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A CLASS OF GENERALIZED INTEGRAL OPERATORS

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Abstract. In this paper, we introduce a class of generalized integral operators that includes Fourier integral operators. We establish some conditions on these operators such that they do not have bounded extension on $L^2(\mathbb{R}^n)$. This permit us in particular to construct a class of Fourier integral operators with bounded symbols in $S^0_{1,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and in $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$ which cannot be extended to bounded operators in $L^2(\mathbb{R}^n)$.

1. Introduction

The integral operators of type

$$A\varphi(x) = \int e^{iS(x,\theta)} a(x,\theta)\mathcal{F}\varphi(\theta)d\theta \tag{1.1}$$

appear naturally for solving the hyperbolic partial differential equations and expressing the C^∞-solution of the associate Cauchy problem's (see e.g. [10, 11]).

If we write formally the expression of the Fourier transform $\mathcal{F}\varphi(\theta)$ in (1.1), we obtain the following Fourier integral operators, so-called canonical transformations,

$$A\varphi(x) = \iint e^{i(S(x,\theta)-y\theta)} a(x,y,\theta)\varphi(y)dyd\theta \tag{1.2}$$

in which appear two C^∞-functions, the phase function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ and the amplitude a called the symbol of the operator A. In the particular case where $S(x,\theta) = x\theta$, one recovers the notion of pseudodifferential operators (see e.g [6, 15]).

Since 1970, many of Mathematicians have been interested to these type of operators: Duistermaat [3], Hörmander [6, 7] Kumano-Go [8], and Fujiwara [2]. We mention also the works of Hasanov [4], and the recent results of Messirdi Senoussaoui [12] and Aiboudi-Messirdi-Senoussaoui [1].

In this paper we study a general class of integral operators including the class of Fourier integral operators, specially we are interested in their continuity on $L^2(\mathbb{R}^n)$.

The continuity of the operator A on $L^2(\mathbb{R}^n)$ is guaranteed if the weight of the symbol a is bounded, if this weight tends to zero then A is compact on $L^2(\mathbb{R}^n)$ (see eg. [12]).

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If the symbol a is only bounded the associated Fourier integral operator A is not necessary bounded on $L^2(\mathbb{R}^n)$. Indeed, in 1998 Hasanov [4] constructed an example of unbounded Fourier integral operators on $L^2(\mathbb{R})$.

Aiboudi-Messirdi-Senoussaoui [1] constructed recently in a class of Fourier integral operators with bounded symbols in the Hörmander class $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$ that cannot be extended to be a bounded operator in $L^2(\mathbb{R}^n), n \geq 1$.

These results of unboundedness was obtained by using the properties of the operators

$$B\varphi(x) = \int_{\mathbb{R}^n} k(z)\varphi((b(x)z + a(x))dz \tag{1.3}$$

on $L^2(\mathbb{R}^n), n \geq 1$, where $k(z) \in S(\mathbb{R}^n)$ (the space of C^∞-functions on $\mathbb{R}^n$, whose derivatives decrease faster than any power of $|x|$ as $|x| \to +\infty$), $a(x)$ and $b(x)$ are real-valued, measurable functions on $\mathbb{R}^n$. Operators of type (1.3) was considered by Hasanov [4] and a slightly different way by Aiboudi Messirdi Senoussaoui [1].

We also give in this paper a generalization of these results since we consider a class of integral operators which is general than thus of type (1.3):

$$C\varphi(x) = \int_{\mathbb{R}^n} K(x,z)\varphi(F(x,z))dz \tag{1.4}$$

where $K(x,z)$ and $F(x,z)$ are real-valued, measurable functions on $\mathbb{R}^{2n}$. The generalized integral operator C includes Hilbert, Mellin and the Fourier-Bros-Iagolnitzer transforms which they has been used by many authors and for many purposes, in particular respectively by Hörmander [5] for the analysis of linear partial differential operators, Robert [13] about the functional calculus of pseudodiffrential operators, Sjöstrand [14] in the area of microlocal and semiclassical analysis and Stein [15] for the study of singular integral operators.

The operators C appears also in the study of the width of the quantum resonances (see e.g. [9]).

We shall discuss in the second section bounded extension problems for the class of operators type C. We give some technical conditions on the functions $K(x,z)$ and $F(x,z)$ such that C do not admit a bounded extension on $L^2(\mathbb{R}^n)$. We also indicate a connection between transformations C and Fourier integral operators.

In the third section, we construct an example of Fourier integral with bounded symbols belongs respectively to $S^0_{1,1}(\mathbb{R}^n)$, (the case $n=1$ is given in [4] and generalized for $n \geq 2$ in [1]), and $\cap_{0<\rho<1} S^0_{\rho,1}$ that cannot be extended as a bounded operator on $L^2(\mathbb{R}^n)$, $n \geq 2$. In the case of the Hörmander symbolic class $S^0_{1,1}(\mathbb{R}^n)$ our constructions are direct and technical.

2. Unboundedness of the generalized integral operators

In this section we construct a class of operators C that cannot be extended to be a bounded operator in $L^2(\mathbb{R}^n)$, $n \geq 1$. We have first an easy boundedness criterion of the operator C.

Proposition 2.1. *Let $F(x,.) \in C^1(\mathbb{R}^n)$, and $K(x,.) \in L^2(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$. Suppose that there exits a function $g(x)$ such that*

$$g(x) > 0, \quad \forall x \in \mathbb{R}^n$$
$$|\det\big(\frac{\partial F(x,z)}{\partial z}\big)| \geq g(x), \quad \forall x, z \in \mathbb{R}^n$$
$$\|K(x,.)\|_{L^2(\mathbb{R}^n)}/\sqrt{g(x)} \in L^2(\mathbb{R}^n)$$

then C is a bounded operator on $L^2(\mathbb{R}^n)$.

Proof. Using Hölder inequality and the change of variable $y = F(x,z)$, it's inverse is denoted $z = G(x,y)$, we obtain for all $\varphi \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}
\|C\varphi\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} K(x,z)\varphi(F(x,z))dz\Big|^2 dx \\
&\leq \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} |K(x,z)\varphi(F(x,z))|dz\Big]^2 dx \\
&\leq \int_{\mathbb{R}^n} \Big[\|K(x,.)\|^2_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\varphi(F(x,z))|^2 dz\Big] dx \\
&= \int_{\mathbb{R}^n} \Big[\|K(x,.)\|^2_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\varphi(y)|^2 \Big|\det\big(\frac{\partial F(x,z)}{\partial z}\big)_{(z=G(x,y))}\Big|^{-1} dy\Big] dx \\
&\leq \|\varphi\|^2_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{\|K(x,.)\|^2_{L^2(\mathbb{R}^n)}}{g(x)} dx
\end{aligned} \tag{2.1}$$

hence C is bounded operator on $L^2(\mathbb{R}^n)$ with $\|C\| \leq M = \|\frac{\|K(x,.)\|_{L^2(\mathbb{R}^n)}}{\sqrt{g(x)}}\|_{L^2(\mathbb{R}^n)}$. ∎

Now we give the main result of this paper. We proof that under some conditions the operator C do not admit a bounded extension on $L^2(\mathbb{R}^n)$.

Theorem 2.2. *Let $\delta \in]0,1[$ and the operator C defined by (1.4) on $L^2(\mathbb{R}^n)$ for $x = (x_1,\dots,x_n) \in]0,\delta[^n$ such that:*

(H1) *For $\varepsilon > 0$ and for all $x \in \mathbb{R}^n$*

$$\{z \in \mathbb{R}^n : |F(x,z)| \leq \varepsilon\} = \prod_{i=1}^n [a_i^-(x,\varepsilon),\ a_i^+(x,\varepsilon)]$$

where $a_i^\pm(x,t)$ are real-measurable functions on $\mathbb{R}^n \times]0,+\infty[$ satisfying
1- for any $p \in \mathbb{N}^*$ and $i \in \{1,\dots,n\}$,

$$\lim_{x_i \to 0^+} a_i^\pm(px, x_i) = \pm\infty$$

2- for any $\lambda \in]0,1[$, $i \in \{1,\dots,n\}$ and $p \in \mathbb{N}^*$, the functions $a_i^+(px,\lambda)$ and $a_i^-(px,\lambda)$ are respectively decreasing and increasing with respect to x in $]0,\delta[^n$.

(H2) *There exists a constant $R > 0$ such that for any $r \geq R$ and for all $x \in]0,\delta[^n$*

$$\Big|\int_{[-r,r]^n} K(x,z)dz\Big| \geq \delta$$

Then the operator C cannot be extended to a bounded operator on $L^2(\mathbb{R}^n)$.

Proof. Let us define the generalized sequence of functions

$$\varphi_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in [-\varepsilon, \varepsilon]^n \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

then $\varphi_\varepsilon \in L^2(\mathbb{R}^n)$ for all $\varepsilon > 0$ and we have

$$C\varphi_\varepsilon(x) = \int_{\prod_{i=1}^n [a_i^-(x,\varepsilon), a_i^+(x,\varepsilon)]} K(x,z)dz$$

Consequently,

$$C\varphi_{\varepsilon_j}(x) = \int_{\prod_{i=1}^n [a_i^-(x,\varepsilon_j), a_i^+(x,\varepsilon_j)]} K(x,z)dz \tag{2.3}$$

where $\varepsilon_j \geq 0$ and $\lim_{j\to+\infty} \varepsilon_j = 0$.

By condition 1 of the the assumption $(H1)$, for any $p \in \mathbb{N}^*$ there exists a number $\varepsilon_p \geq 0$ such that

$$a_i^+(p\Lambda_p, \varepsilon_p) \geq R \tag{2.4}$$

and

$$a_i^-(p\Lambda_p, \varepsilon_p) \leq -R \tag{2.5}$$

for $\Lambda_p = (\varepsilon_p, \varepsilon_p, \dots \varepsilon_p)$, $p\varepsilon_p \leq \delta < 1$ and $i \in \{1, \dots, n\}$.

It follows from (2.4), (2.5) and condition 2 of the assumption (H1) that for $x \in]0, p\varepsilon_p]^n$ and $i \in \{1, \dots, n\}$ we have

$$a_i^+(x, \varepsilon_p) \geq a_i^+(p\Lambda_p, \varepsilon_p) \geq R, \tag{2.6}$$

$$a_i^-(x, \varepsilon_p) \leq a_i^-(p\Lambda_p, \varepsilon_p) \leq -R \tag{2.7}$$

Finally using (H2), (2.3), (2.6) and (2.7) we deduce

$$\|C\varphi_{\varepsilon_p}\|^2_{L^2(\mathbb{R}^n)} \geq \int_{]0,p\varepsilon_p]^n} |C\varphi_{\varepsilon p}(x)|^2 dx \geq \delta^2 p^n \varepsilon_p^n \tag{2.8}$$

If we consider that C has a bounded extension to $L^2(\mathbb{R}^n)$, then by virtue of (2.1) we obtain for $\varphi = \varphi_{\varepsilon_p} \in L^2(\mathbb{R}^n)$

$$\delta^2 p^n \varepsilon_p^n \leq \|C\varphi_{\varepsilon_p}\|^2_{L^2(\mathbb{R}^n)} \leq M^2 \varepsilon_p^n$$

and for any $p \in \mathbb{N}^*$

$$p^n \leq \frac{M^2}{\delta^2}$$

This is a contradiction. Consequently A cannot be a bounded operator in $L^2(\mathbb{R}^n)$. □

Remark 2.3. (1) If in particular $K(x,z) = K(z)$ is independent on x and $F(x,z) = b(x) \circ z + a(x)$, where $K(z)$ is a real-valued measurable function, $b(x), a(x) \in \mathbb{R}^n$ are measurable functions on $\mathbb{R}^n$, we obtain the so-called generalized Hilbert transforms introduced in [4]

(2) The operator C is an Fourier integral operator for an appropriate choice of the functions $K(x,z)$ and $F(x,z)$.

$$\begin{aligned} C\varphi(x) &= \int_{\mathbb{R}^n} K(x,z)\varphi(F(x,z))dz \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{iz.\xi} \mathcal{F}K(x,\xi)\varphi(F(x,z))d\xi dz, \end{aligned}$$

where $\mathcal{F}K(x,\xi)$ is the Fourier transform of the partial function $z \to K(x,z)$. Setting $y = F(x,z)$ and $z = G(x,y)$, we have

$$C\varphi(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{iG(x,y).\xi}\mathcal{F}K(x,\xi)\varphi(y)|\det(\frac{\partial G}{\partial y})|d\xi dy$$

which is a Fourier integral operator with the phase function $\phi(x,y,\xi) = G(x,y).\xi$ and the symbol $p(x,y,\xi) = \mathcal{F}K(x,\xi)|\det(\frac{\partial G}{\partial y})|$ if K and G are infinitely regular with respect to x, y and ξ.

3. A class of unbounded Fourier integral operators on $L^2(\mathbb{R}^n)$

It follows from theorem 2.2 that with an appropriate choice of $K(x,z)$ and $F(x,z)$ we can construct a class of Fourier integral operators which cannot be extended as bounded operators on $L^2(\mathbb{R}^n)$.

An example of unbounded fourier integral operator with a symbol in $S^0_{1,1}(\mathbb{R}\times\mathbb{R})$ and $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n\times\mathbb{R}^n)$ was given respectively in [4] and [1], where if $\rho \in \mathbb{R}$,

$$\begin{gathered} S^0_{\rho,1}(\mathbb{R}^n\times\mathbb{R}^n) = \big\{p \in C^\infty(\mathbb{R}^n\times\mathbb{R}^n) : \forall(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n \exists C_{\alpha,\beta} > 0; \\ |\partial_x^\alpha\partial_\theta^\beta p(x,\theta)| \le C_{\alpha,\beta}\lambda^{-\rho|\beta|+|\alpha|}(\theta) \end{gathered} \tag{3.1}$$

3.1. A class with symbols in $S^0_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$.

Hre, we generalize the example given by Hasanov on $\mathbb{R}$ to high dimensions. Namely, in the same spirit of [8]. we have easily if we get $K(z) \in \mathcal{S}(\mathbb{R}^n)$ and $b \in C^\infty(\mathbb{R}^n,\mathbb{R})$.

Proposition 3.1. *If $K(z) \in \mathcal{S}(\mathbb{R}^n)$ and $b \in C^\infty(\mathbb{R}^n,\mathbb{R})$, then for all $\alpha,\beta \in \mathbb{N}^n$ there exists $C_{\alpha\beta} > 0$ such that*

$$|\partial_x^\alpha\partial_\xi^\beta K(b(x)\xi)| \le C_{\alpha\beta}(1+|\xi|)^{|\alpha|-|\beta|} \tag{3.2}$$

for all $(x,\xi) \in [-1,1]^n\times\mathbb{R}^n$.

Proof. It suffices to use the fact that $K \in \mathcal{S}(\mathbb{R}^n)$ and β is bounded on $[-1,1]^n$. □

Let also $a = (a_1, a_2, \dots, a_n) \in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ such that a, b, K satisfy (H1) and (H2), with

$$\begin{gathered} b(x) > 0 \\ a_i^\pm(x,t) = \frac{\pm t + a_i(x)}{b(x)}, \quad t > 0,\ x \in \mathbb{R}^n \end{gathered} \tag{3.3}$$

Then, for $q(x,\xi) = K(b(x)\xi)$ defined on $[-1,1]^n\times\mathbb{R}^n$, we have

$$|\partial_x^\alpha\partial_\xi^\beta q(x,\xi)| \le C_{\alpha\beta}(1+|\xi)^{|\alpha|-|\beta|}$$

on $[-1,1]^n\times\mathbb{R}^n$, $\alpha,\beta \in \mathbb{N}^n$, $C_{\alpha\beta}$ being constants.

Thus, $q \in S^0_{1,1}([-1,1]^n\times\mathbb{R}^n)$, in particular $q(x,\xi)$ is a well bounded symbol. Take a function $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\eta \subset [-1,1]^n$ and $\eta(x) = 1$ for $x \in [-\delta,\delta]^n$, $\delta < 1$. It is now obvious to see that the function $p(x,\xi) = \eta(x)q(x,\xi) \in S^0_{1,1}(\mathbb{R}^n\times\mathbb{R}^n)$.

Now the Fourier integral operator defined by

$$\begin{aligned} C\varphi(x) &= \int_{\mathbb{R}^{2n}} e^{-i(a(x).\xi+y.\xi)}p(x,\xi)\varphi(\xi)dyd\xi \\ &= \int_{\mathbb{R}^{2n}} e^{-i(a(x).\xi+y.\xi)}\eta(x)K(b(x)\xi)\varphi(\xi)dyd\xi \end{aligned}$$

is of the type (1.4). Indeed, for $s = b(x)\xi$ and $x \in]0, \delta]^n$

$$C\varphi(x) = \int_{\mathbb{R}^{2n}} e^{-i\frac{(a(x)+t).s}{\beta(x)}} K(s) \frac{1}{b^n(x)} \varphi(y) dy ds$$

Finally, if we pose $\frac{a(x)+y}{b(x)} = z$, we have

$$C\varphi(x) = \int \mathcal{F}K(z) \varphi(b(x)z - a(x)) dz$$

By theorem 2.2, we conclude that the operator C cannot be extended as a bounded operator on $L^2(\mathbb{R}^n)$.

3.2. **A class with symbols in $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$.** We describe in this section the results of Aiboudi-Messirdi-Senoussaoui [1], they constructed a class of unbounded Fourier integral operators with a separated variables phase function and a symbol in the Hörmander class $\bigcap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$.

Precisely, let $K \in S(\mathbb{R})$ with $K(t) = 1$ on $[-\delta, \delta]$ and $b(t)$ is continuous function on $[0, 1]$ such that

$$\begin{gathered} b(t) \in C^\infty(]0,1]), \quad b(0) = 0, \quad b'(t) > 0 \text{ in }]0,1] \\ |b^{(k)}(t)| \leq \frac{C_k}{t^k} \text{ in }]0,1], \ k \in \mathbb{N}^*, C_k > 0 \end{gathered} \tag{3.4}$$

$\chi(x), \psi(\xi) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ homogeneous of degree 1. Thus the function

$$q(x,\xi) = e^{-i\chi(x)\psi(\xi)} \prod_{j=1}^n K(b(|x|)|x|\xi_j), \quad \xi = (\xi_1, \dots, \xi_n) \tag{3.5}$$

belongs to $C^\infty([-1,1]^n \times \mathbb{R}^n)$ and satisfies, as in the proposition 3.1, the following estimates

Proposition 3.2. *For all α, β in $\mathbb{N}^n$,*

$$|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \leq C_{\alpha\beta} \frac{(1+|\xi|)^{|\alpha|-|\beta|}}{b((1+|\xi|)^{-1})^{|\beta|}} \tag{3.6}$$

on $[-1,1]^n \times \mathbb{R}^n$ where $C_{\alpha\beta} > 0$.

Now if $\phi(x)$ is a $C_0^\infty(\mathbb{R})$-function such that

$$\begin{gathered} \phi(s) = 1 \quad \text{on } [-\delta, \delta], \ \delta < 1 \\ \operatorname{supp} \phi \subset [-1, 1] \end{gathered}$$

define the global C^∞ symbol on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\begin{gathered} p(x,\xi) = e^{-i\chi(x)\psi(\xi)} \prod_{j=1}^n \phi(x_j) K(b(|x|)|x|\xi_j) \\ x = (x_1, \dots x_n), \quad \xi = (\xi_1, \dots, \xi_n). \end{gathered} \tag{3.7}$$

Then $p(x,\xi) \in \cap_{0<\rho<1} S^0_{\rho,1}(\mathbb{R}^n \times \mathbb{R}^n)$ and the corresponding Fourier integral operator is

$$\begin{aligned} C\varphi(x) &= \int_{\mathbb{R}^n} e^{i\chi(x)\psi(\xi)} p(x,\xi) \mathcal{F}\varphi(\xi) d\xi \\ &= \prod_{j=1}^n \phi(x_j) \int_{\mathbb{R}^n} K(b(|x|)|x|\xi_j) \mathcal{F}\varphi(\xi) d\xi \end{aligned} \tag{3.8}$$

By using an adequate change of variable in the integral (3.8), we have

$$C\varphi(x) = \int_{\mathbb{R}^n} \varphi(b(|x|)|x|z) \prod_{j=1}^{n} \mathcal{F}K(z_j)d\xi, \quad z = (z_1, \dots, z_n) \tag{3.9}$$

which is of the form C in theorem 2.2 where the functions $F(x,z) = b(|x|)|x|z$ and $K(x,z) = \prod_{j=1}^{n} \mathcal{F}K(z_j)$ satisfy (H1) and (H2). Consequently, the operator C cannot be continuously extended on $L^2(\mathbb{R}^n)$.

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