality
$$\|I^\pm(\xi,\cdot)\|_{L^2_\tau} \lesssim \int_{k_1+k_2=k} |k_1k_2| \| |k_1\eta - k\eta_1^\pm|^{-\frac{1}{2}} \widehat{u_0}(k_1,\eta_1^\pm)\widehat{v_0}(k_2,\eta-\eta_1^\pm)\|_{L^2_\tau} dk_1,$$
where, with $\lambda := \sqrt{\frac{k_1k_2a}{k}}$, the square of the last L^2_τ-norm equals
$$\begin{aligned} &\int |k_1\eta - k\eta_1^\pm|^{-1} |\widehat{u_0}(k_1, \frac{k_1}{k}\eta \pm \lambda)\widehat{v_0}(k_2, \frac{k_2}{k}\eta \mp \lambda)|^2 d\tau\\ &\le \frac{2}{|k_1k_2|}\int |\widehat{u_0}(k_1, \frac{k_1}{k}\eta \pm \lambda)\widehat{v_0}(k_2, \frac{k_2}{k}\eta \mp \lambda)|^2 d\lambda, \end{aligned}$$
since $d\tau = 2\frac{\lambda k}{|k_1k_2|}d\lambda = \mp\frac{2}{k_1k_2}(k_1\eta - k\eta_1^\pm)d\lambda$. This gives
$$\|I^\pm(\xi,\cdot)\|_{L^2_\tau} \lesssim \int_{k_1+k_2=k} |k_1k_2|^{1/2} \Big(\int |\widehat{u_0}(k_1, \frac{k_1}{k}\eta \pm \lambda)\widehat{v_0}(k_2, \frac{k_2}{k}\eta \mp \lambda)|^2 d\lambda\Big)^{1/2} dk_1.$$

[2] It is not apparent from the proof, but there is a very simple idea behind Lemma 1.3. If we take the partial Fourier transform $\mathcal{F}_x W(t)u_0(k) = e^{itk^3} e^{i\frac{t}{k}\partial_y^2}\mathcal{F}_x u_0(k)$ with respect to the x-variable only, we obtain a free solution of the linear Schrödinger equation with rescaled time variable $s = \frac{t}{k}$, multiplied by a phase factor of size one. So any space time estimate for the Schrödinger equation should give a corresponding estimate for linearized KP-type equations. This idea was exploitet in [3], [4] to obtain suitable substitutes for Strichartz type estimates for semiperiodic and periodic problems. From this point of view Lemma 1.3 corresponds to the one-dimensional estimate
$$\|D_y^{1/2}(e^{it\partial_y^2}u_0 e^{-it\partial_y^2}v_0)\|_{L^2_{yt}} \lesssim \|u_0\|_{L^2_y}\|v_0\|_{L^2_y}.$$

Using Parseval and again Minkowski's inequality for the L^2_η-norm we arrive at

$$\begin{aligned}
&\|\mathcal{F}_x M(W(t)u_0, W(t)v_0)(k)\|_{L^2_{yt}} \\
&\lesssim \int_{k_1+k_2=k} |k_1k_2|^{1/2}\Big(\int |\widehat{u_0}(k_1, \frac{k_1}{k}\eta \pm \lambda)\widehat{v_0}(k_2, \frac{k_2}{k}\eta \mp \lambda)|^2 d\lambda d\eta\Big)^{1/2} dk_1 \\
&= \int_{k_1+k_2=k} |k_1k_2|^{1/2}\|\widehat{u_0}(k_1,\cdot)\|_{L^2_\eta}\|\widehat{v_0}(k_2,\cdot)\|_{L^2_\eta} dk_1 \\
&\lesssim \|D_x^{1/2}u_0\|_{L^2_{xy}}\|D_x^{1/2}v_0\|_{L^2_{xy}}
\end{aligned}$$

by Cauchy-Schwarz and a second application of Parseval's identity. □

Corollary 1.4. *Let $b > 1/2$. Then*

$$\|M(u,v)\|_{\widehat{L}^1_x L^2_{yt}} \lesssim \|D_x^{1/2}u\|_{X_{0,b}}\|D_x^{1/2}v\|_{X_{0,b}} \tag{1.14}$$

$$\|D_x^{-\frac12}M(u,v)\|_{X_{0,-b}} \lesssim \|u\|_{\widehat{L}^\infty_x L^2_{yt}}\|D_x^{1/2}v\|_{X_{0,b}}. \tag{1.15}$$

Proof. Lemma 1.3 implies (1.14) via the transfer principle, (1.15) is then obtained by duality. In fact, if we fix v and consider the linear map $M_v(u) := M(u,v)$, then its adjoint is given by $M_v^* = M_{\overline{v}}$, and we have $\|\overline{v}\|_{X_{0,b}} = \|v\|_{X_{0,b}}$. □

Now we are prepared for the proof of the central multilinear estimate.

Proof of Theorem 1.2.
1. We use (1.6) with $\lambda = 0$, $\theta = \frac12$ and Hölder to obtain for $b_0 < -\frac14$

$$\begin{aligned}
\|D_x^{1/2}\prod_{j=1}^l u_j\|_{X_{0,b_0}} &\lesssim \|\prod_{j=1}^l u_j\|_{L_x^{\frac43}L^2_{yt}} \\
&\lesssim \|u_1\|_{L^4_xL^2_{yt}}\|u_2\|_{L^4_xL^\infty_{yt}}\|u_3\|_{L^4_xL^\infty_{yt}}\prod_{j\ge4}\|u_j\|_{L^\infty_{xyt}}.
\end{aligned}$$

For the first factor we have by (1.5), again with $\lambda = 0$, $\theta = \frac12$,

$$\|u_1\|_{L^4_xL^2_{yt}} \lesssim \|D_x^{-\frac12}u_1\|_{X_{0,\frac14+}},$$

while the second and third factor are estimated by (1.8)

$$\|u_{2,3}\|_{L^4_xL^\infty_{yt}} \lesssim \|\langle D_x\rangle^{\frac34+}\langle D_x^{-1}D_y\rangle^{\frac12+}u_{2,3}\|_{X_{0,b}},$$

where $b > \frac12$. It is here that the time support assumption is needed. For $j \ge 4$ we use Sobolev embeddings in all variables to obtain for $b > \frac12$

$$\|u_j\|_{L^\infty_{xyt}} \lesssim \|\langle D_x\rangle^{\frac12+}\langle D_y\rangle^{\frac12+}u_j\|_{X_{0,b}}.$$

Summarizing we have for $b_0 < -1/4$, $b > 1/2$,

$$\begin{aligned}
\|D_x^{1/2}\prod_{j=1}^l u_j\|_{X_{0,b_0}} &\lesssim \|D_x^{-\frac12}u_1\|_{X_{0,b}}\|\langle D_x\rangle^{\frac34+}\langle D_x^{-1}D_y\rangle^{\frac12+}u_2\|_{X_{0,b}} \\
&\quad\times\|\langle D_x\rangle^{\frac34+}\langle D_x^{-1}D_y\rangle^{\frac12+}u_3\|_{X_{0,b}}\prod_{j\ge4}\|\langle D_x\rangle^{\frac12+}\langle D_y\rangle^{\frac12+}u_j\|_{X_{0,b}}.
\end{aligned} \tag{1.16}$$

2. Combining the dual version (1.12) of the bilinear estimate with Hölder's inequality, (1.11) and Sobolev embeddings we obtain for $b_1 < -1/2$, $b > 1/2$,

$$\begin{aligned} &\|D_x^{1/2}\prod_{j=1}^{l} u_j\|_{X_{0,b_1}} \\ &\lesssim \|D_x^{1/2}u_3\|_{X_{0,b}}\|u_1u_2\|_{L^2_{xyt}}\prod_{j\geq 4}\|u_j\|_{L^\infty_{xyt}} \\ &\lesssim \|D_x^{-\frac12}u_1\|_{X_{0,b}}\|D_x^{1/2}u_2\|_{X_{0,b}}\|D_x^{1/2}u_3\|_{X_{0,b}}\prod_{j\geq 4}\|\langle D_x\rangle^{\frac12+}\langle D_y\rangle^{\frac12+}u_j\|_{X_{0,b}}. \end{aligned} \tag{1.17}$$

3. Bilinear interpolation involving u_2 and u_3 gives

$$\begin{aligned} \|D_x^{1/2}\prod_{j=1}^{l} u_j\|_{X_{0,b'}} &\lesssim \|D_x^{-\frac12}u_1\|_{X_{0,b}}\|\langle D_x\rangle^{\frac12+\frac{\theta}{4}+}\langle D_x^{-1}D_y\rangle^{\frac{\theta}{2}+}u_2\|_{X_{0,b}} \\ &\quad\times \|\langle D_x\rangle^{\frac12+\frac{\theta}{4}+}\langle D_x^{-1}D_y\rangle^{\frac{\theta}{2}+}u_3\|_{X_{0,b}}\prod_{j\geq 4}\|\langle D_x\rangle^{\frac12+}\langle D_y\rangle^{\frac12+}u_j\|_{X_{0,b}}, \end{aligned}$$

where $0 < \theta \ll 1$ and $b' = \theta b_0 + (1-\theta)b_1$. Now symmetrization via $(l-1)$-linear interpolation among $u_2, \dots, u_l$ yields

$$\|D_x^{1/2}\prod_{j=1}^{l} u_j\|_{X_{0,b'}} \lesssim \|D_x^{-\frac12}u_1\|_{X_{0,b}}\prod_{j\geq 2}\|\langle D_x\rangle^{\alpha_1+}\langle D_y\rangle^{\alpha_2+}\langle D_x^{-1}D_y\rangle^{\alpha_3+}u_j\|_{X_{0,b}}, \tag{1.18}$$

with b, b' as before, $\alpha_1 = \frac12 + \frac{\theta}{2(l-1)}$, $\alpha_2 = \frac{l-3}{2(l-1)}$, and $\alpha_3 = \frac{\theta}{l-1}$. Using $\langle\eta\rangle \leq \langle k\rangle\langle k^{-1}\eta\rangle$, we may replace α_2+ by α_2 in (1.18). Now, for given $s_1 > \frac12$ and $\varepsilon > 0$ we choose θ close enough to zero, so that $\alpha_1 < s_1$ and $\alpha_3 < \varepsilon$, and b_0 (respectively b_1) close enough to $-\frac14$ (respectively to $-\frac12$), so that $b' > -\frac12$. Then, assuming by symmetry that u_1 has the largest frequency with respect to the x-variable (i. e. $|k_1| \geq |k_{2,\dots,l}|$), we obtain

$$\|\partial_x\prod_{j=1}^{l} u_j\|_{X_{s_1+2s_2+\varepsilon,0,0;b'}} \lesssim \|u_1\|_{X_{s_1+2s_2+\varepsilon,0,0;b}}\prod_{j\geq 2}\|u_j\|_{X_{s_1,s_2,\varepsilon;b}}. \tag{1.19}$$

4. The same upper bound holds for $\|\partial_x\prod_{j=1}^{l}u_j\|_{X_{s_1,s_2,\varepsilon;b'}}$, if $|\eta| \leq |k|$ (or even if $|\eta| \leq |k|^2$), where $|k|$ (respectively $|\eta|$) are the frequencies in x (respectively in y) of the whole product. In the case where $|k| \leq 1$ - assuming $\varepsilon \leq \min(s_1, 1)$ and b' sufficiently close to $-\frac12$, the estimate

$$\|\partial_x\prod_{j=1}^{l} u_j\|_{X_{s_1,s_2,\varepsilon;b'}} \lesssim \prod_{j=1}^{l}\|u_j\|_{X_{s_1,s_2,\varepsilon;b}} \tag{1.20}$$

is easily derived by a combination of the standard Strichartz type estimate (1.10) and Sobolev embeddings. So we may henceforth assume $|k| \geq 1$, $|\eta| \geq 1$, and $|k^{-1}\eta| \geq 1$.

One last simple observation concerning the estimation of $\|\partial_x\prod_{j=1}^{l}u_j\|_{X_{s_1,s_2,\varepsilon;b'}}$: If we assume in addition to $|k_1| \geq |k_{2,\dots,l}|$ that u_1 has a large frequency with respect to y; i.e., $|\eta| \lesssim |\eta_1|$, then from (1.18) we also obtain (1.20).

5. It remains to estimate $\|\partial_x \prod_{j=1}^{l} u_j\|_{X_{s_1,s_2,\varepsilon;b'}}$ in the case where $|k_1| \geq |k_{2,\dots,l}|$ (symmetry assumption as before) and $|\eta_1| \ll |\eta|$. By symmetry among $u_{2,\dots,l}$ we may assume in addition that $|\eta_2| \geq |\eta_{1,3,\dots,l}|$ and hence that $|k_2| \ll |k|$, because otherwise previous arguments apply with u_1 and u_2 interchanged. For this distribution of frequencies the symbol of the Fourier multiplier $M(u_1 u_3 \cdots u_l, u_2)$ becomes

$$|k\eta_2 - k_2\eta|^{1/2} \sim |k\eta_2|^{1/2} \gtrsim |k\eta|^{1/2}.$$

Now let $P(u_1,\dots,u_l)$ denote the projection in Fourier space on $\{|\eta_2| \geq |\eta_{1,3,\dots,l}|\} \cap \{\langle k_2\rangle \ll |k| \lesssim k_1\}$. Then by (1.15) we obtain for $b > \frac{1}{2}$, $b_1 < -\frac{1}{2}$

$$\begin{aligned}
&\|D_y^{1/2} P(u_1,\dots,u_l)\|_{X_{0,b_1}} \\
&\lesssim \|D_x^{1/2} u_2\|_{X_{0,b}} \|u_1 u_3 \cdots u_l\|_{\widehat{L}_x^\infty L_{yt}^2} \\
&\lesssim \|D_x^{1/2} u_2\|_{X_{0,b}} \|\langle D_x\rangle^{\frac{1}{2}+}(u_1 u_3)\|_{L_{xyt}^2} \prod_{j\geq 4} \|u_j\|_{\widehat{L}_x^\infty L_{yt}^\infty} \\
&\lesssim \|D_x^{0+} u_1\|_{X_{0,b}} \|\langle D_x\rangle^{\frac{1}{2}+} u_2\|_{X_{0,b}} \|\langle D_x\rangle^{\frac{1}{2}+} u_3\|_{X_{0,b}} \prod_{j\geq 4} \|\langle D_x\rangle^{\frac{1}{2}+}\langle D_y\rangle^{\frac{1}{2}+} u_j\|_{X_{0,b}},
\end{aligned}$$

where besides Sobolev type inequalities we have used (1.11) in the last step. Interpolation with (1.17) gives for $0 \leq \lambda \leq 1$

$$\begin{aligned}
&\|D_x^{\frac{\lambda}{2}} D_y^{\frac{1-\lambda}{2}} P(u_1,\dots,u_l)\|_{X_{0,b_1}} \\
&\lesssim \|D_x^{-\frac{\lambda}{2}+} u_1\|_{X_{0,b}} \|\langle D_x\rangle^{\frac{1}{2}+} u_2\|_{X_{0,b}} \|\langle D_x\rangle^{\frac{1}{2}+} u_3\|_{X_{0,b}} \prod_{j\geq 4} \|\langle D_x\rangle^{\frac{1}{2}+}\langle D_y\rangle^{\frac{1}{2}+} u_j\|_{X_{0,b}}.
\end{aligned}$$

Yet another interpolation, now with (1.16), gives for $0 < \theta \ll 1$, $b' = \theta b_0 + (1-\theta) b_1$, $s_x = \frac{1}{2}(\lambda(1-\theta)+\theta)$ and $s_y = \frac{1}{2}(1-\theta)(1-\lambda)$

$$\begin{aligned}
&\|D_x^{s_x} D_y^{s_y} P(u_1,\dots,u_l)\|_{X_{0,b'}} \\
&\lesssim \|D_x^{-s_x+} u_1\|_{X_{0,b}} \|\langle D_x\rangle^{\frac{1}{2}+\frac{\theta}{4}+}\langle D_x^{-1} D_y\rangle^{\frac{\theta}{2}+} u_2\|_{X_{0,b}} \|\langle D_x\rangle^{\frac{1}{2}+\frac{\theta}{4}+}\langle D_x^{-1} D_y\rangle^{\frac{\theta}{2}+} u_3\|_{X_{0,b}} \\
&\quad \times \prod_{j\geq 4} \|\langle D_x\rangle^{\frac{1}{2}+}\langle D_y\rangle^{\frac{1}{2}+} u_j\|_{X_{0,b}}.
\end{aligned}$$

The next step is to equidistribute the $\langle D_y\rangle$'s on $u_2,\dots,u_l$. Here we must be careful, because the symmetry between u_2 and $u_3,\dots,u_l$ was broken. But since u_2 has the largest y-frequency we may first shift a $\langle D_y\rangle^{\frac{l-3}{2(l-1)}+}$ onto u_2 and then interpolate among $u_3,\dots,u_l$ in order to obtain

$$\begin{aligned}
&\|D_x^{s_x} D_y^{s_y} P(u_1,\dots,u_l)\|_{X_{0,b'}} \\
&\lesssim \|D_x^{-s_x+} u_1\|_{X_{0,b}} \prod_{j\geq 2} \|\langle D_x\rangle^{\beta_1+}\langle D_y\rangle^{\beta_2+}\langle D_x^{-1} D_y\rangle^{\beta_3+} u_j\|_{X_{0,b}},
\end{aligned} \tag{1.21}$$

where $\beta_1 = \frac{1}{2} + \frac{\theta}{4}$, $\beta_2 = \frac{l-3}{2(l-1)}$ and $\beta_3 = \frac{\theta}{2}$. Again we may replace β_2+ by β_2. Now (1.21) is applied to $\langle D_x\rangle^{s_1}\langle D_y\rangle^{s_2}\langle D_x^{-1} D_y\rangle^{\varepsilon}\partial_x P(u_1,\dots,u_l)$, where we can shift the $\langle D_x\rangle^{s_1}$ partly from the product to u_1 and the $\langle D_y\rangle^{s_2}$ partly to u_2. Moreover, since $|k_2| \lesssim |k|$ and $|\eta| \lesssim |\eta_2|$ we have $|k^{-1}\eta| \lesssim |k_2^{-1}\eta_2|$, so that a $\langle D_x^{-1} D_y\rangle^{\varepsilon-\theta}$ may be thrown from the product onto u_2. The result is

$$\begin{aligned}
&\|\partial_x P(u_1,\dots,u_l)\|_{X_{s_1,s_2,\varepsilon;b'}}\\
&\lesssim \|D_x^{s_1+1-\theta-2s_x+}u_1\|_{X_{0,b}}\|\langle D_x\rangle^{\beta_1+}\langle D_y\rangle^{\beta_2+s_2-s_y+\theta}\langle D_x^{-1}D_y\rangle^{\varepsilon-\theta+\beta_3+}u_2\|_{X_{0,b}}\\
&\times\prod_{j\geq 3}\|\langle D_x\rangle^{\beta_1+}\langle D_y\rangle^{\beta_2}\langle D_x^{-1}D_y\rangle^{\beta_3+}u_j\|_{X_{0,b}}.
\end{aligned}$$

Here $\beta_2 \leq s_2$ and by choosing $\theta < \min\big(\varepsilon, \frac{2}{3(l-1)}, s_1-\frac{1}{2}\big)$ and λ such that $s_y = \beta_2+\theta$ we can achieve that

- $s_1+1-\theta-2s_x < s_1+2s_2+\varepsilon$,
- $\beta_2+s_2-s_y+\theta \leq s_2$,
- $\varepsilon-\theta+\beta_3 < \varepsilon$,

as well as $\beta_1 < s_1$, $\beta_3 < \varepsilon$. This gives

$$\|\partial_x P(u_1,\dots,u_l)\|_{X_{s_1,s_2,\varepsilon;b'}} \lesssim \|u_1\|_{X_{s_1+2s_2+\varepsilon,0,0;b}}\prod_{j\geq 2}\|u_j\|_{X_{s_1,s_2,\varepsilon;b}}$$

as desired. □

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