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EXISTENCE OF POSITIVE SOLUTIONS FOR A FOURTH-ORDER MULTI-POINT BEAM PROBLEM ON MEASURE CHAINS

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Abstract. This article concerns the fourth-order multi-point beam problem

$$(EIW^{\Delta\nabla})^{\nabla\Delta}(x) = m(x)f(x, W(x)), \quad x \in [x_1, x_n]_{\mathbb{X}}$$

$$W(\rho^2(x_1)) = \sum_{i=2}^{n-1} a_i W(x_i), \quad W^{\Delta}(\rho^2(x_1)) = 0,$$

$$(EIW^{\Delta\nabla})(\sigma(x_n)) = 0, \quad (EIW^{\Delta\nabla})^{\nabla}(\sigma(x_n)) = \sum_{i=2}^{n-1} b_i (EIW^{\Delta\nabla})^{\nabla}(x_i).$$

Under various assumptions on the functions f and m and the coefficients a_i and b_i we establish the existence of one or two positive solutions for this measure chain boundary value problem using the Green's function approach.

1. Introduction

The aim of this work is to obtain sufficient conditions for the existence of positive solutions of the measure chain fourth-order multi-point boundary value problem composed by the equation

$$(EIW^{\Delta\nabla})^{\nabla\Delta}(x) = m(x)f(x, W(x)) \quad \text{for all } x \in [x_1, x_n]_{\mathbb{X}} \tag{1.1}$$

and the multi-point boundary conditions

$$\begin{gathered} W(\rho^2(x_1)) = \sum_{i=2}^{n-1} a_i W(x_i), \quad W^{\Delta}(\rho^2(x_1)) = 0, \\ (EIW^{\Delta\nabla})(\sigma(x_n)) = 0, \quad (EIW^{\Delta\nabla})^{\nabla}(\sigma(x_n)) = \sum_{i=2}^{n-1} b_i (EIW^{\Delta\nabla})^{\nabla}(x_i), \end{gathered} \tag{1.2}$$

on a measure chain $\mathbb{X}$, $n \geq 4$. The boundary points satisfy $x_1 \in \mathbb{X}_{\kappa^2}$ and $x_n \in \mathbb{X}^{\kappa^2}$ with $\rho^2(x_1) < x_2 < \cdots < x_{n-1} < \sigma(x_n)$, while $f : \mathbb{X}\times[0, \infty) \to [0, \infty)$ is continuous, $I : [\rho(x_1), \sigma(x_n)]_{\mathbb{X}} \to (0, \infty)$ is left-dense continuous and $E > 0$ is constant. The mass function $m : [\rho(x_1), \sigma(x_n)]_{\mathbb{X}} \to [0, \infty)$ is right-dense continuous, not identically zero on $[x_2, x_3]_{\mathbb{X}}$ and the non-negative coefficients a_i and b_i satisfy the non-resonant

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conditions $\sum_{i=2}^{n-1} a_i < 1$ and $\sum_{i=2}^{n-1} b_i < 1$. Physically, the motivation for this fourth-order problem is a nonuniform cantilever beam of length L in transverse vibration such that the left end is clamped and the right end is free with vanishing bending moment and shearing force. Let E be the modulus of elasticity, $I(x)$ the area moment of inertia about the neutral axis and $m(x)$ the mass per unit length of the beam. After separation of variables, the space-variable problem is formulated as

$$\begin{gathered}(EI(x)W''(x))'' = m(x)W(x), \quad \text{for all } x \in [0, L],\\ W(0) = W'(0) = (EIW'')(L) = (EIW'')'(L) = 0\,;\end{gathered} \tag{1.3}$$

see Meirovitch [14, 15].

Throughout this work we assume a working knowledge of measure chains (time scales) and measure chain notation, where any arbitrary nonempty closed subset of $\mathbb{R}$ can serve as a measure chain $\mathbb{X}$. See Hilger [11] for an introduction to measure chains; other excellent sources on delta dynamic equations include [5, 6], and for nabla dynamic equations, see [4]. For more on beam and other fourth-order continuous problems we refer to the recent papers [1, 9, 16, 17, 18], and for functional boundary value problems see [7, 8]. Related to fourth-order dynamic equations, see [2, 3, 12, 19]. However, as far as we know, this is the first time where multi-point boundary conditions as in (1.2) are considered in fourth order nonlinear problems on time scales.

The second section contains some preliminary lemmas needed to evaluate explicitly the unique solution W of a related fourth-order equation, by a Green's function approach, and to prove some properties of W. Section three provides some sufficient conditions on the nonlinearity to obtain the existence and the multiplicity of positive solutions, via index theory in cones. Two examples are referred in the last section, to illustrate the existence of multiple positive solutions.

2. Foundational lemmas

For the related fourth-order multi-point boundary value problem composed by the equation

$$(EIW^{\Delta\nabla})^{\nabla\Delta}(x) = y(x), \quad x \in [x_1, x_n]_{\mathbb{X}}, \tag{2.1}$$

with $y : [x_1, x_n]_{\mathbb{X}} \to \mathbb{R}$ right-dense continuous, and boundary conditions (1.2), it is referred [2, Theorem 7.1], where the Green's function $G(x, s)$ for the corresponding homogeneous equation

$$(EIW^{\Delta\nabla})^{\nabla\Delta}(x) = 0 \tag{2.2}$$

satisfying boundary conditions

$$\begin{gathered}W(\rho^2(x_1)) = W^{\Delta}(\rho^2(x_1)) = 0,\\ (EIW^{\Delta\nabla})(\sigma(x_n)) = (EIW^{\Delta\nabla})^{\nabla}(\sigma(x_n)) = 0,\end{gathered} \tag{2.3}$$

is given, for $(x, s) \in [\rho^2(x_1), \sigma^2(x_n)]_{\mathbb{X}} \times [\rho(x_1), \sigma(x_n)]_{\mathbb{X}}$, by

$$G(x,s) = \begin{cases} \int_{\rho^2(x_1)}^{s} \Big(\int_{\rho^2(x_1)}^{\zeta} \frac{x-\xi}{EI(\xi)} \nabla\xi \Big) \Delta\zeta & s \in [\rho(x_1), x]_{\mathbb{X}},\ x \le \sigma^2(x_n), \\ \int_{\rho^2(x_1)}^{x} \Big(\int_{\rho^2(x_1)}^{\zeta} \frac{s-\xi}{EI(\xi)} \nabla\xi \Big) \Delta\zeta & s \in [x, \sigma(x_n)]_{\mathbb{X}},\ x \ge \rho^2(x_1). \end{cases} \tag{2.4}$$

Example 2.1. Consider the Green's function (2.4) for $\rho^2(x_1) = 0$ and $\sigma^2(x_n) = 1$, with $EI(x) \equiv 1$. Then we have the following continuous and discrete illustrations:

$$\mathbb{X} = \mathbb{R}: \quad G(x,s) = \begin{cases} \frac{s^2(3x-s)}{6} & s \in [0,x],\ x \in [0,1], \\ \frac{x^2(3s-x)}{6} & s \in [x,1],\ x \in [0,1], \end{cases}$$

$$\mathbb{X} = h\mathbb{Z}: \quad G(x,s) = \begin{cases} \frac{s(s-h)(3x-s-h)}{6} & s \in [h,x]_{h\mathbb{Z}},\ x \le 1, \\ \frac{x(x-h)(3s-x-h)}{6} & s \in [x,1-h]_{h\mathbb{Z}},\ x \ge 0, \end{cases}$$

where for $0 < h << 1$ we have $h\mathbb{Z} = \{0, h, 2h, \dots, 1-h, 1\}$.

This Green's function satisfies the following properties.

Lemma 2.2 ([3]). *For all $(x,s) \in [\rho^2(x_1), \sigma^2(x_n)]_{\mathbb{X}} \times [\rho(x_1), \sigma(x_n)]_{\mathbb{X}}$, the Green's function given by* (2.4) *is increasing in x and satisfies*

$$0 \le G(x,s) \le G\big(\sigma^2(x_n), s\big). \tag{2.5}$$

Now we prove an existence and uniqueness result.

Lemma 2.3. *Assume the coefficients a_i and b_i in* (1.2) *are real non-negative numbers that satisfy the non-resonant conditions*

$$0 \le \sum_{i=2}^{n-1} a_i < 1, \quad 0 \le \sum_{i=2}^{n-1} b_i < 1. \tag{2.6}$$

If $y \in C_{rd}[\rho(x_1), \sigma(x_n)]_{\mathbb{X}}$, then the nonhomogeneous dynamic equation (2.1) *with boundary conditions* (1.2) *has a unique solution W defined by*

$$W(x) = \int_{\rho(x_1)}^{\sigma(x_n)} G(x,s)y(s)\Delta s + A(y) + B(y)\int_{\rho^2(x_1)}^{x}\int_{\rho^2(x_1)}^{\zeta} \frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta, \tag{2.7}$$

where $G(x,s)$ is the Green's function (2.4) *related with the boundary value problem* (2.2), (2.3) *and the positive constants $A(y)$ and $B(y)$ are given by*

$$\begin{aligned} A(y) = \Big(1 - \sum_{i=2}^{n-1} a_i\Big)^{-1} \sum_{i=2}^{n-1} a_i \Big(\int_{\rho(x_1)}^{\sigma(x_n)} G(x_i,s)y(s)\Delta s \\ + B(y)\int_{\rho^2(x_1)}^{x_i}\int_{\rho^2(x_1)}^{\zeta} \frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta\Big) \end{aligned} \tag{2.8}$$

and

$$B(y) = \Big(1 - \sum_{i=2}^{n-1} b_i\Big)^{-1} \sum_{i=2}^{n-1} b_i \int_{x_i}^{\sigma(x_n)} y(s)\Delta s. \tag{2.9}$$

Proof. First, we consider equation (2.1) together with conditions

$$\begin{gathered} W(\rho^2(x_1)) = A, \quad W^{\Delta}(\rho^2(x_1)) = 0 \\ (EIW^{\Delta\nabla})(\sigma(x_n)) = 0, \quad (EIW^{\Delta\nabla})^{\nabla}(\sigma(x_n)) = B. \end{gathered} \tag{2.10}$$

It is clear that any solution of problem (2.1), (2.10) can be expressed for some constants A and B as

$$W(x) = u(x) + Av(x) + Br(x),$$

where u is the unique solution of problem value problem (2.1), (2.3), v is the unique solution of (2.2) with boundary conditions

$$v(\rho^2(x_1))=1,\quad v^{\Delta}(\rho^2(x_1))=(EIv^{\Delta\nabla})(\sigma(x_n))=(EIv^{\Delta\nabla})^{\nabla}(\sigma(x_n))=0,$$

and r is the unique solution of (2.2) with boundary conditions

$$(EIr^{\Delta\nabla})^{\nabla}(\sigma(x_n))=1,\quad r(\rho^2(x_1))=r^{\Delta}(\rho^2(x_1))=(EIr^{\Delta\nabla})(\sigma(x_n))=0.$$

One can verify directly that these functions are

$$u(x)=\int_{\rho(x_1)}^{\sigma(x_n)}G(x,s)y(s)\Delta s,\quad v(x)\equiv 1,\quad r(x)=\int_{\rho^2(x_1)}^{x}\int_{\rho^2(x_1)}^{\zeta}\frac{\xi-\sigma(x_n)}{EI(\xi)}\nabla\xi\Delta\zeta.$$

It is clear that $W^{\Delta}(\rho^2(x_1))=0$ and $(EIW^{\Delta\nabla})(\sigma(x_n))=0$. To satisfy the two other boundary conditions in (1.2), we must have at $\sigma(x_n)$ that

$$-B=\Big(1-\sum_{i=2}^{n-1}b_i\Big)^{-1}\sum_{i=2}^{n-1}\int_{x_i}^{\sigma(x_n)}y(s)\Delta s,$$

and at $\rho^2(x_1)$ that

$$A=\sum_{i=2}^{n-1}a_i\Big(\int_{\rho(x_1)}^{\sigma(x_n)}G(x_i,s)y(s)\Delta s+A-B\int_{\rho^2(x_1)}^{x_i}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta\Big).$$

Solving, we arrive at the expression (2.7) for $A(y)$ given in (2.8). □

For problem (2.1),(1.2) the following maximum principle holds.

Lemma 2.4. *Assume that* (2.6) *holds. If* $y\in C_{rd}[\rho(x_1),\sigma(x_n)]_{\mathbb{X}}$ *with* $y\geq 0$ *on* $[\rho(x_1),\sigma(x_n)]_{\mathbb{X}}$*, the unique solution* W *as in* (2.7) *of the problem* (2.1), (1.2) *satisfies* $W(x)\geq 0$ *for* $x\in[\rho^2(x_1),\sigma^2(x_n)]_{\mathbb{X}}$.

Proof. From Lemma 2.3, problem (2.1), (1.2) has a unique solution W given by (2.7) and, by Lemma 2.2, the Green's function (2.4) satisfies $G(x,s)\geq 0$ on the set $[\rho^2(x_1),\sigma^2(x_n)]_{\mathbb{X}}\times[\rho(x_1),\sigma(x_n)]_{\mathbb{X}}$. The result is a direct consequence of assumption (2.6) and the fact that $A(y),B(y)\geq 0$. □

Lemma 2.5. *Assume that* (2.6) *holds. If* $y\in C_{rd}[\rho(x_1),\sigma(x_n)]_{\mathbb{X}}$ *with* $y\geq 0$ *on* $[\rho(x_1),\sigma(x_n)]_{\mathbb{X}}$*, then the unique solution* W *of the time scale boundary value problem* (2.1), (1.2)*, given by* (2.7)*, satisfies*

$$\min_{x\in[x_2,x_3]_{\mathbb{X}}}W(x)=W(x_2)\geq\gamma\|W\|,$$

where

$$\gamma:=\frac{\int_{\rho^2(x_1)}^{x_2}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta}{\int_{\rho^2(x_1)}^{\sigma(x_n)}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma^2(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta}\in(0,1),\tag{2.11}$$

and

$$\|W\|:=\max_{x\in[\rho^2(x_1),\sigma^2(x_n)]_{\mathbb{X}}}W(x)=W(\sigma^2(x_n)).$$

Proof. Using Lemma 2.2 and (2.7), we conclude that for all $x\in[\rho^2(x_1),\sigma^2(x_n)]_{\mathbb{X}}$,

$$W(x)\leq\int_{\rho(x_1)}^{\sigma(x_n)}G\big(\sigma^2(x_n),s\big)y(s)\Delta s+A(y)+B(y)\int_{\rho^2(x_1)}^{\sigma^2(x_n)}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta.$$

For $x \in [x_2, x_3]_{\mathbb{X}}$, from Lemma 2.2 the Green's function (2.4) satisfies

$$\frac{G(x,s)}{G(\sigma^2(x_n),s)} \geq \frac{G(x_2,s)}{G(\sigma^2(x_n),s)} \geq \frac{\int_{\rho^2(x_1)}^{x_2}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta}{\int_{\rho^2(x_1)}^{\sigma(x_n)}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma^2(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta} = \gamma \tag{2.12}$$

for γ as in (2.11), and the constant $A(y)$ in (2.8) satisfies $A(y) \geq \gamma A(y)$ since $\gamma \in (0,1)$ and $A(y) \geq 0$. Thus for $x \in [x_2, x_3]_{\mathbb{X}}$, we have

$$\begin{aligned}
W(x) &= \int_{\rho(x_1)}^{\sigma(x_n)} \frac{G(x,s)}{G(\sigma^2(x_n),s)} G(\sigma^2(x_n),s)y(s)\Delta s + A(y) \\
&\quad + B(y)\int_{\rho^2(x_1)}^{x}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta \\
&\geq \int_{\rho(x_1)}^{\sigma(x_n)} \gamma G(\sigma^2(x_n),s)y(s)\Delta s + \gamma A(y) \\
&\quad + B(y)\int_{\rho^2(x_1)}^{x_2}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta \\
&= \int_{\rho(x_1)}^{\sigma(x_n)} \gamma G(\sigma^2(x_n),s)y(s)\Delta s + \gamma A(y) \\
&\quad + \gamma B(y)\int_{\rho^2(x_1)}^{\sigma(x_n)}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma^2(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta \\
&= \int_{\rho(x_1)}^{\sigma(x_n)} \gamma G(\sigma^2(x_n),s)y(s)\Delta s + \gamma A(y) \\
&\quad + \gamma B(y)\int_{\rho^2(x_1)}^{\sigma^2(x_n)}\int_{\rho^2(x_1)}^{\zeta}\frac{\sigma(x_n)-\xi}{EI(\xi)}\nabla\xi\Delta\zeta \\
&= \gamma W(\sigma^2(x_n)) = \gamma\|W\|.
\end{aligned}$$

This completes the proof. □

3. Existence of Positive Solutions

In this section some criteria are identified whereby the existence of positive solutions to the multi-point boundary value problem (1.1), (1.2) can be established, where $f : \mathbb{X} \times [0,\infty) \to [0,\infty)$ is continuous such that the limits

$$f_0 := \lim_{y\to 0^+}\frac{f(x,y)}{y}, \quad f_\infty := \lim_{y\to\infty}\frac{f(x,y)}{y},$$

exist uniformly for $x \in [x_1, x_n]_{\mathbb{X}}$.

In the sequel it is assumed that the right-dense continuous mass function m satisfies

$$m : [\rho(x_1), \sigma(x_n)]_{\mathbb{X}} \to [0,\infty), \quad \exists\, x_* \in (x_2, x_3)_{\mathbb{X}} : m(x_*) > 0. \tag{3.1}$$

Let $\mathcal{B}$ denote the Banach space $C[\rho^2(x_1), \sigma^2(x_n)]_{\mathbb{X}}$ with the norm

$$\|W\| = \sup_{x\in[\rho^2(x_1),\sigma^2(x_n)]_{\mathbb{X}}} |W(x)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{W \in \mathcal{B} : W(x) \geq 0 \text{ on } [\rho^2(x_1), \sigma^2(x_n)]_{\mathbb{X}},\ W(x) \geq \gamma\|W\| \text{ on } [x_2, x_3]_{\mathbb{X}}\}, \tag{3.2}$$

where γ is given in (2.11). Since W is a solution of (1.1), (1.2) if and only if it satisfies equation (2.7) replacing in this case $y(s)$ by $m(s)f(s, W(s))$, define for $W \in \mathcal{P}$ the operator $\mathcal{L} : \mathcal{P} \to \mathcal{B}$ by

$$\begin{aligned} \mathcal{L}W(x) &= \int_{\rho(x_1)}^{\sigma(x_n)} G(x,s)m(s)f(s,W(s))\Delta s + A(mf(\cdot,W)) + \Big(1 - \sum_{i=2}^{n-1} b_i\Big)^{-1} \\ &\quad \times \Big(\sum_{i=2}^{n-1} b_i \int_{x_i}^{\sigma(x_n)} m(s)f(s,W(s))\Delta s\Big) \int_{\rho^2(x_1)}^{x} \int_{\rho^2(x_1)}^{\zeta} \frac{\sigma(x_n) - \xi}{EI(\xi)} \nabla\xi\Delta\zeta. \end{aligned} \tag{3.3}$$

By Lemmas 2.4 and 2.5, $\mathcal{L} : \mathcal{P} \to \mathcal{P}$. Moreover, $\mathcal{L}$ is completely continuous by a typical application of the Ascoli-Arzela Theorem.

Lemma 3.1 ([10, 13]). *Let P be a cone in a Banach space S and B an open, bounded subset of S with $B_P := B \cap P \neq \emptyset$ and $\overline{B}_P \neq P$. Assume that $L : \overline{B}_P \to P$ is a compact map such that $y \neq Ly$ for $y \in \partial B_P$, and the following results hold:*

- (i) *If $\|Ly\| \leq \|y\|$ for $y \in \partial B_P$, then $i_P(L, B_P) = 1$.*
- (ii) *If there exists an $\eta \in P\backslash\{0\}$ such that $y \neq Ly + \lambda\eta$ for all $y \in \partial B_P$ and all $\lambda > 0$, then $i_P(L, B_P) = 0$.*
- (iii) *Let U be open in P such that $\overline{U}_P \subset B_P$. If $i_P(L, B_P) = 1$ and $i_P(L, U_P) = 0$, then L has a fixed point in $B_P\backslash\overline{U}_P$; the same is true if $i_P(L, B_P) = 0$ and $i_P(L, U_P) = 1$.*

For the cone $\mathcal{P}$ given in (3.2) and any positive real number r, define the convex set

$$P_r := \{W \in \mathcal{P} : \|W\| < r\},$$

and, for γ in (2.11), the set

$$\Omega_r := \{W \in \mathcal{P} : \min_{x \in [x_2, x_3]_{\mathbb{X}}} W(x) < \gamma r\}.$$

Lemma 3.2 ([13]). *The set Ω_r has the following properties:*

- (i) *Ω_r is open relative to $\mathcal{P}$.*
- (ii) *$P_{\gamma r} \subset \Omega_r \subset P_r$.*
- (iii) *$W \in \partial\Omega_r$ if and only if $\min_{x \in [x_2, x_3]_{\mathbb{X}}} W(x) = \gamma r$.*
- (iv) *If $W \in \partial\Omega_r$, then $\gamma r \leq W(x) \leq r$ for $x \in [x_2, x_3]_{\mathbb{X}}$.*

For $G(x, s)$ in (2.4) and $A(y)$ in (2.8) with y replaced by the mass function m, consider the constant K given by

$$\begin{aligned} K &:= \int_{\rho(x_1)}^{\sigma(x_n)} G(\sigma^2(x_n), s)m(s)\Delta s + A(m) \\ &\quad + \Big(1 - \sum_{i=2}^{n-1} b_i\Big)^{-1} \Big(\sum_{i=2}^{n-1} b_i \int_{x_i}^{\sigma(x_n)} m(s)\Delta s\Big) \int_{\rho^2(x_1)}^{\sigma^2(x_n)} \int_{\rho^2(x_1)}^{\zeta} \frac{\sigma(x_n) - \xi}{EI(\xi)} \nabla\xi\Delta\zeta \end{aligned} \tag{3.4}$$

and

$$f_{\gamma r}^r := \min_{W \in [\gamma r, r]} \Big\{ \min_{x \in [x_2, x_3]_{\mathbb{X}}} \frac{f(x, W)}{r} \Big\}, \quad f_0^r := \max_{W \in [0, r]} \Big\{ \max_{x \in [\rho(x_1), \sigma(x_n)]_{\mathbb{X}}} frac f(x, W) r \Big\}.$$

The next two lemmas present sufficient conditions on f to evaluate the index of $\mathcal{L}$.

Lemma 3.3. *Let K be as in (3.4). If $f_0^r < 1/K$ holds, then $i_P(\mathcal{L}, P_r) = 1$.*

Proof. From (2.8),

$$|A(mf(\cdot, W))| \le A(m)\|f(\cdot, W)\|.$$

For $W \in \partial P_r$, by (3.3) and Lemma 2.2,

$$\begin{aligned}
(\mathcal{L}W)(x) &= \int_{\rho(x_1)}^{\sigma(x_n)} G(x,s)m(s)f(s,W(s))\Delta s + A(mf(\cdot,W)) + (1-\sum_{i=2}^{n-1} b_i)^{-1} \\
&\quad \times \Big(\sum_{i=2}^{n-1} b_i \int_{x_i}^{\sigma(x_n)} m(s)f(s,W(s))\Delta s\Big) \int_{\rho^2(x_1)}^{x} \int_{\rho^2(x_1)}^{\zeta} \frac{\sigma(x_n)-\xi}{EI(\xi)} \nabla\xi\Delta\zeta \\
&\le \|f(\cdot,W)\| \Big[\int_{\rho(x_1)}^{\sigma(x_n)} G(\sigma^2(x_n),s)m(s)\Delta s + A(m) + \Big(1-\sum_{i=2}^{n-1} b_i\Big)^{-1} \\
&\quad \times \Big(\sum_{i=2}^{n-1} b_i \int_{x_i}^{\sigma(x_n)} m(s)\Delta s\Big) \int_{\rho^2(x_1)}^{\sigma^2(x_n)} \int_{\rho^2(x_1)}^{\zeta} \frac{\sigma(x_n)-\xi}{EI(\xi)} \nabla\xi\Delta\zeta \Big] \\
&< (r/K)K = r = \|W\|.
\end{aligned}$$

It follows that for $W \in \partial P_r$, $\|\mathcal{L}W\| < \|W\|$. By Lemma 3.1 (i), $i_P(\mathcal{L}, P_r) = 1$. $\square$

Lemma 3.4. *Let*

$$M^{-1} := \int_{x_2}^{x_3} G(x_2,s)m(s)\Delta s. \tag{3.5}$$

If the inequality $f_{\gamma r}^r > M\gamma$ is satisfied, then $i_P(\mathcal{L}, \Omega_r) = 0$.

Proof. Let $\eta(x) \equiv 1$ for $x \in [\rho^2(x_1), \sigma^2(x_n)]_{\mathbb{X}}$, so that $\eta \in \partial P_1$. Suppose there exist $W_* \in \partial\Omega_r$ and $\lambda_* \ge 0$ such that $W_* = \mathcal{L}W_* + \lambda_*\eta$. Then for $x \in [x_2, x_3]_{\mathbb{X}}$,

$$\begin{aligned}
W_*(x) &= (\mathcal{L}W_*)(x) + \lambda_*\eta(x) \\
&\ge \int_{x_2}^{x_3} G(x,s)m(s)f(s,W_*(s))\Delta s + \lambda_* \\
&> M\gamma r \int_{x_2}^{x_3} G(x_2,s)m(s)\Delta s + \lambda_* = \gamma r + \lambda_*,
\end{aligned}$$

with γ given in (2.11), and, by Lemma 3.2 (iv), this contradiction is obtained: $\gamma r > \gamma r + \lambda_*$. Consequently, $W_* \ne \mathcal{L}W_* + \lambda_*\eta$ for $W_* \in \partial\Omega_r$ and $\lambda_* \ge 0$, so, by Lemma 3.1 (ii), $i_P(\mathcal{L}, \Omega_r) = 0$. $\square$

Theorem 3.5. *Let γ, K and M be as given in (2.11), (3.4) and (3.5), respectively. Assume that one of the following assumptions holds:*
there exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < \gamma c_2$ and $c_2 < c_3$ such that

(H1) $f_0^{c_1}, f_0^{c_3} \le 1/K$, $f_{\gamma c_2}^{c_2} > M\gamma$

or there exist constants $c_1, c_2, c_3 \in \mathbb{R}$ with $0 < c_1 < c_2 < \gamma c_3$ such that

(H2) $f_{\gamma c_1}^{c_1}, f_{\gamma c_3}^{c_3} \ge M\gamma$, $f_0^{c_2} < 1/K$.

Then the multi-point problem (1.1), (1.2) has two positive solutions in $\mathcal{P}$, given by (3.2).

Proof. Assume (H2) holds (the case for (H1) is similar and is omitted). We show that either $\mathcal{L}$ has a fixed point in $\partial\Omega_{c_1}$ or in $P_{c_2}\backslash\overline{\Omega}_{c_1}$. From Lemma 3.4 , if $W \neq \mathcal{L}W$ for $W \in \partial\Omega_{c_1} \cup \partial\Omega_{c_3}$, then $i_P(\mathcal{L}, \Omega_{c_1}) = 0$ and $i_P(\mathcal{L}, \Omega_{c_3}) = 0$. Since $f_0^{c_2} \leq 1/K$ and $W \neq \mathcal{L}W$ for $W \in \partial P_{c_2}$, Lemma 3.3 implies that $i_P(\mathcal{L}, P_{c_2}) = 1$. By Lemma 3.2 (ii), $\Omega_{c_1} \subset P_{c_1} \subset P_{c_2}$. From Lemma 3.1 (iii), $\mathcal{L}$ has a fixed point in $P_{c_2}\backslash\overline{\Omega}_{c_1}$. In the same way $P_{c_2} \subset P_{\gamma c_3} \subset \Omega_{c_3}$ and $\mathcal{L}$ has a fixed point in $\Omega_{c_3}\backslash\overline{P}_{c_2}$. □

For $a \in \{0^+, \infty\}$ define

$$f_{Wa} := \liminf_{W\to a} \min_{x\in[x_2,x_3]_{\mathbb{X}}} \frac{f(x,W)}{W}, \quad f_W^a := \limsup_{W\to a} \max_{x\in[\rho(t_1),\sigma(x_n)]_{\mathbb{X}}} \frac{f(x,W)}{W}.$$

Corollary 3.6. *Suppose there exists a positive constant c such that either one the following to conditions holds:*

(H1') $0 \leq f_W^0$, $f_W^\infty < 1/K$, $f_{\gamma c}^c > M\gamma$;
(H2') $M < f_{W0}$, $f_{W\infty} \leq \infty$, $f_0^c < 1/K$.

Then problem (1.1), (1.2) *has two positive solutions in* $\mathcal{P}$.

Proof. Since (H1') implies (H1) and (H2') implies (H2), the result follows. □

The proofs of the following two results are similar to those given above and are omitted.

Theorem 3.7. *Assume that there exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < \gamma c_2$ *such that*

(H3) $f_0^{c_1} \leq 1/K$ *and* $f_{\gamma c_2}^{c_2} \geq M\gamma$,

or that there exist constants $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < c_2$ *such that*

(H4) $f_{\gamma c_1}^{c_1} \geq M\gamma$ *and* $f_0^{c_2} \leq 1/K$.

Then problem (1.1), (1.2) *has a positive solution.*

Corollary 3.8. *Suppose either one of the following conditions holds:*

(H3') $0 \leq f_W^0 < 1/K$ *and* $M\gamma < f_{W\infty} \leq \infty$;
(H4') $0 \leq f_W^\infty < 1/K$ *and* $M\gamma < f_{W0} \leq \infty$.

Then problem (1.1), (1.2) *has a positive solution.*

4. Examples

In the first example, for γ, K, and M given by (2.11), (3.4), and (3.5), respectively, assume positive constants $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1 < \gamma c_2$, $c_2 < c_3$ and

$$\frac{c_1}{K} \leq M\gamma c_2 + \delta \leq \frac{c_3}{K},$$

for some $\delta > 0$. Consider a particular case of equation (1.1) given by

$$(EIW^{\Delta\nabla})^{\nabla\Delta}(x) = m(x)f(W) \quad \text{for all} \quad x \in [x_1, x_n]_{\mathbb{X}}, \tag{4.1}$$

where

$$f(W) = \begin{cases} \frac{1}{K}W & \text{if } W \in [0, c_1], \\ \frac{M\gamma c_2+\delta-\frac{c_1}{K}}{\gamma c_2 - c_1}(W - c_1) + \frac{c_1}{K} & \text{if } W \in [c_1, \gamma c_2], \\ \frac{\frac{c_3}{K}-M\gamma c_2-\delta}{c_3-\gamma c_2}(W - c_3) + \frac{c_3}{K} & \text{if } W \geq \gamma c_2. \end{cases}$$

As f satisfies assumption (H1), by Theorem 3.5, problem (4.1), (1.2) has two positive solutions.

For the second example consider, on the time scale $\mathbb{X} = [0,1]$, the boundary value problem composed by the equation

$$W^{(4)}(x) = x\left(\frac{x}{5} + (W(x))^2\right), \quad \text{for } x \in \mathbb{X}, \tag{4.2}$$

with the boundary conditions

$$\begin{gathered} W(0) = 0.2W\Big(\frac{1}{3}\Big) + 0.5W\Big(\frac{2}{3}\Big), \\ W'(0) = 0, \quad W''(1) = 0, \\ W'''(1) = 0.1W'''\Big(\frac{1}{3}\Big) + 0.3W'''\Big(\frac{2}{3}\Big). \end{gathered} \tag{4.3}$$

In fact this is a particular case of the initial problem (1.1), (1.2), with $EI(x) \equiv 1$, $m(x) = x$, $f(x, W(x)) = \frac{x}{5} + (W(x))^2$, $n = 4$, $\rho(x) = x$, $\sigma(x) = x$, $x_2 = \frac{1}{3}$ and $x_3 = \frac{2}{3}$. Applying the Green's function given in Example 2.1, then $K = 0.72921$, $\gamma = \frac{14}{27}$ and $M = \frac{2916}{11}$.

For $c_1 = \frac{1}{2070}$, $c_2 = 1$ and $c_3 = 552$ assumption (H2) holds and, by Theorem 3.5, problem (4.2), (4.3) has two positive solutions in the cone

$$\mathcal{P} = \big\{W \in C([0,1]) : W(x) \geq 0 \text{ on } [0,1] \text{ and } W(x) \geq \frac{14}{27}||W|| \text{ on } [\frac{1}{3}, \frac{2}{3}]\big\}.$$

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