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EXISTENCE OF SOLUTIONS TO P-LAPLACE EQUATIONS WITH LOGARITHMIC NONLINEARITY

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Abstract. This article concerns the the nonlinear elliptic equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \log u^{p-1} + \lambda f(x,u)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ and $u = 0$ on $\partial\Omega$. By means of a double perturbation argument, we obtain a nonnegative solution.

1. Introduction

In this paper we consider the existence of solutions to the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \log u^{p-1} + \lambda f(x,u), \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$, λ is a positive parameter. Equations of this form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction-diffusion theory [12], non-Newtonian fluid theory [1,13], non-Newtonian filtration [11,21] and the turbulent flow of a gas in a porous medium [6]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo-plastics. If $p = 2$, they are Newtonian fluids. When $p = 2$, the existence of bounded positive solutions were proved by Deng [3]. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to be verified. The main differences between $p = 2$ and $p \neq 2$ can be found in [8,9]. In recent years, the existence and uniqueness of the positive solutions for the quasilinear eigenvalue problem

$$\begin{gathered} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(x,u) = 0 \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

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with $\lambda > 0$, $p > 1$ on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ have been studied by many authors see [9,10,19] and the references therein when f is strictly increasing on $\mathbb{R}^+$, $f(0) = 0, \lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^{\mu}$, $0 < \mu < p-1$, α_1 and $\alpha_2 > 0$. It was shown in [10] that there exists at least two positive solutions for equation (1.2) when $\lambda > 0$ is sufficiently large. If $\liminf_{s\to 0^+}(f(s)/(s^{p-1})) > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/(s^{p-1}))' < 0$ holds for all $s > 0$, it was proved in [9] that the problem (1.2) has a unique positive solution when λ is sufficiently large.

For $p = 2$, some results to a semilinear elliptic equation with logarithmic nonlinearity

$$\begin{gathered} -\Delta u = \log u + h(x)u^q, \quad \text{in } B_R, \\ u > 0, \quad \text{in } B_R, \\ u = 0, \quad \text{on } \partial B_R; \end{gathered} \tag{1.3}$$

and

$$\begin{gathered} -\Delta u = \chi_{\{u>0\}}(\log u + \lambda\ f(x,u)), \quad \text{in } \Omega, \\ u \geq 0, \quad \text{in; } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

have been extensively studied. (See, for example, [15,19] and their references.) In [19], the authors obtained a positive radial solution $u \in C^2(\overline{B}_R\backslash\{0\})\bigcap C(\overline{B}_R)$ of (1.3) by means of a double perturbation argument. In [15], the authors study the problem (1.4), which obtain a maximal solution $u_\lambda \geq 0$ for every $\lambda > 0$ and prove its global regularity $C^{1,\gamma}(\overline{\Omega})$. Motivated by the results of the above cited papers, we shall attempt to treat such equation (1.1), the results of the semilinear equations are extended to the quasilinear ones. We can find the related results for $p = 2$ in [15]. In this paper, the authors obtained the maximal solution $u_\lambda \geq 0$ for every $\lambda > 0$ and proved its global regularity $C^{1,\gamma}(\overline{\Omega})$. Our strategy in the study of (1.1) is to use the sub-super solution method and the mountain pass lemma.

The paper is organized as follows. In section 2, we obtain a subsolution of (1.1) by adopting a double perturbation argument. Section 3 is dedicated to prove the existence of a supersolution of (1.1) by the mountain pass lemma. In section 4, we shall use the results of Section 2 and 3 to obtain a solution for the problem (1.1) by using the sub-super solution method which proves our main result. Some regularity properties of the solution of (1.1) are studied in section 5.

In this problem, the function f satisfies the following hypothesis:

- (H1) $f : \Omega \times [0, +\infty)$ is measurable in $x \in \Omega$ with f is continuous;
- (H2) f is nondecreasing, $f \neq 0$;
- (H3) $\lim_{s\to\infty} f(x,s)/s^{\beta} = 0$, $f(x,s)/s^{\beta}$ is decreasing where $0 < \beta < p-1$ (with respect to s) uniformly in $x \in \Omega$.

2. Subsolutions for (1.1)

In this section we obtain a subsolution of (1.1). We begin by considering the family of perturbed problems

$$\begin{gathered} -\operatorname{div}(|\nabla u^{\varepsilon}|^{p-2}\nabla u^{\varepsilon}) = \log\frac{(u^{\varepsilon})^p + \varepsilon u^{\varepsilon} + \varepsilon}{u^{\varepsilon} + \varepsilon} + \lambda f(x,u^{\varepsilon}), \quad \text{in } \Omega, \\ u^{\varepsilon} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

We show that solutions to these problems converge to a subsolution of (1.1). For $0 < \varepsilon < 1$, the solutions u^ε of (2.1) are a priori bounded, independently of ε. From [14], we give the following comparison principle which will be used to obtain a subsolution. (the proof can be found in [14,16])

Lemma 2.1. *Let $g(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable for x and nondecreasing for t, let $u, v \in W^{1,p}(\Omega)$ satisfy*

$$-\Delta_p u + g(x,u) \leq -\Delta_p v + g(x,v)(x \in \Omega).$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ on Ω.

Lemma 2.2. *Suppose f satisfies* (H1), (H3). *For $0 < \varepsilon \leq 1$, let u^ε be a solution of (2.1), then there exists a constant $C_1 > 0$, such that $\sup_{0<\varepsilon<1} \|u^\varepsilon\|_{L^\infty} \leq C_1$.*

Proof. We denote

$$h_\varepsilon(s) = \log \frac{s^p + \varepsilon s + \varepsilon}{s + \varepsilon}.$$

Assume by contradiction that there exists a sequence $\varepsilon_j \to 0$ as $j \to \infty$, and $\|u^{\varepsilon j}\|_{L^\infty} \to \infty$ as $j \to \infty$, where $u^{\varepsilon j}$ solves (2.1), for each $j \in \mathbb{N}$, we set

$$\alpha_j = \|u^{\varepsilon j}\|_{L^\infty}, \quad \beta_j = \inf_{s \geq 0} h_{\varepsilon j}(s), \quad \Omega_j = |\beta_j|\Omega, \quad \widetilde{x} = x/|\beta_j|$$

and define

$$U^{\varepsilon j}(x) = u^{\varepsilon j}(\widetilde{x})/\alpha_j, \quad x \in \Omega_j,$$

clearly, $\|U^{\varepsilon j}\|_{L^\infty(\Omega_j)} = 1$ for all $j \in \mathbb{N}$. On the other hand

$$-\operatorname{div}(|\nabla U^{\varepsilon j}(\widetilde{x})|^{p-2}\nabla U^{\varepsilon j}(\widetilde{x})) = \frac{h_{\varepsilon j}(u^{\varepsilon j}(\widetilde{x})) + \lambda f(\widetilde{x}, u^{\varepsilon j}(\widetilde{x}))}{(\alpha_j)^{p-1}|\beta_j|^{p-1}}$$

As a result, $\|U^{\varepsilon j}\|_{C(\overline{\Omega}_j)} \to 0$ as $j \to \infty$, which contradicts $\|U^{\varepsilon j}\|_{L^\infty(\Omega_j)} = 1$. □

We shall prove that (2.1) has a solution. First we find a supersolution which is independent on ε. Clearly $\underline{u} = 0$ is a subsolution of (2.1). Then our solution $u^\varepsilon \geq 0$.

Lemma 2.3. *Suppose f satisfies* (H1)–(H3)*, then for each $\lambda > 0$, there is a supersolution $\bar{u}$ of (2.1) for $0 < \varepsilon < 1$.*

Proof. First consider the solution Y of the problem

$$\begin{gathered} -\operatorname{div}(|\nabla Y|^{p-2}\nabla Y) = 1, \quad \text{in } \Omega, \\ Y = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

Since Y is bounded in Ω, we choose $\theta > 0$ such that $\theta\|Y\|_{L^\infty} \leq 1$, next we fix $M > 0$ and $c_1 > 0$ in a such way that (see (H_3)), $f(x,u) \leq \theta u^\beta$ for all $u \geq M$ and $f(x,u) \leq c_1$ for all $u \leq M$. In fact we may choose $\theta < \frac{\beta+1}{2pC(N,p)^{\beta+1}}$. We fix $k > 0$ such that

$$k^{p-1} - \log(k^{p-1}\|Y\|_{L^\infty}^{p-1} + 1) \geq \lambda\theta k^{p-1}\|Y\|_{L^\infty}^{p-1}$$

and

$$k^{p-1} - \log(M^{p-1} + 1) \geq c_1,$$

setting $\bar{u^\varepsilon} = \bar{u} = kY$, we obtain a supersolution of (2.1) for all $0 < \varepsilon < 1$. Indeed, recall the definition of h_ε, if $u \geq M$, we have

$$\begin{aligned}
-\Delta_p \bar{u} - h_\varepsilon(\bar{u}) &= k^{p-1} - h_\varepsilon(\bar{u})\\
&\geq k^{p-1} - \log(\bar{u}^{p-1} + 1)\\
&= k^{p-1} - \log(k^{p-1}Y^{p-1} + 1)\\
&\geq k^{p-1} - \log(k^{p-1}\|Y\|_{L^\infty}^{p-1} + 1)\\
&\geq \lambda\theta k^{p-1}\|Y\|_{L^\infty}^{p-1}\\
&\geq \lambda\theta \bar{u}^{p-1} \geq \lambda\theta\bar{u}^\beta\\
&\geq \lambda f(x, \bar{u}).
\end{aligned}$$

Whenever $\bar{u} \leq M$, we obtain

$$\begin{aligned}
-\Delta_p \bar{u} - h_\varepsilon(\bar{u}) &= k^{p-1} - h_\varepsilon(\bar{u}) \geq k^{p-1} - \log(\bar{u}^{p-1} + 1)\\
&\geq k^{p-1} - \log(M^{p-1} + 1) \geq c_1\\
&\geq \lambda f(x, \bar{u}).
\end{aligned}$$

Consequently, $\bar{u}^\varepsilon = kY$ is a supersolution of (2.1) for all $\varepsilon > 0$. $\square$

Lemma 2.4. *Let $0 < \varepsilon < \varepsilon_0$ and $\lambda > 0$ be fixed. Then the problem (2.1) has a solution $u^\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$ be fixed and

$$F_\varepsilon(x, u) = \log\frac{u^p + \varepsilon u + \varepsilon}{u + \varepsilon} + \lambda f(x, u) + a_\varepsilon u$$

where the constant a_ε is fixed in such a way that $u \to F_\varepsilon(x, u)$ is increasing on $[\underline{u}^\varepsilon, \bar{u}^\varepsilon]$ uniformly in $x \in \Omega$. Starting with $u_0 = \bar{u}^\varepsilon$, we define the sequence $\{u_n\}$ of (unique) solution of the problem

$$\begin{gathered}
-\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + a_\varepsilon u_n = F_\varepsilon(x, u_{n-1}), \quad \text{in } \Omega,\\
u_n = 0, \quad \text{on } \partial\Omega,
\end{gathered}\tag{2.3}$$

Then we have $\underline{u}^\varepsilon \leq \dots, \leq u_{n+1} \leq u_n \dots \leq u_0 = \bar{u}^\varepsilon$. In fact, it follows by the comparison principle in lemma 2.1 applied to the problem

$$\begin{gathered}
-\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) + a_\varepsilon u_0 \geq -\operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) + a_\varepsilon u_1, \quad \text{in } \Omega,\\
u_0 \geq u_1, \quad \text{on } \partial\Omega,
\end{gathered}\tag{2.4}$$

that $u_0 \geq u_1 \geq 0$. Similarly, $u^\varepsilon \leq u_1$ in Ω. There is a function u^ε defined by pointwise limit

$$u^\varepsilon(x) = \lim_{n\to\infty} u_n(x), x \in \Omega.$$

By a standard bootstrap argument, we may take the $\lim n \to \infty$, so we conclude that u satisfies (2.1). $\square$

Lemma 2.5. *The pointwise $u(x) = \lim_{\varepsilon\to 0} u^\varepsilon(x)(x \in \Omega)$ is the subsolution of (1.1), in other words*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx + \int_\Omega (-\log u^{p-1})\varphi dx \leq \int_\Omega \lambda f(x, u)\varphi dx \tag{2.5}$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$ in Ω.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$ in Ω, $\lambda > 0$ and recall the definition of h_ε. For each $0 < \varepsilon < \varepsilon_0$, we have

$$\int_\Omega |\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon \nabla\varphi dx = \int_\Omega h(u^\varepsilon)\varphi + \int_\Omega \lambda f(x,u^\varepsilon)\varphi dx \tag{2.6}$$

The dominated convergence theorem implies

$$\lim_{\varepsilon\to 0}\int_\Omega \lambda f(x,u^\varepsilon)\psi dx = \int_\Omega \lambda f(x,u)\varphi dx \tag{2.7}$$

Analogously,

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon\nabla\varphi dx = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi dx \tag{2.8}$$

Since

$$\liminf_{\varepsilon\to 0} -h_\varepsilon(u^\varepsilon) \geq -\log(u^\varepsilon)^{p-1},$$

from the Fatou's Lemma, it follows that

$$\liminf_{\varepsilon\to 0}\int_\Omega -h_\varepsilon(u^\varepsilon)\varphi dx \geq \int_\Omega -\log(u^\varepsilon)^{p-1}\varphi dx \tag{2.9}$$

Letting $\varepsilon \to 0$ in (2.6) and using (2.7)-(2.8), we obtain (2.5). □

3. Supersolutions for (1.1)

In this section we use that that

(F1) $\log u^{p-1} \leq u^{q-1}$ for all $u > 0, q > p$.

As in lemma 2.3, we only consider the case $u \geq M$ and $f(x,u) \leq \theta u^\beta, 0 < \beta < p-1$. In fact, when $u \leq M$ and $\lambda f(x,u) \leq c_1$, we can easily show that the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^{q-1} + C_1, \quad \text{in } \Omega,\\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

has a solution $\beta_0(x)$. Obviously, $\beta_0(x)$ is the supersolution of (1.1). Next we consider a supersolution of (1.1) which comes from the mountain pass lemma. We consider the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^{q-1} + \lambda\theta u^\beta, \quad \text{in } \Omega,\\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

Lemma 3.1 (Mountain pass lemma)**.** *Let E be a Banach space and $I \in C^1(E,R)$ satisfy the Palais-Smale condition. Assume also that:*

1. *$I(0) = 0$;*
2. *There exists constant $r, a > 0$ such that $I(u) \geq a$ if $\|u\| = r$;*
3. *There exists an element $v \in H$ with $\|v\| > r, I(v) \leq 0$.*

Define

$$\Gamma := \{g \in C[0,1]; H : g(0) = 0, g(1) = 1\}.$$

Then $c = \inf_{g\in\Gamma}\max_{0\leq t\leq 1} I[g(t)]$ is a critical value of I.

In the following, we define the space $D^{1,p}(\Omega)$ as the closure of the set $C_c^\infty(\Omega)$ with the norm

$$\|u\|_{D^{1,p}(\Omega)} = \Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p}.$$

Lemma 3.2. *There exists a solution u of the problem (3.2).*

To prove the existence of a solution of (3.2), we will apply the mountain pass lemma to the energy functional

$$J(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\frac{1}{q}\int_\Omega u^q dx-\frac{\lambda\theta}{\beta+1}\int_\Omega u^{\beta+1}dx \tag{3.3}$$

The facts that $D^{1,p}(\Omega)$ is a Banach space (reflexive) and that $J\in C^1(D^{1,p}(\Omega),\mathbb{R})$ satisfies the Palais-Smale condition are basic results (see [2]). It remains to see the two following points to prove that the functional J has a mountain pass geometry:

(C1) There exists $R>0$ and $a>0$ such that $\|u\|_{D^{1,p}(\Omega)}=R$ implies $J(u)\geq a$;
(C2) There exists $u_0\in D^{1,p}(\Omega)$ such that $\|u_0\|_{D^{1,p}(\Omega)}>R$ and $J(u_0)<a$.

Proof of Lemma 3.2. Let $\varepsilon=\int_\Omega|\nabla u|^p dx<1$,

$$J(u)=\frac{\varepsilon}{p}-\frac{1}{q}\int_\Omega u^q dx-\frac{\lambda\theta}{\beta+1}\int_\Omega u^{\beta+1}dx.$$

By Hölder inequality and Sobolev embeddings, we arrive to

$$J(u)\geq\frac{\varepsilon}{p}-\frac{1}{q}C(N,p)^q\varepsilon^q-\frac{\lambda\theta}{\beta+1}C(N,p)^{\beta+1}\varepsilon^{\beta+1}$$

where $C(N,p)$ is the Sobolev constant. We can also choose ε as small as we want such that

$$\frac{C(N,p)^q}{q}\varepsilon^q<\frac{\lambda\theta}{\beta+1}C(N,p)^{\beta+1}\varepsilon^{\beta+1}\quad(q>p>\beta+1). \tag{3.4}$$

So

$$\begin{aligned}J(u)&\geq\frac{\varepsilon}{p}-\frac{2\lambda\theta}{\beta+1}C(N,p)^{\beta+1}\varepsilon^{\beta+1}\\&\geq\frac{\varepsilon}{p}-\frac{2\lambda\theta}{\beta+1}C(N,p)^{\beta+1}\varepsilon\\&=\varepsilon(\frac{1}{p}-\frac{2\lambda\theta}{\beta+1}C(N,p)^{\beta+1}).\end{aligned}$$

Finally, when

$$\theta<(\beta+1)(2pC(N,p)^{\beta+1})^{-1}$$

if we take two constant $R=\varepsilon>0$ and $a=\varepsilon(\frac{1}{p}-\frac{2\lambda\theta}{\beta+1}C(N,p)^{\beta+1})>0$, the functional J satisfies the condition (C1).

Let $u\in C_0^\infty(\Omega)$ fixed such that $u>0$ in Ω, $u\geq 0$ on $\partial\Omega$.

$$J(ku)=\frac{k^p}{p}\int_\Omega|\nabla u|^p dx-\frac{k^q}{q}\int_\Omega u^q dx-\frac{\lambda\theta k^{\beta+1}}{\beta+1}\int_\Omega u^{\beta+1}dx \tag{3.5}$$

for all $k>0$. As $q>p>1$, we obtain $J(ku)\to-\infty$ when $k\to\infty$. So putting $u^0=ku$, there exists some k great enough that $\|u^0\|_{D^{1,p}(\Omega)}>R$ and $J(u^0)<a$ which are exactly satisfying the condition (C2). Thus we have a solution $\beta(x)$ of the problem (3.2) by the mountain pass lemma. It is easy to show that it is the supersolution of (1.1). $\square$

4. Solution for (1.1)

We have obtained a solution for problem (3.2), noted $\beta(x) = \overline{u}$, but affirming that solution is the corresponding supersolution of the subsolution of (1.1), it remains to prove that it is greater than the subsolution $\underline{u}$.

Lemma 4.1 (A comparison principle, [14, Thm 4.1]). *Suppose ψ_1 and ψ_2 satisfies $\psi_1(x,z) \le \psi_2(x,z)$ and let ψ_1 (or ψ_2) satisfy*

(F2) *For each $x \in \Omega$, the function $t \mapsto f(x,t)t^{1-p}$ is decreasing on $(0,\infty)$.*

Furthermore, let $u, v \in W^{1,p}(\Omega)$ with $u \in L^\infty(\Omega), u > 0, v > 0$ on Ω be such that

$$-\Delta_p u \le \psi_1(x,u) \text{ and } -\Delta_p v \ge \psi_2(x,v) \quad \text{on } \Omega.$$

If $u \le v$ on $\partial\Omega$ and $\psi_1(x,u)$ (or $\psi_2(x,u)$) belongs to $L^1(\Omega)$, then $u \le v$ on Ω.

Lemma 4.2. *$\underline{u} < \overline{u}$ in Ω.*

Proof. From section 2 and section 3, we know that

$$-\Delta_p \underline{u} \le \log \underline{u}^{p-1} + \lambda f(x,\underline{u})$$

and

$$-\Delta_p \overline{u} \ge \log \overline{u}^{p-1} + \lambda f(x,\overline{u})$$

in weak sense. From (F1) and (F2), we know that

$$\log u^{p-1} + \lambda f(x,u) < u^{q-1} + \lambda\theta u^\beta$$

and

$$-\Delta_p \overline{u} \ge \overline{u}^{q-1} + \lambda\theta\overline{u}^\beta.$$

Furthermore, $\frac{\log u^{p-1} + \lambda f(x,u)}{u^{p-1}}$ is decreasing on $u \in (0,\infty)$ uniformly in $x \in \Omega$ and $\underline{u} \le \overline{u}$ on $\partial\Omega$, by lemma 4.1, we get $\underline{u} \le \overline{u}$ on Ω, but we clearly know that $\underline{u} \ne \overline{u}$, so $\underline{u} < \overline{u}$ on Ω.

Next we use the sub and super solution from section 2 and section 3 ($\underline{u}$ and $\overline{u}$ respectively) to obtain a solution for (1.1). Define the function

$$G(x,u) = \log u^{p-1} + \lambda f(x,u) + b(x)u, u > 0$$

where we choose b in such a way that the function $u \mapsto G(x,u)$ is increasing in u on $[\underline{u}, \overline{u}]$ for all $x \in \Omega$. ◻

Theorem 4.3. *There exists a solution for (1.1).*

Proof. As noted above we start with $u_0 = \overline{u}$. We define the sequence $\{u_n\}$ of (unique) solution of the problems

$$\begin{gathered} -\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + bu_n = G(x,u_{n-1}), \quad \text{in } \Omega \\ u_n = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{4.1}$$

we apply the comparison principle in lemma 2.1 to the problem

$$\begin{gathered} -\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) + bu_0 \ge -\operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) + bu_1, \quad \text{in } \Omega \\ u_0 \ge u_1, \quad \text{on } \partial\Omega \end{gathered} \tag{4.2}$$

it follows that $u_0 \ge u_1 \ge 0$, similarly, $\underline{u} \le u_1$ in Ω. So $\underline{u} \le \dots, \le u_{n+1} \le u_n \dots \le u_0 = \bar{u}$. There is a function u defined by pointwise limits

$$u(x) = \lim_{n\to\infty} u_n(x), x \in \Omega.$$

We see that $\underline{u} \le u \le \overline{u}, x \in \Omega$. By a standard bootstrap argument, we may take the $\lim n \to \infty$. The function $u(x)$ is in fact a solution of (1.1). □

5. Regularity Properties of the Solution

In this section, we study some regularity properties of the solution to (1.1). Firstly, we state the following lemma, due to DiBenedetto [5], which is the local regularity for the elliptic equation.

Lemma 5.1. *Let $u \in W^{1,p}_{\rm loc}(\Omega) \bigcap L^{\infty}_{\rm loc}(\Omega)$ be a local weak solution of $-\Delta_p u = b(x,r)$ in Ω, an open domain in $\mathbb{R}^N$, where $b(x,r)$ is measurable in $x \in \Omega$ and continuous in $r \in \mathbb{R}$ such that $|b(x,r)| \le \gamma$ on $\Omega \times \mathbb{R}$. Given a sub-domain compact $\Omega' \subset\subset \Omega$, there exists positive constants C_0, C_1 and $\alpha \in (0,1)$, depending only upon $N, p, \gamma, M = ess\sup_{\Omega'} |u|$ and $\operatorname{dist}(\Omega', \Omega)$ such that $\|\nabla u(x)\|_{\infty,\Omega'} \le C_0$ and $x \mapsto \nabla u(x)$ is locally Hölder continuous in Ω'; i.e.,*

$$|u_{x_i}(x) - u_{x_i}(y)| \le C_1 |x-y|^{\alpha}, \quad x, y \in \Omega', i = 1, 2, \dots, N \tag{5.1}$$

Theorem 5.2. *Assume f satisfies* (H1–H2). *For the solution u of (1.1) there holds:*

1. *$u \in C^{1,\alpha}(\Omega)$ where $0 < \alpha < 1$;*
2. *There exists $\underline{\lambda} > 0$ such that, for each $\lambda \ge \underline{\lambda}$, the solution to (1.1) is positive in Ω;*
3. *Let λ_1 be the first eigenvalue of $-\Delta_p$ in $W^{1,p}_0(\Omega)$. There exists $\theta > 0$ such that, if $\lambda_1(\Omega) < \theta$, then $u > 0$ for all $\lambda > 0$.*

Proof. (1) Since we have got the weak solution of (1.1), $u \in W^{1,p}_0(\Omega)$. From the interior $C^{1,\alpha}$ estimate in lemma 5.1, we conclude that $|\nabla u| \in C^{\alpha}(\Omega)$ for some $\alpha \in (0,1)$ and we find that $u \in C^{1,\alpha}(\Omega)$ for $\alpha \in (0,1)$.

(2) We just need to find a strictly positive subsolution. Let Y be the solution of (2.2) and ϕ be the solution of the following problem

$$\begin{gathered} -\operatorname{div}(|\nabla \phi|^{p-2}\nabla \phi) = \lambda f(x, \delta^{\nu}(x)), \quad \text{in } \Omega, \\ \phi = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{5.2}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function independently of λ, and $\nu > 1$ will be fixed latter. Since $f(x, \delta^{\nu}(x))$ is not identically zero in Ω, there exists a constant $C > 0$ such that $\phi \ge 2C\|Y\|_{L^\infty}$. We set $v := \phi - C\|Y\|_{L^\infty}$ and $\underline{u} := kv^{\nu}$, where $k > 0$ to be fixed accordingly. We choose $\Omega' \subset \Omega$ and $\eta_1, \eta_2 > 0$ such that

$$|\nabla v|^p \ge \eta_1 > 0, \quad \text{in} \quad \Omega \backslash \Omega', \quad v \ge \eta_2 > 0 \quad \text{in } \Omega'.$$

Since

$$\begin{aligned} &\log(kv^{\nu})^{p-1} - (k\nu)^{p-1}(\nu-1)(p-1)v^{(\nu-1)(p-1)-1}|\nabla v|^p \\ &\le \log(kv^{\nu})^{p-1} - (k\nu)^{p-1}(\nu-1)(p-1)v^{(\nu-1)(p-1)-1}\eta_1 \\ &\le 0 \quad \text{in } \Omega\backslash\Omega'. \end{aligned}$$

we obtain $\underline{u} = kv^{\nu}$ is strictly positive subsolution for $\underline{\lambda} \le (k\nu)^{p-1}\|v\|^{(\nu-1)(p-1)}|_{L^\infty}$, which proves (2).

(3) Similarly as in the above proof, we need to find a positive subsolution for (1.1) with $\lambda = 0$. Thus, let Y be the solution of (2.2) and φ_1 be the first eigenfunction associated with λ_1. There exists a constant $C > 0$ such that $\varphi_1 \ge 2C\|Y\|_{L^\infty}$. We

set $v := \phi - C\|Y\|_{L^\infty}$ and $\underline{u} := kv^\nu$, where $k > 0$ to be fixed accordingly. Then if $\nu > 1$, we have

$$\begin{aligned}
-\Delta_p \underline{u} &= -(k\nu)^{p-1}(\nu-1)(p-1)v^{(\nu-1)(p-1)-1}|\nabla v|^p \\
&\quad + (k\nu)^{p-1}v^{(\nu-1)(p-1)}\lambda_1|v + C\|Y\|_{L^\infty}|^{p-2}(v + C\|Y\|_{L^\infty}) \\
&\le -(k\nu)^{p-1}(\nu-1)(p-1)v^{(\nu-1)(p-1)-1}\eta_1 \\
&\quad + (k\nu)^{p-1}v^{(\nu-1)(p-1)}\lambda_1(v + C\|Y\|_{L^\infty})^{p-1} \\
&\le (k\nu)^{p-1}v^{(\nu-1)(p-1)}\big[\lambda_1(\|v\|_{L^\infty} + C\|Y\|_{L^\infty})^{p-1} - \frac{(\nu-1)(p-1)\eta_1}{\|v\|_{L^\infty} + C\|Y\|_{L^\infty}}\big].
\end{aligned}$$

Suppose that

$$\lambda_1 < \frac{(\nu-1)(p-1)\eta_1}{\|v\|_{(L^\infty} + C\|Y\|_{L^\infty})^p},$$

then

$$(k\nu)^{p-1}v^{(\nu-1)(p-1)}[\lambda_1(\|v\|_{L^\infty} + C\|Y\|_{L^\infty})^{p-1} - \frac{(\nu-1)(p-1)\eta_1}{\|v\|_{L^\infty} + C\|Y\|_{L^\infty}}] \to -\infty$$

So

$$\begin{aligned}
&-\operatorname{div}(|\nabla \underline{u}|^{p-2}\nabla \underline{u}) \\
&\le (k\nu)^{p-1}v^{(\nu-1)(p-1)}[\lambda_1(\|v\|_{L^\infty} + C\|Y\|_{L^\infty})^{p-1} - \frac{(\nu-1)(p-1)\eta_1}{\|v\|_{L^\infty} + C\|Y\|_{L^\infty}}] \\
&\le \log(kv^\nu)
\end{aligned}$$

for some $k > 0$. The proof is complete. □

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