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SECOND-ORDER BOUNDARY ESTIMATES FOR SOLUTIONS TO SINGULAR ELLIPTIC EQUATIONS

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ABSTRACT. Let $\Omega \subset R^N$ be a bounded smooth domain. We investigate the effect of the mean curvature of the boundary $\partial\Omega$ in the behaviour of the solution to the homogeneous Dirichlet boundary value problem for the singular semilinear equation $\Delta u + f(u) = 0$. Under appropriate growth conditions on $f(t)$ as t approaches zero, we find an asymptotic expansion up to the second order of the solution in terms of the distance from x to the boundary $\partial\Omega$.

1. INTRODUCTION

In this article, we study the Dirichlet problem

$$\begin{gathered}\Delta u + f(u) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and $f(t)$ is a decreasing and positive smooth function in $(0, \infty)$, which approaches infinity as $t \to 0$. Equation (1.1) arises in problems of heat conduction and in fluid mechanics.

Problems with singular data are discussed in many papers; see, for instance, [8, 9, 10, 12, 16, 17] and references therein. Let $f(t) = t^{-\gamma}$. For $\gamma > 0$, in [7] it is shown that there exists a positive solution continuous up to the boundary $\partial\Omega$. For $\gamma > 1$, in [13] it is shown that the solution u satisfies

$$0 < c_1 \leq u(x)(\delta(x))^{-\frac{2}{1+\gamma}} \leq c_2,$$

where $\delta = \delta(x)$ denotes the distance of x from the boundary $\partial\Omega$. Actually, in [7] and in [13] the more general equation $\Delta u + p(x)u^{-\gamma} = 0$ with $p(x) > 0$ is discussed. For equation (1.1) in case $1 < \gamma < 3$, in [6] it is shown that there exists a constant $B > 0$ such that

$$\Big| u(x) - \Big(\frac{\gamma + 1}{\sqrt{2(\gamma - 1)}} \delta \Big)^{\frac{2}{1+\gamma}} \Big| < B\delta^{\frac{2\gamma}{\gamma+1}}.$$

For $\gamma > 3$, in [15] it is proved that

$$\Big| u(x) - \Big(\frac{\gamma + 1}{\sqrt{2(\gamma - 1)}} \delta \Big)^{\frac{2}{1+\gamma}} \Big| < B\delta^{\frac{\gamma+3}{\gamma+1}}.$$

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In [1], for $\gamma > 3$, it is proved that

$$u(x) = \Big(\frac{\gamma+1}{\sqrt{2(\gamma-1)}}\delta\Big)^{\frac{2}{1+\gamma}}\Big[1 + \frac{N-1}{3-\gamma}K\delta + o(\delta)\Big],$$

where $K = K(x)$ stands for the mean curvature of the surface $\{x \in \Omega : \delta(x) = \text{constant}\}$.

In this article we extend the latter estimate to more general nonlinearities. More precisely, assume

$$\frac{f'(t)F(t)}{f^2(t)} = \frac{\gamma}{1-\gamma} + O(1)t^\beta, \quad F(t) = \int_t^1 f(\tau)d\tau \tag{1.2}$$

where $\gamma > 3$, $\beta > 0$ and $O(1)$ denotes a bounded quantity as $t \to 0$. In addition, we suppose there is M finite such that for all $\theta \in (1/2, 2)$ and for t small we have

$$\frac{|f''(\theta t)|t^2}{f(t)} \le M. \tag{1.3}$$

An example which satisfies these conditions is $f(t) = t^{-\gamma} + t^{-\nu}$ with $0 < \nu < \gamma$. Here $\beta = \min[\gamma - \nu, \gamma - 1]$.

When $\phi(\delta)$ is defined as

$$\int_0^{\phi(\delta)} (2F(t))^{-1/2} dt = \delta \tag{1.4}$$

and $\gamma > 3$, we prove that

$$u(x) = \phi(\delta)\Big[1 + \frac{N-1}{3-\gamma}K\delta + O(1)\delta^{\sigma+1}\Big], \tag{1.5}$$

where σ is any number such that $0 < \sigma < \min[\frac{\gamma-3}{\gamma+1}, \frac{2\beta}{\gamma+1}]$. Note that ϕ is a solution to the one dimensional problem

$$\phi'' + f(\phi) = 0, \quad \phi(0) = 0.$$

The estimate (1.5) shows that the expansion of $u(x)$ in terms of δ has the first part which is independent of the geometry of the domain, and the second part which depends on the mean curvature of the boundary as well as on γ. For $1 < \gamma < 3$, the first part of the expansion is still independent of the geometry of the domain, but it is not possible to find the second part in terms of the mean curvature even in the special case $f(t) = t^{-\gamma}$, see [1] for details.

We observe that similar results are known for boundary blow-up problems, see [3, 4, 5]. In [3] the problem

$$\Delta u = f(u), \quad u \to \infty \quad \text{as } x \to \partial\Omega$$

is discussed under the assumption that $f(t)$ is positive, increasing and satisfying

$$\frac{f'(t)\mathcal{F}(t)}{f^2(t)} = \frac{p}{1+p} + O(1)t^{-\beta}, \quad \mathcal{F}(t) = \int_0^t f(\tau)d\tau,$$

where $p > 1$, $\beta > 0$ and $O(1)$ is a bounded quantity as $t \to \infty$. Under some additional conditions for f, in [3] it is shown that

$$u(x) = \Phi(\delta)\Big[1 + \frac{N-1}{p+3}K\delta + o(\delta)\Big], \tag{1.6}$$

where Φ is defined as
$$\int_{\Phi(\delta)}^{\infty} (2F(t))^{-1/2} dt = \delta.$$
Note that Φ satisfies
$$\Phi'' = f(\Phi), \quad \Phi(0) = \infty.$$
We underline that (1.6) holds for $p > 1$, in contrast to (1.5) which holds when $\gamma > 3$. Moreover, when $f(t) = t^p$ with p close to one, it is possible to find other terms in the expansion (1.6). For example, the third term depends on the mean curvature K as well as on its gradient ∇K (see [2]). Instead, the estimate (1.5) cannot be improved for any value of γ because $0 < \sigma < 1$.

2. Preliminary results

Let $f(t)$ be a decreasing and positive smooth function in $(0, \infty)$, which approaches infinity as $t \to 0$. Assume condition (1.2) with $\gamma > 1$ and $\beta > 0$. Let us rewrite (1.2) as
$$(F(t))^{\frac{1}{1-\gamma}} \Big(\frac{(F(t))^{\frac{\gamma}{\gamma-1}}}{f(t)} \Big)' = O(1) t^{\beta}. \tag{2.1}$$
Since by [14, Lemma 2.1],
$$\lim_{t \to 0} \frac{F(t)}{f(t)} = 0, \tag{2.2}$$
integration by parts on $(0, t)$ of (2.1) yields
$$\frac{F(t)}{t f(t)} = \frac{1}{\gamma - 1} + O(1) t^{\beta}. \tag{2.3}$$
Using the latter estimate and (1.2) again we find
$$\frac{t f'(t)}{f(t)} = -\gamma + O(1) t^{\beta}. \tag{2.4}$$
Let us write (2.4) as
$$\frac{f'(t)}{f(t)} = -\frac{\gamma}{t} + O(1) t^{\beta - 1}.$$
Integration over $(t, 1)$ yields
$$\log \frac{f(1)}{f(t)} = \log t^{\gamma} + O(1).$$
Therefore, we can find two positive constants C_1, C_2 such that
$$C_1 t^{-\gamma} < f(t) < C_2 t^{-\gamma}, \quad \forall t \in (0, 1). \tag{2.5}$$
Since $F(t) = \int_t^1 f(\tau) d\tau$, using (2.5) we find two positive constants C_3, C_4 such that
$$C_3 t^{1-\gamma} < F(t) < C_4 t^{1-\gamma}, \quad \forall t \in (0, 1/2). \tag{2.6}$$

Lemma 2.1. *Let $A(\rho, R) \subset \mathbb{R}^N$, $N \geq 2$, be the annulus with radii ρ and R centered at the origin, let $f(t) > 0$ smooth, decreasing for $t > 0$ and such that $f(t) \to \infty$ as $t \to 0$. Assume condition (1.2) with $\gamma > 3$. If $u(x)$ is a solution to problem (1.1) in $\Omega = A(\rho, R)$ and $v(r) = u(x)$ for $r = |x|$, then*
$$v(r) > \phi(R - r) - C \frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}} (R - r), \quad \tilde{r} < r < R, \tag{2.7}$$

and

$$v(r) < \phi(r-\rho) + C\phi'(r-\rho)\frac{\int_v^1 (F(t))^{1/2}dt}{F(v)}(r-\rho), \quad \rho < r < \overline{r}, \tag{2.8}$$

where ϕ is defined as in (1.4), $\rho < \overline{r} \le \tilde{r} < R$ *and C is a suitable positive constant.*

Proof. If $\Omega = A(\rho, R)$, the corresponding solution $u(x)$ to problem (1.1) is radial. With $v(r) = u(x)$ for $r = |x|$ we have

$$v'' + \frac{N-1}{r}v' + f(v) = 0, \quad v(\rho) = v(R) = 0. \tag{2.9}$$

It is easy to show that there is r_0 such that $v(r)$ is increasing for $\rho < r < r_0$ and decreasing for $r_0 < r < R$, with $v'(r_0) = 0$. Multiplying (2.9) by v' and integrating over (r_0, r) we find

$$\frac{(v')^2}{2} + (N-1)\int_{r_0}^r \frac{(v')^2}{s}ds = F(v) - F(v_0), \quad v_0 = v(r_0). \tag{2.10}$$

By (2.6), $F(t) \to \infty$ as $t \to 0$. Therefore, $F(v(r)) \to \infty$ as $r \to R$, and (2.10) implies that

$$|v'| < 2(F(v))^{1/2}, \quad r \in (r_1, R). \tag{2.11}$$

As a consequence we have

$$\int_{r_0}^r \frac{(v')^2}{s}ds \le \frac{2}{r_0}\int_{r_0}^r (F(v))^{1/2}(-v')ds = \frac{2}{r_0}\int_v^{v_0}(F(t))^{1/2}dt. \tag{2.12}$$

Since

$$\int_v^{v_0}(F(t))^{1/2}dt \le (F(v))^{1/2}v_0, \tag{2.13}$$

using (2.12) we find

$$\lim_{r\to R}\frac{\int_{r_0}^r \frac{(v')^2}{s}ds}{F(v)} = \lim_{r\to R}\frac{\int_v^{v_0}(F(t))^{1/2}dt}{F(v)} = 0.$$

On the other hand, by (2.10) we find

$$\frac{(v')^2}{2F(v)} = 1 - \frac{(N-1)\int_{r_0}^r \frac{(v')^2}{s}ds + F(v_0)}{F(v)}.$$

The above equation yields

$$\frac{-v'}{(2F(v))^{1/2}} = 1 - \Gamma(r), \tag{2.14}$$

where

$$\Gamma(r) = 1 - \Big[1 - \frac{(N-1)\int_{r_0}^r \frac{(v')^2}{s}ds + F(v_0)}{F(v)}\Big]^{1/2}.$$

Let us write equation (2.9) as

$$(r^{N-1}v')'(r^{N-1}v') + r^{2N-2}f(v)v' = 0.$$

Integration over (r_0, r) yields

$$\frac{(r^{N-1}v')^2}{2} \ge r_0^{2N-2}[F(v) - F(v_0)],$$

from which we find that $|v'| > c(F(v))^{1/2}$ for some positive constant c. Hence,

$$\int_{r_0}^{r} \frac{(v')^2}{s} ds > \frac{c}{r_0} \int_{r_0}^{r} (F(v))^{1/2}(-v')ds = \frac{c}{r_0} \int_{v}^{v_0} (F(t))^{1/2}dt.$$

By using the estimate (2.6) for $F(t)$, the latter inequality implies that $\int_{r_0}^{r} \frac{(v')^2}{s} ds \to \infty$ as $r \to R$. As a consequence, we have

$$(N-1)\int_{r_0}^{r} \frac{(v')^2}{s} ds + F(v_0) > 0$$

for r close to R. Since for $0 < \epsilon < 1$, we have $1 - [1-\epsilon]^{1/2} < \epsilon$. Using (2.12) we find a constant M such that

$$0 \le \Gamma(r) \le \frac{\frac{2(N-1)}{r_0}\int_{v}^{v_0}(F(t))^{1/2}dt + F(v_0)}{F(v)} \le M\frac{\int_{v}^{v_0}(F(t))^{1/2}dt}{F(v)}. \tag{2.15}$$

The inverse function of ϕ is

$$\psi(s) = \int_0^s \frac{1}{(2F(t))^{1/2}} dt.$$

Integration of (2.14) over (r, R) yields

$$\psi(v) = R - r - \int_r^R \Gamma(s)ds,$$

from which we find

$$v(r) = \phi\big(R - r - \int_r^R \Gamma(s)ds\big). \tag{2.16}$$

By (2.16) we have

$$v(r) = \phi(R-r) - \phi'(\omega)\int_r^R \Gamma(s)ds, \tag{2.17}$$

with

$$R - r > \omega > R - r - \int_r^R \Gamma(s)ds.$$

Since

$$\phi'(\omega) = (2F(\phi(\omega)))^{1/2},$$

and since the function $t \to F(\phi(t))$ is decreasing we have

$$\phi'(\omega) < \Big(2F\big(\phi(R - r - \int_r^R \Gamma(s)ds)\big)\Big)^{1/2} = (2F(v))^{1/2},$$

where (2.16) has been used in the last step. Hence, by (2.17) we find

$$v(r) > \phi(R-r) - (2F(v))^{1/2}\int_r^R \Gamma(s)ds.$$

Using (2.15) we also have

$$v(r) > \phi(R-r) - (2F(v))^{1/2}M\int_r^R \frac{\int_v^{v_0}(F(t))^{1/2}dt}{F(v)} ds. \tag{2.18}$$

We claim that the function

$$r \to \frac{\int_{v(r)}^{v_0}(F(t))^{1/2}dt}{F(v(r))}$$

is decreasing for r close to R. This is equivalent to show that the function

$$s \to (F(s))^{-1} \int_s^{v_0} (F(t))^{1/2} dt$$

is increasing for s close to 0. We have

$$\begin{aligned}\frac{d}{ds}\Big[(F(s))^{-1} \int_s^{v_0} (F(t))^{1/2} dt\Big] &= (F(s))^{-2} f(s) \int_s^{v_0} (F(t))^{1/2} dt - (F(s))^{-1/2} \\ &= (F(s))^{-1/2} \Big[\frac{\int_s^{v_0} (F(t))^{1/2} dt}{(F(s))^{\frac{3}{2}} (f(s))^{-1}} - 1\Big].\end{aligned}$$

By (2.5) and (2.6) it follows that $\int_s^{v_0} (F(t))^{1/2} dt$ and $(F(s))^{\frac{3}{2}} (f(s))^{-1}$ tend to ∞ as $s \to 0$. Therefore, we can use de l'Hôpital rule and condition (1.2) to find

$$\lim_{s\to 0} \frac{\int_s^{v_0} (F(t))^{1/2} dt}{(F(s))^{\frac{3}{2}} (f(s))^{-1}} - 1 = \lim_{s\to 0} \frac{2}{3 + 2(F(s))(f(s))^{-2} f'(s)} - 1 = \frac{\gamma + 1}{\gamma - 3}.$$

Hence, since $\gamma > 3$, the function $(F(s))^{-1/2} \int_s^{v_0} (F(t))^{1/2} dt$ is increasing for s close to 0, and the claim follows. Using this fact, inequality (2.7) follows from (2.18).

To prove (2.8), let us write equation (2.10) as

$$\frac{(v')^2}{2} = F(v) - F(v_0) + (N-1) \int_r^{r_0} \frac{(v')^2}{s} ds, \tag{2.19}$$

with $\rho < r < r_0$. Note that, since $(v'(r))^2 \to \infty$ as $r \to \rho$ and $v'' > 0$, we have [14, Lemma 2.1]

$$\lim_{r\to\rho} \frac{\int_r^{r_0} \frac{(v')^2}{t} dt}{(v')^2} = 0.$$

Hence, (2.19) implies $0 < v' < 2(F(v))^{1/2}$ for r near to ρ. As a consequence we have

$$\int_r^{r_0} \frac{(v')^2}{s} ds \le \frac{2}{\rho} \int_r^{r_0} (F(v))^{1/2} (v') ds = \frac{2}{\rho} \int_v^{v_0} (F(t))^{1/2} dt.$$

Since $\int_v^{v_0} (F(t))^{1/2} dt \le (F(v))^{1/2} v_0$, the latter estimate implies that

$$\lim_{r\to\rho} \frac{\int_r^{r_0} \frac{(v')^2}{s} ds}{F(v)} = 0.$$

Using (2.10) again we find

$$\frac{(v')^2}{2F(v)} = 1 + \frac{(N-1)\int_r^{r_0} \frac{(v')^2}{s} ds - F(v_0)}{F(v)}.$$

The above equation yields

$$\frac{v'}{(2F(v))^{1/2}} = 1 + \Gamma(r), \tag{2.20}$$

where

$$\Gamma(r) = \Big[1 + \frac{(N-1)\int_r^{r_0} \frac{(v')^2}{s} ds - F(v_0)}{F(v)}\Big]^{1/2} - 1.$$

Arguing as in the previous case one proves that $\int_r^{r_0} \frac{(v')^2}{s} ds \to \infty$ as $r \to \rho$, and

$$(N-1) \int_r^{r_0} \frac{(v')^2}{s} ds - F(v_0) > 0$$

for r close to ρ. Since for $\epsilon > 0$ we have $[1+\epsilon]^{1/2} - 1 < \epsilon$, the function $\Gamma(r)$ satisfies (2.15) (possibly with a different constant M). Integration of (2.20) over (ρ, r) yields

$$\psi(v) = r - \rho + \int_\rho^r \Gamma(s) ds,$$

from which we find

$$v(r) = \phi(r-\rho) + \phi'(\omega_1) \int_\rho^r \Gamma(s) ds, \tag{2.21}$$

with

$$r - \rho < \omega_1 < r - \rho + \int_\rho^r \Gamma(s) ds.$$

Since $\phi'(s)$ is decreasing, $\phi'(\omega_1) < \phi'(r-\rho)$. This estimate and (2.21) imply

$$v(r) < \phi(r-\rho) + \phi'(r-\rho) \int_\rho^r \Gamma(s) ds. \tag{2.22}$$

We have shown that the function $(F(s))^{-1/2} \int_s^{v_0} (F(t))^{1/2} dt$ is increasing for s close to 0. As a consequence, since $v(r)$ is increasing for $\rho < r$, the function

$$r \to \frac{\int_{v(r)}^{v_0} (F(t))^{1/2} dt}{F(v(r))}$$

is increasing for r close to ρ. Hence, inequality (2.8) follows from (2.22) and (2.15). The lemma is proved. $\square$

Corollary 2.2. *Assume the same notation and assumptions of Lemma 2.1. Given $\epsilon > 0$ there are r_1 and r_2 such that*

$$\phi(R-r) > v(r) > (1-\epsilon)\phi(R-r), \quad r_1 < r < R, \tag{2.23}$$
$$\phi(r-\rho) < v(r) < (1+\epsilon)\phi(r-\rho), \quad \rho < r < r_2. \tag{2.24}$$

Proof. By (2.14) we have

$$\frac{-v'}{(2F(v))^{1/2}} < 1.$$

Integrating over (r, R) we find $\psi(v) < R - r$, from which the left hand side of (2.23) follows. By (2.7) we have

$$v(r) > \Big[1 - C \frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}} \frac{R-r}{\phi(R-r)}\Big] \phi(R-r).$$

Since $F(t)$ is decreasing we have

$$\frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}} \le 1.$$

Moreover, putting $R - r = \psi(s)$ we have

$$\lim_{r\to R} \frac{R-r}{\phi(R-r)} = \lim_{s\to 0} \frac{\psi(s)}{s} \le \lim_{s\to 0} (2F(s))^{-1/2} = 0.$$

The right hand side of (2.23) follows from these estimates. By (2.20) we have

$$\frac{v'}{(2F(v))^{1/2}} > 1.$$

Integrating over (ρ, r) we find $\psi(v) > r-\rho$, from which the left hand side of (2.24) follows. By (2.8) we have

$$v(r) < \Big[1 + C\phi'(r-\rho)\frac{\int_v^1 (F(t))^{1/2}dt}{F(v)}\frac{r-\rho}{\phi(r-\rho)}\Big]\phi(r-\rho).$$

We find

$$\lim_{r\to\rho}\frac{\int_v^1 (F(t))^{1/2}dt}{F(v)} \le \lim_{r\to\rho}\frac{1}{(F(v))^{1/2}} = 0.$$

Moreover, putting $r-\rho = \psi(s)$ we have

$$\frac{(r-\rho)\phi'(r-\rho)}{\phi(r-\rho)} = \frac{\psi(s)(2F(s))^{1/2}}{s} \le 1.$$

The right hand side of (2.24) follows from these estimates. The corollary is proved. $\square$

Lemma 2.3. *If* (1.2) *holds with* $\gamma > 1$ *and if* $\phi(\delta)$ *is defined as in* (1.4), *then*

$$\frac{\phi'(\delta)}{\delta f(\phi(\delta))} = \frac{\gamma+1}{\gamma-1} + O(1)(\phi(\delta))^{\beta}, \tag{2.25}$$

$$\frac{\phi(\delta)}{\delta\phi'(\delta)} = \frac{\gamma+1}{2} + O(1)(\phi(\delta))^{\beta}, \tag{2.26}$$

$$\frac{\phi(\delta)}{\delta^2 f(\phi(\delta))} = \frac{(\gamma+1)^2}{2(\gamma-1)} + O(1)(\phi(\delta))^{\beta}, \tag{2.27}$$

$$\phi(\delta) = O(1)\delta^{\frac{2}{\gamma+1}}, \tag{2.28}$$

where $O(1)$ *is a bounded quantity as* $\delta \to 0$.

Proof. By the relation

$$-1 - 2\Big(\frac{\gamma}{1-\gamma} + O(1)t^{\beta}\Big) = \frac{\gamma+1}{\gamma-1} + O(1)t^{\beta},$$

using (1.2) we have

$$-1 - 2F(t)f'(t)(f(t))^{-2} = \frac{\gamma+1}{\gamma-1} + O(1)t^{\beta},$$

where $O(1)$ is bounded as $t \to 0$. Multiplying by $(2F(t))^{-1/2}$ we find

$$-(2F(t))^{-1/2} - (2F(t))^{1/2}f'(t)(f(t))^{-2} = \frac{\gamma+1}{\gamma-1}(2F(t))^{-1/2} + O(1)t^{\beta}(2F(t))^{-1/2},$$

and

$$\big((2F(t))^{1/2}(f(t))^{-1}\big)' = \frac{\gamma+1}{\gamma-1}(2F(t))^{-1/2} + O(1)t^{\beta}(2F(t))^{-1/2}. \tag{2.29}$$

Using (2.5) and (2.6) we find that $(2F(t))^{1/2}(f(t))^{-1} \to 0$ as $t \to 0$. Hence, integrating (2.29) on $(0, s)$ we obtain

$$(2F(s))^{1/2}(f(s))^{-1} = \frac{\gamma+1}{\gamma-1}\int_0^s \big(2F(t)\big)^{-1/2}dt + O(1)\int_0^s t^{\beta}(2F(t))^{-1/2}dt, \tag{2.30}$$

where $O(1)$ is bounded as $s \to 0$. Since

$$\int_0^s t^{\beta}(2F(t))^{-1/2}dt \le s^{\beta}\int_0^s (2F(t))^{-1/2}dt,$$

equation (2.30) implies

$$\frac{(2F(s))^{1/2}}{f(s)} = \frac{\gamma+1}{\gamma-1}\int_0^s \big(2F(t)\big)^{-1/2}dt + O(1)s^\beta \int_0^s (2F(t))^{-1/2}dt.$$

Putting $s = \phi(\delta)$ and recalling that $\phi'(\delta) = (2F(\phi(\delta)))^{1/2}$, estimate (2.25) follows.

Recall that (1.2) implies (2.3). Hence, we have

$$\begin{gathered}
\frac{tf(t)}{2F(t)} = \frac{\gamma-1}{2} + O(1)t^\beta, \\
\frac{2F(t)+tf(t)}{2F(t)} = \frac{\gamma+1}{2} + O(1)t^\beta, \\
\frac{(2F(t))^{-1/2} + tf(t)(2F(t))^{-3/2}}{(2F(t))^{-1/2}} = \frac{\gamma+1}{2} + O(1)t^\beta, \\
\big(t(2F(t))^{-1/2}\big)' = \frac{\gamma+1}{2}(2F(t))^{-1/2} + O(1)t^\beta(2F(t))^{-1/2}.
\end{gathered}$$

By (2.6), $t(2F(t))^{-1/2} \to 0$ as $t \to 0$. Hence, integrating over $(0, s)$ we find

$$s(2F(s))^{-1/2} = \frac{\gamma+1}{2}\psi(s) + O(1)\int_0^s t^\beta (2F(t))^{-1/2}dt = \frac{\gamma+1}{2}\psi(s) + O(1)s^\beta\psi(s),$$

where

$$\psi(s) = \int_0^s (2F(t))^{-1/2}dt.$$

Since $\psi'(s) = (2F(s))^{-1/2}$ we find

$$\frac{s\psi'(s)}{\psi(s)} = \frac{\gamma+1}{2} + O(1)s^\beta.$$

Putting $s = \phi(\delta)$ and noting that ϕ is the inverse function of ψ we get (2.26). By (2.25) we have

$$\frac{\phi(\delta)}{\delta^2 f(\phi(\delta))} = \frac{\phi(\delta)\Big(\frac{\gamma+1}{\gamma-1} + O(1)(\phi(\delta))^\beta\Big)}{\delta\phi'(\delta)}.$$

Using the latter estimate and (2.26) we get

$$\frac{\phi(\delta)}{\delta^2 f(\phi(\delta))} = \Big(\frac{\gamma+1}{2} + O(1)(\phi(\delta))^\beta\Big)\Big(\frac{\gamma+1}{\gamma-1} + O(1)(\phi(\delta))^\beta\Big),$$

from which (2.27) follows. Estimate (2.28) follows immediately from (2.6). The lemma is proved. □

3. Main result

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain, and let $f(t) > 0$ smooth, decreasing for $t > 0$ and such that $f(t) \to \infty$ as $t \to 0$. Assume condition (1.2) with $\gamma > 3$ and $\beta > 0$. If $u(x)$ is a solution to problem (1.1), then*

$$\phi(\delta)\big(1 - C\delta\big) < u(x) < \phi(\delta)\big(1 + C\delta\big), \tag{3.1}$$

where ϕ is defined as in (1.4), $\delta = \delta(x)$ denotes the distance from x to $\partial\Omega$ and C is a suitable positive constant.

Proof. If $P \in \partial\Omega$ we can consider a suitable annulus of radii ρ and R contained in Ω and such that its external boundary is tangent to $\partial\Omega$ in P. If $v(x)$ is the solution of problem (1.1) in this annulus, by using the comparison principle for elliptic equations we have $u(x) \geq v(x)$ for x belonging to the annulus. Choose the origin in the center of the annulus and put $v(x) = v(r)$ for $r = |x|$. By Lemma 2.1 we have

$$v(r) > \phi(R-r) - C_1 \frac{\int_v^1 (F(t))^{1/2} dt}{(F(v))^{1/2}} (R-r), \quad \tilde{r} < r < R. \tag{3.2}$$

Since $\gamma > 3$, by (2.6) we find that $\int_t^1 (F(\tau))^{1/2} d\tau$ and $t(F(t))^{1/2}$ approach infinite as t approaches zero. Using de l'Hôpital rule and (2.3) (which follows from (1.2)) we find

$$\lim_{t\to 0} \frac{\int_t^1 (F(\tau))^{1/2} d\tau}{t(F(t))^{1/2}} = \lim_{t\to 0} \frac{-(F(t))^{1/2}}{(F(t))^{1/2} - \frac{tf(t)}{2(F(t))^{1/2}}} = \lim_{t\to 0} \frac{1}{-1 + \frac{tf(t)}{2F(t)}} = \frac{2}{\gamma - 3}. \tag{3.3}$$

Therefore, (3.2) implies

$$v(r) > \phi(R-r) - C_2 v(r)(R-r).$$

The latter estimate and the left hand side of (2.23) yield

$$v(r) > \phi(R-r)\big(1 - C_2(R-r)\big).$$

For x near $\partial\Omega$ we have $\delta = R - r$; therefore, the latter estimate and the inequality $u(x) \geq v(x)$ yield the left hand side of (3.1).

Consider a new annulus of radii ρ and R containing Ω and such that its internal boundary is tangent to $\partial\Omega$ in P. If $w(x)$ is the solution of problem (1.1) in this annulus, by using the comparison principle for elliptic equations we have $u(x) \leq w(x)$ for x belonging to Ω. Choose the origin in the center of the annulus and put $w(x) = w(r)$ for $r = |x|$. By Lemma 2.1 with w in place of v we have

$$w(r) < \phi(r-\rho) + C_1(r-\rho)\phi'(r-\rho) \frac{\int_w^1 (F(t))^{1/2} dt}{F(w)}, \quad \rho < r < \overline{r}. \tag{3.4}$$

Using (3.3) we can find a constant C_2 such that

$$\frac{\int_w^1 (F(t))^{1/2} dt}{F(w)} \leq C_2 \frac{w}{(F(w))^{1/2}}.$$

Since $\phi' = (2F(\phi))^{1/2}$, (3.4) and the previous inequality yield

$$w(r) < \phi(r-\rho) + C_3(r-\rho)\Big(\frac{F(\phi)}{F(w)}\Big)^{1/2} w. \tag{3.5}$$

By (2.24) with w in place of v and with $\epsilon = 1$, and by (2.3) we find

$$\Big(\frac{F(\phi)}{F(w)}\Big)^{1/2} w \leq \Big(\frac{F(\phi)}{F(2\phi)}\Big)^{1/2} 2\phi \leq C_4\phi.$$

Insertion of this estimate into (3.5) yields

$$w(r) < \phi(r-\rho)\big(1 + C_5(r-\rho)\big). \tag{3.6}$$

For x near to $\partial\Omega$ we have $\delta = r - \rho$; therefore, estimate (3.6) and the inequality $u(x) \leq w(x)$ yield the right hand side of (3.1). The lemma is proved. $\square$

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded smooth domain, and let $f(t) > 0$ smooth, decreasing for $t > 0$ and such that $f(t) \to \infty$ as $t \to 0$. Assume condition (1.2) with $\gamma > 3$ and $\beta > 0$. If $u(x)$ is a solution to (1.1), then*

$$\phi(\delta)\Big[1 + \frac{N-1}{3-\gamma}K\delta - C\delta^{1+\sigma}\Big] < u(x) < \phi(\delta)\Big[1 + \frac{N-1}{3-\gamma}K\delta + C\delta^{1+\sigma}\Big], \tag{3.7}$$

where ϕ is defined as in (1.4), $K = K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = constant\}$, σ is a number such that $0 < \sigma < \min[\frac{\gamma-3}{\gamma+1}, \frac{2\beta}{\gamma+1}]$, and C is a suitable constant.

Proof. We look for a super solution of the form

$$w(x) = \phi(\delta)\big(1 + A\delta + \alpha\delta^{1+\sigma}\big),$$

where

$$A = \frac{H}{3-\gamma}, \quad H = (N-1)K \tag{3.8}$$

and α is a positive constant to be determined. We have

$$w_{x_i} = \phi'\delta_{x_i}\big(1 + A\delta + \alpha\delta^{1+\sigma}\big) + \phi\big(A_{x_i}\delta + A\delta_{x_i} + \alpha(1+\sigma)\delta^\sigma\delta_{x_i}\big).$$

Recalling that

$$\sum_{i=1}^N \delta_{x_i}\delta_{x_i} = 1, \quad \sum_{i=1}^N \delta_{x_i x_i} = -H,$$

we find that

$$\begin{aligned}
\Delta w &= \phi''\big(1 + A\delta + \alpha\delta^{1+\sigma}\big) - \phi' H\big(1 + A\delta + \alpha\delta^{1+\sigma}\big) \\
&\quad + 2\phi'\big(\nabla A \cdot \nabla\delta\delta + A + \alpha(1+\sigma)\delta^\sigma\big) \\
&\quad + \phi\big(\Delta A\ \delta + 2\nabla A \cdot \nabla\delta - AH + \alpha\sigma(1+\sigma)\delta^{\sigma-1} - \alpha(1+\sigma)\delta^\sigma H\big).
\end{aligned} \tag{3.9}$$

By (1.4) we find $\phi'' = -f(\phi)$. Using this equation together with (2.25) and (2.27), by (3.9) we find that

$$\begin{aligned}
\Delta w &= f(\phi)\Big[-1 - A\delta - \alpha\delta^{1+\sigma} - \Big(\frac{\gamma+1}{\gamma-1} + O(1)\phi^\beta\Big)\delta H\big(1 + A\delta + \alpha\delta^{1+\sigma}\big) \\
&\quad + 2\Big(\frac{\gamma+1}{\gamma-1} + O(1)\phi^\beta\Big)\delta\big(\nabla A \cdot \nabla\delta\ \delta + A + \alpha(1+\sigma)\delta^\sigma\big) \\
&\quad + \Big(\frac{(\gamma+1)^2}{2(\gamma-1)} + O(1)\phi^\beta\Big)\delta^2\big(\Delta A\ \delta + 2\nabla A \cdot \nabla\delta - AH + \alpha\sigma(1+\sigma)\delta^{\sigma-1} \\
&\quad - \alpha(1+\sigma)\delta^\sigma H\big)\Big].
\end{aligned} \tag{3.10}$$

This means that we can find suitable constants C_i such that

$$\begin{aligned}
\Delta w &< f(\phi)\Big[-1 - \Big(A + \frac{\gamma+1}{\gamma-1}(H-2A)\Big)\delta + C_1\phi^\beta\delta + C_2\delta^2 \\
&\quad + \alpha\delta^{1+\sigma}\Big(-1 + 2(1+\sigma)\frac{\gamma+1}{\gamma-1} + \sigma(1+\sigma)\frac{(\gamma+1)^2}{2(\gamma-1)} + C_3\phi^\beta + C_4\delta\Big)\Big].
\end{aligned} \tag{3.11}$$

On the other hand, using Taylor's expansion we have

$$f(w) = f(\phi)\Big[1 + \phi\frac{f'(\phi)}{f(\phi)}(A\delta + \alpha\delta^{1+\sigma}) + \phi^2\frac{f''(\overline{\phi})}{f(\phi)}(A\delta + \alpha\delta^{1+\sigma})^2\Big], \tag{3.12}$$

with $\overline{\phi}$ between ϕ and $\phi(1+A\delta+\alpha\delta^{1+\sigma})$. After α is fixed, we consider only points $x\in\Omega$ such that

$$-\frac{1}{2}<A\delta+\alpha\delta^{1+\sigma}<1. \tag{3.13}$$

This means that $1/2<1+A\delta+\alpha\delta^{1+\sigma}<2$. Therefore, the term $\overline{\phi}$ which appears in (3.12) satisfies $\overline{\phi}=\theta\phi$, with $1/2<\theta<2$. Using (1.3) and (2.4) (which follows from (1.2)), by (3.12) we find that

$$f(w)=f(\phi)\Big[1-\gamma A\delta-\alpha\gamma\delta^{1+\sigma}+O(1)\phi^{\beta}\delta+O(1)\delta^2+O(1)\alpha\phi^{\beta}\delta^{1+\sigma}+O(1)(\alpha\delta^{1+\sigma})^2\Big].$$

Using this equality, we can take suitable positive constants C_i such that

$$\begin{aligned}-f(w)>f(\phi)\Big[&-1+\gamma A\delta+\alpha\gamma\delta^{1+\sigma}-C_5\phi^{\beta}\delta-C_6\delta^2-C_7\alpha\phi^{\beta}\delta^{1+\sigma}\\&-C_8(\alpha\delta^{1+\sigma})^2\Big].\end{aligned} \tag{3.14}$$

Since by (3.8),

$$-\Big(A+\frac{\gamma+1}{\gamma-1}(H-2A)\Big)=\gamma A,$$

by (3.11) and (3.14), we have

$$\Delta w<-f(w) \tag{3.15}$$

provided that

$$\begin{aligned}&C_1\phi^{\beta}\delta+C_2\delta^2+\alpha\delta^{1+\sigma}\Big(-1+2(1+\sigma)\frac{\gamma+1}{\gamma-1}+\sigma(1+\sigma)\frac{(\gamma+1)^2}{2(\gamma-1)}+C_3\phi^{\beta}+C_4\delta\Big)\\&<\alpha\gamma\delta^{1+\sigma}-C_5\phi^{\beta}\delta-C_6\delta^2-C_7\alpha\phi^{\beta}\delta^{1+\sigma}-C_8(\alpha\delta^{1+\sigma})^2.\end{aligned}$$

Rearranging terms,

$$\begin{aligned}&(C_1+C_5)\phi^{\beta}\delta^{-\sigma}+(C_2+C_6)\delta^{1-\sigma}\\&<\alpha\Big(\gamma+1-2(1+\sigma)\frac{\gamma+1}{\gamma-1}-\sigma(1+\sigma)\frac{(\gamma+1)^2}{2(\gamma-1)}-(C_3+C_7)\phi^{\beta}\\&\quad-C_4\delta-C_8\alpha\delta^{1+\sigma}\Big).\end{aligned} \tag{3.16}$$

Using (2.28) we find

$$\phi^{\beta}\delta^{-\sigma}<C\delta^{\frac{2\beta}{\gamma+1}-\sigma}.$$

Since $\sigma<\frac{2\beta}{\gamma+1}$ we have $\phi^{\beta}\delta^{-\sigma}\to 0$ as $\delta\to 0$. Moreover, since $\sigma<\frac{\gamma-3}{\gamma+1}$, we also have $\delta^{1-\sigma}\to 0$ as $\delta\to 0$, and

$$\gamma+1-2(1+\sigma)\frac{\gamma+1}{\gamma-1}-\sigma(1+\sigma)\frac{(\gamma+1)^2}{2(\gamma-1)}=\frac{(\gamma+1)^2}{2(\gamma-1)}(\sigma+2)\Big(\frac{\gamma-3}{\gamma+1}-\sigma\Big)>0.$$

Hence, we can take α_0 large and δ_0 small so that (3.13) and (3.16) hold for $\alpha\geq\alpha_0$, $\delta\leq\delta_0$ with $\alpha\delta^{1+\sigma}\leq\alpha_0\delta_0^{1+\sigma}$.

Let us show now that we can choose δ_1 so that $u(x)\leq w(x)$ for $\delta(x)=\delta_1$. Using Lemma 3.1 we have

$$w(x)-u(x)\geq\phi(\delta)\Big[\frac{(N-1)K}{3-\gamma}\delta+\alpha\delta^{1+\sigma}-C\delta\Big].$$

Take α_0 and δ_0 so that (3.13) and (3.16) hold, and put $q = \alpha_0\delta_0^{1+\sigma}$. Decrease δ and increasing α according to $\alpha\delta^{1+\sigma} = q$ until

$$\frac{(N-1)K}{3-\gamma}\delta + q - C\delta > 0$$

for $\delta(x) = \delta_1$. Then $w(x) > u(x)$ for $\delta(x) = \delta_1$. By (3.15) and the comparison principle [11, Theorem 10.1], it follows that $u(x) \le w(x)$ in $\{x \in \Omega : \delta(x) < \delta_1\}$.

We look for a sub solutions of the form

$$v(x) = \phi(\delta)\big(1 + A\delta - \alpha\delta^{1+\sigma}\big),$$

where A and σ are the same as before and α is a positive constant to be determined. Instead of (3.10) now we have

$$\begin{aligned}\Delta v = f(\phi)\Big[&-1 - A\delta + \alpha\delta^{1+\sigma} - \Big(\frac{\gamma+1}{\gamma-1} + O(1)\phi^\beta\Big)\delta H\big(1 + A\delta - \alpha\delta^{1+\sigma}\big)\\ &+ 2\Big(\frac{\gamma+1}{\gamma-1} + O(1)\phi^\beta\Big)\delta\big(\nabla A\cdot\nabla\delta\delta + A - \alpha(1+\sigma)\delta^\sigma\big)\\ &+ \Big(\frac{(\gamma+1)^2}{2(\gamma-1)} + O(1)\phi^\beta\Big)\delta^2\big(\Delta A\delta + 2\nabla A\cdot\nabla\delta - AH - \alpha\sigma(1+\sigma)\delta^{\sigma-1}\\ &+ \alpha(1+\sigma)\delta^\sigma H\big)\Big].\end{aligned} \tag{3.17}$$

This means that we can find suitable constants C_i (not necessarily the same as before) such that

$$\begin{aligned}\Delta v > f(\phi)\Big[&-1 - \Big(A + \frac{\gamma+1}{\gamma-1}(H-2A)\Big)\delta - C_1\phi^\beta\delta - C_2\delta^2\\ &- \alpha\delta^{1+\sigma}\Big(-1 + 2(1+\sigma)\frac{\gamma+1}{\gamma-1} + \sigma(1+\sigma)\frac{(\gamma+1)^2}{2(\gamma-1)} + C_3\phi^\beta + C_4\delta\Big)\Big].\end{aligned} \tag{3.18}$$

After α is fixed, we consider only points $x \in \Omega$ such that

$$-\frac{1}{2} < A\delta - \alpha\delta^{1+\sigma} < 1. \tag{3.19}$$

Using Taylor's expansion, now we find

$$-f(v) < f(\phi)\Big[-1 + \gamma A\delta - \alpha\gamma\delta^{1+\sigma} + C_5\phi^\beta\delta + C_6\delta^2 + C_7\alpha\phi^\beta\delta^{1+\sigma} + C_8(\alpha\delta^{1+\sigma})^2\Big]. \tag{3.20}$$

Recalling that

$$-\Big(A + \frac{\gamma+1}{\gamma-1}(H-2A)\Big) = \gamma A,$$

by (3.18) and (3.20) we have

$$\Delta v > -f(v) \tag{3.21}$$

when

$$\begin{aligned}&- C_1\phi^\beta\delta - C_2\delta^2\\ &- \alpha\delta^{1+\sigma}\Big(-1 + 2(1+\sigma)\frac{\gamma+1}{\gamma-1} + \sigma(1+\sigma)\frac{(\gamma+1)^2}{2(\gamma-1)} + C_3\phi^\beta + C_4\delta\Big)\\ &> -\alpha\gamma\delta^{1+\sigma} + C_5\phi^\beta\delta + C_6\delta^2 + C_7\alpha\phi^\beta\delta^{1+\sigma} + C_8(\alpha\delta^{1+\sigma})^2.\end{aligned}$$

Rearranging terms,

$$
\begin{aligned}
&(C_1 + C_5)\phi^\beta \delta^{-\sigma} + (C_2 + C_6)\delta^{1-\sigma} \\
&< \alpha\Big(\gamma + 1 - 2(1+\sigma)\frac{\gamma+1}{\gamma-1} - \sigma(1+\sigma)\frac{(\gamma+1)^2}{2(\gamma-1)} - (C_3 + C_7)\phi^\beta \\
&\quad - C_4\delta - C_8\alpha\delta^{1+\sigma}\Big),
\end{aligned}
\tag{3.22}
$$

which is the same as (3.16) (possibly with different constants). Therefore, we can take δ small and α large in order to satisfy this inequality. Take α_0 and δ_0 so that (3.19) and (3.22) hold, and put $q = \alpha_0\delta_0^{1+\sigma}$. Using Lemma 3.1,

$$
v(x) - u(x) \le \phi(\delta)\Big[\frac{(N-1)K}{3-\gamma}\delta - \alpha\delta^{1+\sigma} + C\delta\Big].
$$

Decrease δ and increasing α according to $\alpha\delta^{1+\sigma} = q$ until

$$
\frac{(N-1)K}{3-\gamma}\delta - q + C\delta < 0
$$

for $\delta(x) = \delta_2$. Then $v(x) < u(x)$ for $\delta(x) = \delta_2$. By (3.21) and the usual comparison principle it follows that $v(x) \le u(x)$ in $\{x \in \Omega : \delta(x) < \delta_2\}$. The theorem follows. $\square$

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