Electronic Journal of Differential Equations, Vol. 2010(2010), No. 01, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INITIAL-BOUNDARY VALUE PROBLEMS FOR QUASILINEAR DISPERSIVE EQUATIONS POSED ON A BOUNDED INTERVAL

ANDREI V. FAMINSKII, NIKOLAI A. LARKIN

Abstract. This paper studies nonhomogeneous initial-boundary value problems for quasilinear one-dimensional odd-order equations posed on a bounded interval. For reasonable initial and boundary conditions we prove existence and uniqueness of global weak and regular solutions. Also we show the exponential decay of the obtained solution with zero boundary conditions and right-hand side, and small initial data.

1. Introduction

This work concerns global well-posedness of nonhomogeneous initial-boundary value problems for general odd-order quasilinear partial differential equations

$$u_t + (-1)^{l+1}\partial_x^{2l+1}u + \sum_{j=0}^{2l} a_j \partial_x^j u + uu_x = f(t,x), \tag{1.1}$$

where $l \in \mathbb{N}$, a_j are real constants. This class of equations includes well-known Korteweg-de Vries and Kawahara equations which model the dynamics of long small-amplitude waves in various media [3, 30, 42].

Our study is motivated by physics and numerics and our main goal is to formulate a correct nonhomogeneous initial-boundary value problem for (1.1) in a bounded interval and to prove the existence and uniqueness of global in time weak and regular solutions in a large scale of Sobolev spaces as well as to study decay of solutions while $t \to \infty$. Dispersive equations such as KdV and Kawahara equations have been developed for unbounded regions of wave propagations, however, if one is interested in implementing numerical schemes to calculate solutions in these regions, there arises the issue of cutting off a spatial domain approximating unbounded domains by bounded ones. In this occasion some boundary conditions are needed to specify the solution. Therefore, precise mathematical analysis of boundary value problems in bounded domains for general dispersive equations is welcome and attracts attention of specialists in the area of dispersive equations, especially KdV and BBM equations, [2, 5, 8, 7, 9, 10, 11, 12, 13, 14, 18, 17, 21, 22, 23, 24, 26, 27, 28, 29, 33, 34, 35, 36, 39, 45, 46]. Cauchy problem for dispersive equations of

2000 *Mathematics Subject Classification.* 35M20, 35Q72.
Key words and phrases. Nonlinear boundary value problems; odd-order differential equations; existence and uniqueness.

Submitted October 2, 2009. Published January 5, 2010.
A. V. Faminskii was supported by grant 06-01-00253 from RFBR.

high orders was successfully explored by various authors, [4, 15, 16, 21, 31, 41, 44]. On the other hand, we know few published results on initial-boundary value problems posed on a finite interval for general nonlinear odd-order dispersive equations, such as the Kawahara equation, see [19, 20, 37].

In this paper we study an initial-boundary value problem for (1.1) in a rectangle $Q_T = (0,T) \times (0,1)$ with initial data

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{1.2}$$

and boundary data

$$\partial_x^j u(t,0) = \mu_j(t), \quad j = 0,\dots,l-1, \tag{1.3}$$

$$\partial_x^j u(t,1) = \nu_j(t), \quad j = 0,\dots,l,\ t \in (0,T). \tag{1.4}$$

Well-posedness of such a problem for a linearized version of (1.1) with homogeneous initial and boundary data (1.2)–(1.4) was established in [40]. It should be noted that imposed boundary conditions are reasonable at least from mathematical point of view, see comments in [19].

The theory of global solvability of dispersive equations is based on conservation laws, the first one – in L^2. Let $u(t,x)$ be a sufficiently smooth and decaying while $|x| \to \infty$ solution of an initial value problem for (1.1) (where $a_{2j} = 0$, $f \equiv 0$), then

$$\int_{\mathbb{R}} u^2\,dx = \text{const.}$$

The analogous equality can be written for problem (1.1)–(1.4) in the case of zero boundary data. In the general case one has to make this data zero with the help of a certain auxiliary function. In the present paper we construct a solution of an initial-boundary value problem for the linear homogeneous equation

$$u_t + (-1)^{l+1}\partial_x^{2l+1} u = 0 \tag{1.5}$$

with the same initial and boundary data (1.2)–(1.4) and use it as such an auxiliary function. This idea gives us an opportunity to establish our existence results for (1.1) under natural assumptions on boundary data (see Remark 2.11 below).

Another important fact is extra smoothing of solutions in comparison with initial data. In a finite domain it was first established for the KdV equation in [33, 10, 11] based on multiplication of the equation by $(1+x)u$ and consequent integration. In our case, we also have an extra smoothing effect. Roughly speaking, if $u_0 \in H^{(2l+1)k}(0,1)$, then $u \in L^2(0,T; H^{(2l+1)k+l}(0,1))$.

It has been shown in [35, 36] that the KdV equation is implicitly dissipative. This means that for small initial data the energy decays exponentially as $t \to +\infty$ without any additional damping terms in the equation. Moreover, the energy decays even for the modified KdV equation with a linear source term, [36]. In the present paper we prove that this phenomenon takes place for general dispersive equations of odd-orders.

The paper has the following structure. Section 2 contains main notations and definitions. The main results of the paper on well-posedness of the considered problem are also formulated in this section. In Section 3 we study the aforementioned initial-boundary value problem for equation (1.5). Section 4 is devoted to the corresponding problem for a complete linear equation. In Section 5 local well-posedness of the original problem is established. Section 6 contains global a priori estimates. Finally, the decay of small solutions, while $t \to +\infty$, is studied in Section 7.

2. Notation and statement of main results

For any space of functions, defined on the interval $(0,1)$, we omit the symbol $(0,1)$, for example, $L^p = L^p(0,1)$, $H^k = H^k(0,1)$, $C_0^\infty = C_0^\infty(0,1)$ etc. Define linear differential operators in L^2 with constant coefficients

$$P_0 \equiv \sum_{j=0}^{2l} a_j \partial_x^j, \quad P \equiv (-1)^{l+1}\partial_x^{2l+1} + P_0.$$

The main assumption on P_0 is the following.

Definition 2.1. We say that the operator P_0 satisfies Assumption A if either

$$(-1)^j a_{2j} \geq 0, \quad j = 1,\dots,l,$$

or there is a natural number $m \leq l$ such, that

$$(-1)^m a_{2m} > 0 \quad \text{and} \quad a_{2j} = 0, \quad j = m+1,\dots,l.$$

Lemma 2.2. *Assumption A is equivalent to the following property: There exists a constant $c_0 \geq 0$ such that for any function $\varphi \in H^{2l+1}$, $\varphi(0) = \dots = \varphi^{(l-1)}(0) = 0$, $\varphi(1) = \dots = \varphi^{(l-1)}(1) = 0$,*

$$(P_0\varphi, \varphi) \geq -c_0\|\varphi\|_{L^2}^2 \tag{2.1}$$

(here and further $(\cdot,\cdot)$ denotes scalar product in L^2).

Proof. Sufficiency of Assumption A is obvious (in the second case by virtue of the Ehrling inequality, [1]).

In order to prove necessity, assume that there exists a natural $m \leq l$ such that $a_{2j} = 0, \quad j = m+1,\dots,l$. Consider a set of functions

$$\varphi_\lambda(x) \equiv \lambda^{1/2-m}\varphi(\lambda x)$$

for certain $\varphi \in C_0^\infty$, $\varphi \not\equiv 0$, and $\lambda \geq 1$ and write down (2.1) for φ_λ:

$$(P_0\varphi_\lambda, \varphi_\lambda) = \sum_{j=0}^{m} (-1)^j a_{2j}\lambda^{2(j-m)}\|\varphi^{(j)}\|_{L^2}^2 \geq -c_0\lambda^{-2m}\|\varphi\|_{L^2}^2.$$

It follows for $\lambda \to +\infty$ that $(-1)^m a_{2m} \geq 0$. □

Lemma 2.3. *If the operator P_0 satisfies Assumption A, then for any function φ as in Lemma 2.2*

$$(P\varphi, \varphi) \geq -c_0\|\varphi\|_{L^2}^2 - \frac{1}{2}\big(\varphi^{(l)}(1)\big)^2. \tag{2.2}$$

Moreover, for certain positive constants c_1, c_2

$$(P\varphi, (1+x)\varphi) \geq c_1\|\varphi^{(l)}\|_{L^2}^2 - c_2\|\varphi\|_{L^2}^2 - \big(\varphi^{(l)}(1)\big)^2. \tag{2.3}$$

Proof. Inequality (2.2) is obvious because

$$(-1)^{l+1}(\varphi^{(2l+1)}, \varphi) = -\frac{1}{2}\big(\varphi^{(l)}\big)^2\big|_0^1.$$

Then integration by parts yields

$$(-1)^{l+1}(\varphi^{(2l+1)}, (1+x)\varphi) = \frac{2l+1}{2}\|\varphi^{(l)}\|_{L^2}^2 + \frac{1}{2}\big(\varphi^{(l)}(0)\big)^2 - \big(\varphi^{(l)}(1)\big)^2,$$
$$a_{2l}(\varphi^{(2l)}, (1+x)\varphi) = (-1)^l a_{2l}\|(1+x)^{1/2}\varphi^{(l)}\|_{L^2}^2 \geq 0;$$

and for $j \le l-1$ again with the use of the Ehrling inequality

$$\begin{aligned} &a_{2j+1}(\varphi^{(2j+1)},(1+x)\varphi)+a_{2j}(\varphi^{(2j)},(1+x)\varphi)\\ &=(-1)^{j+1}\frac{2j+1}{2}a_{2j+1}\|\varphi^{(j)}\|^2_{L^2}+(-1)^j a_{2j}\|(1+x)^{1/2}\varphi^{(j)}\|^2_{L^2}\\ &\ge -\delta\|\varphi^{(l)}\|^2_{L^2}-c(\delta)\|\varphi\|^2_{L^2}, \end{aligned}$$

where $\delta>0$ can be chosen arbitrary small, we obtain (2.3). $\square$

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be respectively the direct and inverse Fourier transforms of a function f. For $s\in\mathbb{R}$ define the fractional order Sobolev space

$$H^s(\mathbb{R})=\big\{f:\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\widehat{f}(\xi)]\in L_2(\mathbb{R})\big\}$$

and for a certain interval $I\subset\mathbb{R}$ let $H^s(I)$ be a space of restrictions on I of functions from $H^s(\mathbb{R})$. Define also

$$H^s_0(I)=\big\{f\in H^s(\mathbb{R}):\operatorname{supp} f\subset\overline{I}\big\}.$$

If ∂I is a finite part of the boundary of the interval I, then for $s\in(k+1/2,k+3/2)$, where $k\ge 0$ – integer,

$$H^s_0(I)=\big\{f\in H^s(I):f^{(j)}\big|_{\partial I}=0,\ j=0,\dots,k\big\}.$$

Note, that $H^s_0(I)=H^s(I)$ for $s\in[0,1/2)$.

If $\mathcal{X}$ is a certain Banach (or full countable-normed) space, define by $C_b(\overline{I};\mathcal{X})$ a space of continuous bounded mappings from $\overline{I}$ to $\mathcal{X}$. Let

$$\begin{gathered} C^k_b(\overline{I};\mathcal{X})=\big\{f(t):\partial^j_t f\in C_b(\overline{I};\mathcal{X}),\ j=0,\dots,k\big\},\\ C^\infty_b(\overline{I};\mathcal{X})=\big\{f(t):\partial^j_t f\in C_b(\overline{I};\mathcal{X}),\ \forall j\ge 0\big\}. \end{gathered}$$

If I is a bounded interval, the index b is omitted.

The symbol $L^p(I;\mathcal{X})$ is used in the usual sense for the space of Bochner measurable mappings from I to $\mathcal{X}$, summable with order p (essentially bounded if $p=+\infty$).

Next we introduce some special functional spaces.

Definition 2.4. For integer $k\ge 0$, $T>0$ and an interval (bounded or unbounded) $I\subset\mathbb{R}$ define

$$\begin{aligned} X_k((0,T)\times I)=\{u(t,x):\partial^n_t u\in C([0,T];H^{(2l+1)(k-n)}(I))\\ \cap L^2(0,T;H^{(2l+1)(k-n)+l}(I)),\ n=0,\dots,k\}, \end{aligned}$$

$$\begin{aligned} &M_k((0,T)\times I)\\ &=\{f(t,x):\partial^k_t f\in L^2(0,T;H^{-l}(I)),\partial^n_t f\in C([0,T];H^{(2l+1)(k-n-1)}(I))\\ &\quad\cap L^2(0,T;H^{(2l+1)(k-n)-l-1}(I)),\ n=0,\dots,k-1\}. \end{aligned}$$

Obviously,

$$\|P_0 u\|_{M_k((0,T)\times I))}\le c\|u\|_{X_k((0,T)\times I))}. \tag{2.4}$$

In fact, we construct solutions to problem (1.1)–(1.4) in the spaces $X_k(Q_T)$ for the right parts of equation (1.1) in the spaces $M_k(Q_T)$. To describe properties of boundary functions μ_j, ν_j we use the following functional spaces.

Definition 2.5. Let $s \geq 0$, $m = l-1$ or $m = l$, define

$$\mathcal{B}_s^m(0,T) = \prod_{j=0}^m H^{s+(l-j)/(2l+1)}(0,T).$$

Also we use auxiliary subsets of $\mathcal{B}_s^m(0,T)$:

$$\mathcal{B}_{s0}^m(0,T) = \prod_{j=0}^m H_0^{s+(l-j)/(2l+1)}(\mathbb{R}_+)\big|_{(0,T)}, \quad \mathbb{R}_+ = (0,+\infty).$$

For the study of properties of equation (1.5) we need more sophisticated spaces than X_k.

Definition 2.6. For $s \geq 0$, $I \subset \mathbb{R}$ define

$$\begin{aligned} Y_s((0,T)\times I) =&\{u(t,x) : \partial_t^n u \in C([0,T]; H^{(2l+1)(s-n)}(I)),\ n = 0,\dots,[s], \\ &\partial_x^j u \in C_b(\overline{I}; H^{s+(l-j)/(2l+1)}(0,T)),\ j = 0,\dots,[(2l+1)s]+l\}. \end{aligned}$$

Obviously, $Y_k(Q_T) \subset X_k(Q_T)$. The spaces Y_s originate from internal properties of the linear operator $\partial_t + (-1)^{l+1}\partial_x^{2l+1}$. In fact, consider an initial value problem in a strip $\Pi_T = (0,T)\times\mathbb{R}$ for (1.5) with the initial data (1.2). This problem was studied in [31]. In particular, if $u_0 \in H^{(2l+1)s}(\mathbb{R})$, then for any $T > 0$ there exists a solution of (1.5), (1.2), $S(t,x;u_0)$, given by the formula

$$S(t,x;u_0) = \mathcal{F}_x^{-1}[e^{i\xi^{2l+1}t}\widehat{u}_0(\xi)](x). \tag{2.5}$$

For this solution for any $t \in \mathbb{R}$ and integer $0 \leq n \leq s$, $0 \leq j \leq (2l+1)(s-n)$

$$\|\partial_t^n \partial_x^j S(t,\cdot;u_0)\|_{L^2(\mathbb{R})} = \|u_0^{((2l+1)n+j)}\|_{L^2(\mathbb{R})}, \tag{2.6}$$

and for any $x \in \mathbb{R}$ and integer $0 \leq j \leq (2l+1)s + l$

$$\|D_t^{s+(l-j)/(2l+1)}\partial_x^j S(\cdot,x;u_0)\|_{L^2(\mathbb{R})} = c(l)\|D_x^{(2l+1)s}u_0\|_{L^2(\mathbb{R})}, \tag{2.7}$$

where the symbol D^s denotes the Riesz potential of the order $-s$. In particular, the traces of $\partial_x^j S$ for $x=0$, $j=0,\dots,m=l-1$, and $x=1$, $j=0,\dots,m=l$ lie in $\mathcal{B}_s^m(0,T)$. To formulate compatibility conditions for the original problem we now introduce certain special functions.

Definition 2.7. Let $\Phi_0(x) \equiv u_0(x)$ and for natural n

$$\Phi_n(x) \equiv \partial_t^{n-1}f(0,x) - P\Phi_{n-1}(x) - \sum_{m=0}^{n-1}\binom{n-1}{m}\Phi_m(x)\Phi'_{n-m-1}(x).$$

Now we can present the main results of this paper.

Theorem 2.8 (local well-posedness)**.** *Let the operator P_0 satisfy Assumption A. Let $u_0 \in H^{(2l+1)k}(0,1)$, $(\mu_0,\dots,\mu_{l-1}) \in \mathcal{B}_k^{l-1}(0,T)$, $(\nu_0,\dots,\nu_l) \in \mathcal{B}_k^l(0,T)$, $f \in M_k(Q_T)$ for some $T > 0$ and integer $k \geq 0$. Assume also that $\mu_j^{(n)}(0) = \Phi_n^{(j)}(0)$, $j = 0,\dots,l-1$, $\nu_j^{(n)}(0) = \Phi_n^{(j)}(1)$, $j=0,\dots,l$, for $0 \leq n \leq k-1$. Then there exists $t_0 \in (0,T]$ such that* (1.1)–(1.4) *is well-posed in $X_k(Q_{t_0})$.*

Theorem 2.9 (global well-posedness)**.** *Let the hypothesis of Theorem 2.8 be satisfied and, in addition, if $k=0$, then $f \in L^1(0,T;L^2)$, and if $l=1$, $k=0$, then $\mu_0,\nu_0 \in H^{1/3+\varepsilon}(0,T)$ for a certain $\varepsilon > 0$. Then* (1.1)–(1.4) *is well-posed in $X_k(Q_T)$.*

Remark 2.10. A problem is well-posed in the space X_k, if there exists a unique solution $u(t,x)$ in this space and the map $(u_0,(\mu_0,\dots,\mu_{l-1}),(\nu_0,\dots,\nu_l),f)\mapsto u$ is Lipschitz continuous on any ball in the corresponding norms.

Remark 2.11. Properties (2.7) of the solution S to the initial-value problem (1.5), (1.2) show that the smoothness conditions on the boundary data in our results are natural (with the only exception in the case $l=1$, $k=0$ for global results) because they originate from the properties of the operator $\partial_t+(-1)^{l+1}\partial_x^{2l+1}$.

Remark 2.12. All these well-posedness results can be easily generalized for an equation of (1.1) type with a nonlinear term $g(u)u_x$, where a sufficiently smooth function g has not more than a linear growth rate.

3. Linear problem for a homogeneous equation

The goal of this section is to construct solutions to an initial-boundary value problem in Q_T for equation (1.5) with initial and boundary data (1.2)–(1.4) in the spaces $Y_s(Q_T)$. Uniqueness will be discussed in the next section for more general linear equations.

In what follows, we need simple properties of roots $r_m(\lambda,\varepsilon)$, $m=0,\dots,2l$ of an algebraic equation

$$r^{2l+1}=(-1)^l(\varepsilon+i\lambda),\quad \varepsilon\ge 0,\ \lambda\in\mathbb{R},\ (\lambda,\varepsilon)\ne(0,0). \tag{3.1}$$

An enumeration of these roots can be chosen such that they are continuous with respect to (λ,ε), $r_m(-\lambda,\varepsilon)=\overline{r_m(\lambda,\varepsilon)}$,

$$\operatorname{Re} r_m<0,\quad m=0,\dots,l-1;\quad \operatorname{Re} r_m>0,\quad m=l,\dots,2l-1; \tag{3.2}$$

$$\operatorname{Re} r_{2l}>0,\quad \varepsilon>0;\quad \operatorname{Re} r_{2l}=0,\quad \varepsilon=0. \tag{3.3}$$

It is obvious that for any m and $j\ne m$

$$|r_m|=(\lambda^2+\varepsilon^2)^{1/(4l+2)},\quad |r_m-r_j|=c(l,m,j)(\lambda^2+\varepsilon^2)^{1/(4l+2)}. \tag{3.4}$$

Denote $r_m(\lambda)\equiv r_m(\lambda,0)$, then

$$|\operatorname{Re} r_m(\lambda)|=c(l,m)|\lambda|^{1/(2l+1)},\quad m=0,\dots,2l-1;\quad r_{2l}(\lambda)=i\lambda^{1/(2l+1)}. \tag{3.5}$$

To construct the desired solutions to the problem in a bounded rectangle, we first consider corresponding problems in half-strips and start with a problem in a right one: $\Pi_T^+=(0,T)\times\mathbb{R}_+$.

Lemma 3.1. *Let $u_0\in H^{(2l+1)s}(\mathbb{R}_+)$, $(\mu_0,\dots,\mu_{l-1})\in\mathcal{B}_s^{l-1}(0,T)$ for some $T>0$ and $s\ge 0$ such that $s+\frac{l-j}{2l+1}-\frac12$ is non-integer for any $j=0,\dots,l-1$. Assume also that $\mu_j^{(n)}(0)=(-1)^{nl}u_0^{((2l+1)n+j)}(0)$ for $n=0,\dots,[s+\frac{l-j}{2l+1}-\frac12]$, $j=0,\dots,l-1$. Then there exists a solution to problem (1.5), (1.2), (1.3) $u(t,x)\in Y_s(\Pi_T^+)$ such that*

$$\|u\|_{Y_s(\Pi_T^+)}\le c(T,l,s)\Big(\|u_0\|_{H^{(2l+1)s}(\mathbb{R}_+)}+\|(\mu_0,\dots,\mu_{l-1})\|_{\mathcal{B}_s^{l-1}(0,T)}\Big). \tag{3.6}$$

Proof. We construct the desired solution in the form

$$u(t,x)=S(t,x;u_0)+w(t,x), \tag{3.7}$$

where u_0 is extended to the whole real line $\mathbb{R}$ in the same class $H^{(2l+1)s}$ with an equivalent norm, the function S is defined by formula (2.5) and $w(t,x)$ is a solution to the problem in Π_T^+ for equation (1.5) with zero initial data (1.2) and boundary data

$$\partial_x^j w(t,0) = \sigma_j(t), \quad j = 0,\dots,l-1, \tag{3.8}$$

where $\sigma_j(t) \equiv \mu_j(t) - \partial_x^j S(t,0;u_0)$. Note that, by virtue of (2.7) and compatibility conditions, $(\sigma_0,\dots,\sigma_{l-1}) \in \mathcal{B}_{s0}^{l-1}(0,T)$.

Assume at first that all functions $\sigma_j \in C_0^\infty(\mathbb{R}_+)$. In this case, according to [43], there exists a solution $w(t,x)$ and $w \in C^\infty([0,T];H^\infty(\mathbb{R}_+))$ for any $T>0$. Moreover, if $\sigma_j(t) = 0$ for $t \ge T_0 > 0$ and all j, then it is easy to show that for $t \ge T_0$ and all integer $n \ge 0$

$$\frac{d}{dt}\|\partial_t^n w\|_{L^2(\mathbb{R}_+)} \le 0, \tag{3.9}$$

whence with the use of (1.5) itself one can prove $w \in C_b^\infty(\overline{\mathbb{R}}_+;H^\infty(\mathbb{R}_+))$.

Therefore, for any $p = \varepsilon + i\lambda$, where $\varepsilon > 0$, we can define the Laplace transform

$$\widetilde{w}(p,x) \equiv \int_{\mathbb{R}_+} e^{-pt} w(t,x)\,dt. \tag{3.10}$$

The function $\widetilde{w}(p,x)$ is a solution to the problem

$$p\widetilde{w}(p,x) + (-1)^{l+1}\partial_x^{2l+1}\widetilde{w}(p,x) = 0, \quad x \ge 0, \tag{3.11}$$

$$\partial_x^j \widetilde{w}(p,0) = \widetilde{\sigma}_j(p) \equiv \int_{\mathbb{R}_+} e^{-pt}\sigma_j(t)\,dt, \quad j = 0,\dots,l-1. \tag{3.12}$$

Since $\widetilde{w}(p,x) \to 0$ as $x \to +\infty$, it follows from (3.2)–(3.4) that

$$\widetilde{w}(p,x) = \sum_{m=0}^{l-1}\sum_{k=0}^{l-1} c_{mk}(l)(\lambda^2+\varepsilon^2)^{-k/(4l+2)} e^{r_m(\lambda,\varepsilon)x}\widetilde{\sigma}_k(\varepsilon+i\lambda).$$

Using the formula of inversion of the Laplace transform and passing to the limit as $\varepsilon \to +0$, we find

$$\begin{aligned} w(t,x) &= \sum_{m=0}^{l-1}\sum_{k=0}^{l-1} c_{mk}(l)\mathcal{F}_t^{-1}\Big[|\lambda|^{-k/(2l+1)} e^{r_m(\lambda)x}\widehat{\sigma}_k(\lambda)\Big](t) \\ &\equiv \sum_{m=0}^{l-1}\sum_{k=0}^{l-1} c_{mk}(l) w_{mk}(t,x). \end{aligned} \tag{3.13}$$

Now consider the integral

$$\mathcal{I}_m(t,x) \equiv \int_{\mathbb{R}} e^{i\lambda t + r_m(\lambda)x} f(\lambda)\,d\lambda, \quad m = 0,\dots,l-1,$$

and establish that, uniformly with respect to $t \in \mathbb{R}$,

$$\|\mathcal{I}_m(t,\cdot)\|_{L^2(\mathbb{R}_+)} \le c(l,m)\big\||\lambda|^{l/(2l+1)} f(\lambda)\big\|_{L^2(\mathbb{R})}. \tag{3.14}$$

The proof of (3.14) is based on the following fundamental inequality from [6]: If a continuous function $\gamma(\xi)$ satisfies an inequality $\operatorname{Re}\gamma(\xi) \le -\varepsilon|\xi|$ for some $\varepsilon > 0$ and all $\xi \in \mathbb{R}$, then

$$\Big\|\int_{\mathbb{R}} e^{\gamma(\xi)x} f(\xi)\,d\xi\Big\|_{L^2(\mathbb{R}_+)} \le c(\varepsilon)\|f\|_{L^2(\mathbb{R})}. \tag{3.15}$$

Changing variables $\xi = \lambda^{1/(2l+1)}$, we derive from (3.15) (since $\operatorname{Re} r_m(\xi^{2l+1}) = -c(l,m)|\xi|$)

$$\|\mathcal{I}_m\|_{L^2(\mathbb{R}_+)} \le c(l,m)\|\xi^{2l} f(\xi^{2l+1})\|_{L^2(\mathbb{R})}$$

which proves (3.14).

Applying (3.14) to (3.13) yields, by virtue of (3.3), (3.4), that uniformly with respect to $t \in \mathbb{R}$ for $n = 0, \dots, [s]$, $j = 0, \dots, [(2l+1)(s-n)]$

$$\|\partial_t^n \partial_x^j w_{mk}(t, \cdot)\|_{L^2(\mathbb{R}_+)} \le c(l, m, s) \|\sigma_k\|_{H^{n+(l+j-k)/(2l+1)}(\mathbb{R})}. \tag{3.16}$$

Next, let

$$\sigma_{k0} \equiv \mathcal{F}^{-1}\left[\widehat{\sigma}_k(\lambda)\chi(\lambda)\right], \quad \sigma_{k1} \equiv \sigma_k - \sigma_{k0},$$

where χ is the characteristic function of the interval $(-1, 1)$, and represent w_{mk} as a sum of two corresponding functions w_{mk0} and w_{mk1}. Then, by virtue of (3.16), uniformly with respect to $x \ge 0$ for $j = 0, \dots, [(2l+1)s] + l$

$$\begin{aligned} \|\partial_x^j w_{mk0}(\cdot, x)\|_{H^{s+(l-j)/2l+1)}(0,T)} &\le c(T) \|w_{mk0}\|_{C^{[s]+2}([0,T]; H^{[(2l+1)s]+l+1}(\mathbb{R}_+))} \\ &\le c(l, s, T) \|\sigma_{k0}\|_{H^{2s+4}(\mathbb{R})} \\ &\le c_1(l, s, T) \|\sigma_k\|_{L^2(\mathbb{R})} \end{aligned} \tag{3.17}$$

and since $\operatorname{Re} r_m(\lambda) \le 0$,

$$\|\partial_x^j w_{mk1}(\cdot, x)\|_{H^{s+(l-j)/(2l+1)}(\mathbb{R})} \le c(l, k) \|\sigma_k\|_{H^{(s+(l-k)/(2l+1)}(\mathbb{R})}. \tag{3.18}$$

Combining (2.6), (2.7), (3.7), (3.8), (3.13), (3.16)–(3.18) we derive (3.6) in the smooth case and via closure in the general case. $\square$

Corollary 3.2. *Let $J(t, x; \sigma_0, \dots, \sigma_{l-1})$ denotes the solution to the problem in Π_T^+ for equation* (1.5) *with zero initial data and boundary data* (3.8) *(where w must be substituted by u) constructed in Lemma 3.1 . Then J is infinitely differentiable for $x > 0$; and for any $x_0 > 0$ and integer $n, j \ge 0$*

$$\sup_{x \ge x_0} |\partial_t^n \partial_x^j J(t, x)| \le c(l, n, j, x_0^{-1}) \sum_{m=0}^{l-1} \|\sigma_m\|_{L^2(0,T)}. \tag{3.19}$$

Proof. From representation (3.13) we obtain

$$\partial_t^n \partial_x^j w_{mk}(t, x) = \mathcal{F}_t^{-1}\left[(i\lambda)^n r_m^j(\lambda) |\lambda|^{-k/(2l+1)} e^{r_m(\lambda)x} \widehat{\sigma}_k(\lambda)\right](t),$$

where, by virtue of (3.2) and (3.5),

$$\operatorname{Re} r_m(\lambda) x \le -c(l, m) |\lambda|^{1/(2l+1)} x_0, \quad |\lambda|^{n + \frac{j-k}{2l+1}} e^{-c(l,m)|\lambda|^{1/(2l+1)} x_0} \in L^2(\mathbb{R}).$$

$\square$

Now consider (1.5), (1.2) in a half-strip $\Pi_T^- = (0, T) \times \mathbb{R}_-$, $\mathbb{R}_- = (-\infty, 0)$, with boundary data

$$\partial_x^j u(t, 0) = \nu_j(t), \quad j = 0, \dots, l. \tag{3.20}$$

Lemma 3.3. *Let $u_0 \in H^{(2l+1)s}(\mathbb{R}_-)$, $(\nu_0, \dots, \nu_l) \in \mathcal{B}_s^l(0, T)$ for some $T > 0$ and $s \ge 0$ such that $s + \frac{l-j}{2l+1} - \frac{1}{2}$ is non-integer for any $j = 0, \dots, l$. Assume also that $\nu_j^{(n)}(0) = (-1)^{nl} u_0^{((2l+1)n+j)}(0)$ for $n = 0, \dots, [s + \frac{l-j}{2l+1} - \frac{1}{2}]$, $j = 0, \dots, l$. Then there exists a solution to problem* (1.5), (1.2), (3.20), *$u(t, x) \in Y_s(\Pi_T^-)$, such that*

$$\|u\|_{Y_s(\Pi_T^-)} \le c(T, l, s) \left(\|u_0\|_{H^{(2l+1)s}(\mathbb{R}_-)} + \|(\nu_0, \dots, \nu_l)\|_{\mathcal{B}_s^l(0,T)}\right). \tag{3.21}$$

Proof. The scheme of the proof repeats the one of Lemma 3.1. The desired solution is constructed in the form (3.7), where w is a solution to the problem in Π_T^- for equation (1.5) with zero initial data and boundary data

$$\partial_x^j w(t,0) = \sigma_j(t) \equiv \nu_j(t) - \partial_x^j S(t,0;u_0), \quad j = 0,\dots,l. \tag{3.22}$$

By virtue of compatibility conditions, $(\sigma_0,\dots,\sigma_l) \in \mathcal{B}^l_{s0}(0,T)$.

Assuming temporarily that all functions $\sigma_j \in C_0^\infty(\mathbb{R}_+)$, results of [43] provide that there exists a solution to this problem $w \in C_b^\infty(\overline{\mathbb{R}}^t_+; H^\infty(\mathbb{R}_-))$ (where inequality (3.9) transforms into a corresponding equality). The Laplace transform $\widetilde{w}(p,x)$, given by formula (3.10), satisfies (3.11) for $x \le 0$ and $(l+1)$ boundary conditions (3.12). Using the properties of the roots of (3.1), by analogy with (3.13), one can easily derive

$$\begin{aligned} w(t,x) &= \sum_{m=l}^{2l}\sum_{k=0}^{l} c_{mk}(l)\mathcal{F}_t^{-1}\big[|\lambda|^{-k/(2l+1)}e^{r_m(\lambda)x}\widehat{\sigma}_k(\lambda)\big](t) \\ &\equiv \sum_{m=l}^{2l}\sum_{k=0}^{l} c_{mk}(l) w_{mk}(t,x). \end{aligned} \tag{3.23}$$

Similarly to (3.16) for $m = l,\dots,2l-1$; $n = 0,\dots,[s]$; $j = 0,\dots,[(2l+1)(s-n)]$; uniformly with respect to $t \in \mathbb{R}$

$$\|\partial_t^n \partial_x^j w_{mk}(t,\cdot)\|_{L^2(\mathbb{R}_-)} \le c(l,m,s)\|\sigma_k\|_{H^{n+(l+j-k)/(2l+1)}(\mathbb{R})}.$$

For $m = 2l$, changing variables $\xi = \lambda^{1/(2l+1)}$ and using (2.5) and (3.5), we find

$$w_{(2l)k} = (2l+1)S(t,x;\mathcal{F}_x^{-1}\left[|\xi|^{2l-k}\widehat{\sigma}_k(\xi^{2l+1})\right]),$$

and, by virtue of (2.6), uniformly with respect to $t \in \mathbb{R}$

$$\begin{aligned} \|\partial_t^n \partial_x^j w_{(2l)k}(t,\cdot)\|_{L^2(\mathbb{R})} &= c(l)\||\xi|^{(2l+1)n+2l+j-k}\widehat{\sigma}_k(\xi^{2l+1})\|_{L^2(\mathbb{R})} \\ &\le c_1(l)\|\sigma_k\|_{H^{n+(l+j-k)/(2l+1)}(\mathbb{R})}. \end{aligned}$$

Since $\operatorname{Re} r_k(\lambda) \ge 0$, $m = l,\dots,2l$, similarly to (3.17), (3.18), one can obtain that for $j = 0,\dots,[(2l+1)s]+l$ uniformly with respect to $x \le 0$

$$\|\partial_x^j w_{mk}(\cdot,x)\|_{H^{s+(l-j)/(2l+1)}(0,T))} \le c(l,s,T,k)\|\sigma_k\|_{H^{s+(l-k)/(2l+1)}(\mathbb{R})}.$$

The end of the proof is the same as in Lemma 3.1. □

Now we pass to a problem on a bounded rectangle.

Lemma 3.4. *Let $u_0 \in H^{(2l+1)s}$, $(\mu_0,\dots,\mu_{l-1}) \in \mathcal{B}^{l-1}_s(0,T)$, $(\nu_0,\dots,\nu_l) \in \mathcal{B}^l_s(0,T)$ for some $T > 0$ and $s \ge 0$ such that $s + \frac{l-j}{2l+1} - \frac12$ is non-integer for any $j = 0,\dots,l$. Assume also that $\mu_j^{(n)}(0) = (-1)^{nl}u_0^{((2l+1)n+j)}(0)$, $j = 0,\dots,l-1$, $\nu_j^{(n)}(0) = (-1)^{nl}u_0^{((2l+1)n+j)}(1)$, $j = 0,\dots,l$, for $n = 0,\dots,[s+\frac{l-j}{2l+1}-\frac12]$. Then there exists a solution to problem (1.5), (1.2)–(1.4), $u(t,x) \in Y_s(Q_T)$, and the following inequality holds:*

$$\begin{aligned} \|u\|_{Y_s(Q_T)} \le c(T,l,s)\Big(&\|u_0\|_{H^{(2l+1)s}} \\ &+ \|(\mu_0,\dots,\mu_{l-1})\|_{\mathcal{B}^{l-1}_s(0,T)} + \|(\nu_0,\dots,\nu_l)\|_{\mathcal{B}^l_s(0,T)}\Big). \end{aligned} \tag{3.24}$$

Proof. We construct the desired solution in the form

$$u(t,x) = w(t,x) + v(t,x), \tag{3.25}$$

where $w(t,x)$ is a solution to an initial boundary-value problem in $\Pi^-_{T,1} = (0,T) \times (-\infty, 1)$ for equation (1.5) with initial and boundary conditions (1.2), (1.4) written for the function w, (u_0 is extended here to the half-line $(-\infty,1)$ in the same class $H^{(2l+1)s}$ with an equivalent norm). According to Lemma 3.3, the solution $w \in Y_s(\Pi^-_{T,1})$ exists and

$$\|w\|_{Y_s(\Pi^-_{T,1})} \le c(T,l,s)\big(\|u_0\|_{H^{(2l+1)s}} + \|(\nu_0,\dots,\nu_l)\|_{\mathcal{B}^l_s(0,T)}\big). \tag{3.26}$$

Let

$$\alpha_j(t) \equiv \mu_j(t) - \partial^j_x w(t,0), \quad j = 0,\dots,l-1.$$

It follows from (3.26)

$$\begin{aligned}\|(\alpha_0,\dots,\alpha_{l-1})\|_{\mathcal{B}^{l-1}_s(0,T)} \le c(T,l,s)\Big(&\|u_0\|_{H^{(2l+1)s}} + \|(\mu_0,\dots,\mu_{l-1})\|_{\mathcal{B}^{l-1}_s(0,T)} \\ &+ \|(\nu_0,\dots,\nu_l)\|_{\mathcal{B}^l_s(0,T)}\Big)\end{aligned} \tag{3.27}$$

and $(\alpha_0,\dots,\alpha_{l-1}) \in \mathcal{B}^{l-1}_{s0}(0,T)$ by virtue of the compatibility conditions in the point $(0,0)$.

Consider the following problem for the function v, in Q_T,

$$v_t + (-1)^{l+1}\partial^{2l+1}_x v = 0, \tag{3.28}$$

$$v(0,x) = 0, \quad x \in (0,1), \tag{3.29}$$

$$\partial^j_x v(t,0) = \alpha_j(t), \quad j = 0,\dots,l-1, \tag{3.30}$$

$$\partial^j_x v(t,1) = 0, \quad j = 0,\dots,l, \quad t \in (0,T). \tag{3.31}$$

To construct a solution, we consider the function $J(t,x;\sigma_0,\dots,\sigma_{l-1}) \in Y_s(\Pi^+_T)$ as in Corollary 3.2 for a certain set of functions $(\sigma_0,\dots,\sigma_{l-1}) \in \mathcal{B}^{l-1}_{s0}(0,T)$. Let

$$\beta_j(t) \equiv \partial^j_x J(t,1;\sigma_0,\dots,\sigma_{l-1}), \quad j = 0,\dots,l.$$

Due to (3.19), for any $\delta \in (0,T]$

$$\|(\beta_0,\dots,\beta_l)\|_{\mathcal{B}^l_s(0,\delta)} \le c(l,s)\delta^{1/2}\|(\sigma_0,\dots,\sigma_{l-1})\|_{\mathcal{B}^{l-1}_s(0,\delta)}.$$

Moreover, $(\beta_0,\dots,\beta_l) \in \mathcal{B}^l_{s0}(0,\delta)$.

Consider in the half-strip $\Pi^-_{\delta,1}$ a problem of the (1.5), (1.2), (1.4) type, where $u_0 \equiv 0$, $\nu_j \equiv -\beta_j$ for $j = 0,\dots,l$. It follows again from Lemma 3.3 that a solution to this problem $V \in Y_s(\Pi^-_{\delta,1})$ exists and, in particular, if

$$\gamma_j(t) \equiv \partial^j_x V(t,0), \quad j = 0,\dots,l-1,$$

then $(\gamma_0,\dots,\gamma_{l-1}) \in \mathcal{B}^{l-1}_{s0}(0,\delta)$ and

$$\begin{aligned}\|(\gamma_0,\dots,\gamma_{l-1})\|_{\mathcal{B}^{l-1}_s(0,\delta)} &\le c(T,l,s)\|(\beta_0,\dots,\beta_l)\|_{\mathcal{B}^l_s(0,\delta)} \\ &\le c_1(T,l,s)\delta^{1/2}\|(\sigma_0,\dots,\sigma_{l-1})\|_{\mathcal{B}^{l-1}_s(0,\delta)}.\end{aligned} \tag{3.32}$$

Consider a linear operator $\Gamma : (\sigma_0,\dots,\sigma_{l-1}) \mapsto (\gamma_0,\dots,\gamma_{l-1})$ in the space $\mathcal{B}^{l-1}_{s0}(0,\delta)$. For small $\delta = \delta(T,l,s)$ estimate (3.32) provides that the operator $(E+\Gamma)$ is invertible (E is the identity operator) and setting

$$\sigma_j(t) \equiv (E+\Gamma)^{-1}\alpha_j(t), \quad j = 0,\dots,l-1,$$

we obtain the desired solution to problem (3.28)–(3.31),

$$v(t,x) \equiv J(t,x;\sigma_0,\dots,\sigma_{l-1}) + V(t,x),$$

where

$$\|v\|_{Y_s(Q_\delta)} \le c(T,l,s)\|(\alpha_0,\dots,\alpha_{l-1})\|_{\mathcal{B}_s^{l-1}(0,T)}. \tag{3.33}$$

Thus, the solution $u(t,x)$ to problem (1.5), (1.2)–(1.4) in the rectangle Q_δ has been constructed and, according to (3.26), (3.27), (3.33), estimated in the space $Y_s(Q_\delta)$ by the right part of (3.24). Moving step by step (δ is constant), we obtain the desired solution in the whole rectangle Q_T. □

Remark 3.5. The idea to construct solutions in a bounded rectangle from solutions in half-strips for the linearized KdV equation goes back to the paper [29], but the method of study of these problems in the infinite domains in [29] differs from the one used here. The method of the present paper is analogous to [25].

4. Complete linear problem

In this section we consider an initial-boundary value problem in Q_T for the equation

$$u_t + Pu = f(t,x) \tag{4.1}$$

with initial and boundary conditions (1.2)–(1.4). First of all we introduce auxiliary functions necessary for compatibility conditions.

Definition 4.1. Let $\widetilde{\Phi}_0(x) \equiv u_0(x)$ and for natural n

$$\widetilde{\Phi}_n(x) \equiv \partial_t^{n-1} f(0,x) - P\widetilde{\Phi}_{n-1}(x).$$

Remark 4.2. It is easy to see that

$$\widetilde{\Phi}_n = (-1)^n P^n u_0 + \sum_{m=0}^{n-1} (-1)^{n-m-1} P^{n-m-1} \partial_t^m f\big|_{t=0}.$$

Lemma 4.3. *Let the operator P_0 satisfies Assumption A. Let $u_0 \in H^{(2l+1)k}$, $(\mu_0,\dots,\mu_{l-1}) \in \mathcal{B}_k^{l-1}(0,T)$, $(\nu_0,\dots,\nu_l) \in \mathcal{B}_k^l(0,T)$, $f \in M_k(Q_T)$ for some $T>0$ and integer $k \ge 0$. Assume also that $\mu_j^{(n)}(0) = \widetilde{\Phi}_n^{(j)}(0)$, $j=0,\dots,l-1$, $\nu_j^{(n)}(0) = \widetilde{\Phi}_n^{(j)}(1)$, $j=0,\dots,l$, for $n=0,\dots,k-1$. Then there exists a unique solution to problem* (4.1), (1.2)–(1.4) *$u(t,x) \in X_k(Q_T)$ and for any $t_0 \in (0,T]$*

$$\begin{aligned}\|u\|_{X_k(Q_{t_0})} \le c(T,l,k)\Big(&\|u_0\|_{H^{(2l+1)k}} + \|f\|_{M_k(Q_{t_0})} \\ &+ \|(\mu_0,\dots,\mu_{l-1})\|_{\mathcal{B}_k^{l-1}(0,T)} + \|(\nu_0,\dots,\nu_l)\|_{\mathcal{B}_k^l(0,T)}\Big).\end{aligned} \tag{4.2}$$

For any natural $n \le k$ the function $\partial_t^n u \in X_{k-n}(Q_T)$ is a solution to a problem of the (4.1), (1.2)–(1.4) *type, where u_0, μ_j, ν_j, f are substituted by $\widetilde{\Phi}_n$, $\mu_j^{(n)}$, $\nu_j^{(n)}$, $\partial_t^n f$.*

Proof. It is sufficient to prove this lemma for $k=0$ and $k=1$. The cases $k \ge 2$ are similar to $k=1$. First consider the case $k=0$. Let $\psi(t,x) \in Y_0(Q_T)$ be a solution to problem (1.5), (1.2)–(1.4) for the same u_0, μ_j, ν_j constructed in Lemma 3.4. Consider the initial-boundary value problem in Q_T for the equation

$$v_t + Pv = f - P_0\psi \equiv F \tag{4.3}$$

with zero initial and boundary conditions of the (1.2)–(1.4) type. By virtue of (2.4), $F \in M_0(Q_T)$ with an appropriate estimate of its norm in $M_0(Q_{t_0})$ by the right part of (4.2). Define

$$w(t,x) \equiv v(t,x)e^{-\lambda t}, \quad \lambda \geq c_0,$$

where c_0 is the constant from (2.1). Then (4.3) transforms into

$$w_t + (P + \lambda E)w = e^{-\lambda t}F \equiv F_1. \tag{4.4}$$

Consider the corresponding initial-boundary value problem for the function w as an abstract Cauchy problem in L^2

$$w_t = \mathcal{A}w + F_1, \quad w\big|_{t=0} = 0, \tag{4.5}$$

where $\mathcal{A} = -(P + \lambda E)$ is the closed linear operator in L^2 with the domain

$$D(\mathcal{A}) = \big\{g \in H^{2l+1} : g^{(j)}(0) = g^{(j)}(1) = g^{(l)}(1) = 0,\ j = 0, \dots, l-1\big\}.$$

The adjoint operator $\mathcal{A}^*$ is defined as

$$\mathcal{A}^* = -(P^* + \lambda E),$$

where P^* is the formally adjoint operator of P, with the domain

$$D(\mathcal{A}^*) = \big\{g^* \in H^{2l+1} : g^{*(j)}(0) = g^{*(j)}(1) = g^{*(l)}(0) = 0, \quad j = 0, \dots, l-1\big\}.$$

It is easy to see that, by virtue of (2.1), both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative which means

$$(\mathcal{A}g, g) \leq 0, \quad (\mathcal{A}^*g^*, g^*) \leq 0.$$

Assume F smooth, for example $F \in C^1([0,T]; H^{2l+1})$. By [38, Corollaries 4.4, Chapter 1 and 2.10 of Chapter 4], $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contraction in L^2 and the Cauchy problem (4.5) has a unique strong solution $w \in C([0,T]; H^{2l+1}) \cap C^1([0,T]; L^2)$. Consequently, the initial-boundary value problem for (4.3) with zero initial and boundary conditions has a unique solution $v(t,x)$ in the same class.

Multiplying (4.3) by $2(1+x)v(t,x)$ and integrating over Q_t, $t \in (0,T]$, we find that by virtue of (2.3)

$$\|v(t,\cdot)\|^2_{L^2} + c_1\|\partial_x^l v\|^2_{L^2(Q_t)} \leq c\|v\|^2_{L^2(Q_t)} + c\|F\|^2_{L^2(0,t;H^{-l})}. \tag{4.6}$$

This estimate gives an opportunity to construct an appropriate solution $v \in X_0(Q_T)$ to the considered problem in the general case $F \in L^2(0,T;H^{-l}) = M_0(Q_T)$ via closure and then, by the formula

$$u(t,x) = v(t,x) + \psi(t,x),$$

a solution to problem (4.1), (1.2)–(1.4) in the same space $X_0(Q_T)$ with the estimate (4.2) for $k = 0$.

For $k = 1$, we consider the initial-boundary value problem, in Q_T,

$$z_t + Pz = f_t, \tag{4.7}$$

$$z(0,x) = \widetilde{\Phi}_1(x), \quad x \in (0,1), \tag{4.8}$$

$$\partial_x^j z(t,0) = \mu_j'(t), \quad j = 0, \dots, l-1, \tag{4.9}$$

$$\partial_x^j z(t,1) = \nu_j'(t), \quad j = 0, \dots, l, \quad t \in (0,T). \tag{4.10}$$

Note that $\widetilde{\Phi}_1 \in L^2$, hence the hypothesis of Lemma 4.1 are satisfied for this problem in the case $k = 0$. Consider the solution $z \in X_0(Q_T)$ of (4.7)–(4.10) and define

$$u(t,x) \equiv u_0(x) + \int_0^t z(\tau,x)\,d\tau.$$

Using the compatibility conditions, it is easy to show that the function $u(t,x)$ is a solution to the original problem (4.1), (1.2)–(1.4) and $u, u_t \in X_0(Q_T)$. Expressing from the equation (4.1) the derivative

$$\partial_x^{2l+1} u = (-1)^{l+1}(f - u_t - P_0 u)$$

and using the Ehrling inequality, one can easily obtain that $\partial_x^{2l+1} u \in X_0(Q_T)$, thus to construct the desired solution $u \in X_1(Q_T)$ with estimate (4.2) for $k = 1$.

Uniqueness of the considered problem in $L^2(Q_T)$ can be proved via the Holmgren principle from the existence in $X_1(Q_T)$ of a solution to the adjoint problem

$$\begin{gathered}\varphi_t - P^*\varphi = f \in C_0^\infty(Q_T),\\ \varphi\big|_{t=T} = 0,\\ \partial_x^j\varphi\big|_{x=0} = 0, \quad j = 0,\dots,l,\\ \partial_x^j\varphi\big|_{x=1} = 0, \quad j = 0,\dots,l-1\end{gathered}$$

which follows by simple change of variables from the already established existence of a solution in the same space to the original problem. □

Corollary 4.4. *Let the hypothesis of Lemma 4.3 be satisfied for $k = 0$, $\mu_j = \nu_j \equiv 0$ for $j \le l-1$. Let $u \in X_0(Q_T)$ be a solution to corresponding problem* (4.1), (1.2)–(1.4). *Then for any $t \in (0,T]$*

$$\int_0^1 u^2(t,x)\,dx \le \int_0^1 u_0^2\,dx + c\int_0^t\int_0^1 u^2\,dxd\tau + 2\int_0^t (f,u)\,d\tau + \int_0^t \nu_l^2\,d\tau. \quad (4.11)$$

Proof. In the smooth case it follows from (2.2) and in the general case can be obtained via closure. □

Remark 4.5. It was shown in [40] that in the case of zero initial and boundary data for $f \in L_2(Q_T)$ the solution to the problem (4.1), (1.2)–(1.4), $u(t,x) : u \in L^2(0,T;H^{2l})$.

Properties of solutions to linear problems estimated in Lemma 4.3 are enough for our next purposes except the case $l = 1$, $k = 0$.

Consider now an algebraic equation related to the complete linear equation in the case $l = 1$

$$r^3 + a_2 r^2 + a_1 r + a_0 = -i\lambda, \quad \lambda \in \mathbb{R}\setminus\{0\}. \quad (4.12)$$

Then there exists $\lambda_0 > 0$ (without loss of genarality we assume that $\lambda_0 \ge 1$) such that for $|\lambda| \ge \lambda_0$ there exist two roots $r_0(\lambda)$ and $r_1(\lambda)$ with properties similar to (3.2)–(3.5), namely, for certain constants $\widetilde{c} > 0$, $\widetilde{c}_1 > 0$ and $|\lambda| \ge \lambda_0 \ge 1$

$$\operatorname{Re} r_0(\lambda) \le -\widetilde{c}|\lambda|^{1/3}, \quad \operatorname{Re} r_1(\lambda) \ge \widetilde{c}|\lambda|^{1/3}, \quad |r_j(\lambda)| \le \widetilde{c}_1|\lambda|^{1/3}, \quad j = 0,1. \quad (4.13)$$

Let

$$\mu_{00}(t) \equiv \mathcal{F}_t^{-1}\left[\chi_{\lambda_0}(\lambda)\widehat{\mu}_0(\lambda)\right](t), \quad \nu_{00}(t) \equiv \mathcal{F}_t^{-1}\left[\chi_{\lambda_0}(\lambda)\widehat{\nu}_0(\lambda)\right](t),$$

where χ_{λ_0} is the characteristic function of the interval $(-\lambda_0,\lambda_0)$,

$$\mu_{01}(t) \equiv \mu_0(t) - \mu_{00}(t), \quad \nu_{01}(t) \equiv \nu_0(t) - \nu_{00}(t).$$

Let

$$\begin{aligned} \psi(t,x) \equiv \Big(\mu_{00}(t) + \mathcal{F}_t^{-1}\big[e^{r_0(\lambda)x}\widehat{\mu}_{01}(\lambda)\big](t)\Big)\,\eta(1-x) \\ + \Big(\nu_{00}(t) + \mathcal{F}_t^{-1}\big[e^{r_1(\lambda)(x-1)}\widehat{\nu}_{01}(\lambda)\big](t)\Big)\,\eta(x), \end{aligned} \tag{4.14}$$

where η is a certain smooth "cut-off" function, namely, $\eta \ge 0$, $\eta' \ge 0$, $\eta(x) = 0$ for $x \le 1/4$, $\eta(x) = 1$ for $x \ge 3/4$. Note that $\psi(t,0) \equiv \mu(t)$, $\psi(t,1) \equiv \nu(t)$.

Lemma 4.6. *Let $\mu_0, \nu_0 \in H^{1/3+\varepsilon}(0,T)$ for some $\varepsilon > 0$. Then*

$$\psi \in Y_0(Q_T), \quad \psi_x \in L^2(0,T;L^\infty), \quad \psi_t + P(\partial_x)\psi \in C^\infty(\overline{Q}_T)$$

(with corresponding estimates).

Proof. The fact that $\psi \in Y_0(Q_T)$ is established similarly to (3.16), (3.18) with the use of inequalities (4.13) (it is sufficient to assume here that $\mu_0, \nu_0 \in H^{1/3}(0,T)$).

Next, similarly to Corollary 3.2 the function $J_0 \equiv \mathcal{F}_t^{-1}\big[e^{r_0 x}\widehat{\mu}_{01}\big]$ is infinitely differentiable for $x > 0$ and by virtue of (4.12) satisfies the homogeneous equation (4.1). The same properties are valid for $J_1 \equiv \mathcal{F}_t^{-1}\big[e^{r_1(x-1)}\widehat{\nu}_{01}\big]$ if $x < 1$. Therefore, $\psi_t + P(\partial_x)\psi \in C^\infty(\overline{Q}_T)$ since $\operatorname{supp}\eta' \subset [1/4, 3/4]$ (here it is sufficient to assume that $\mu_0, \nu_0 \in L^2(0,T)$).

Finally, for any integer $j \ge 0$

$$\begin{aligned} \|\partial_x^j J_0\|_{L^2(\mathbb{R}\times\mathbb{R}_+)} &= \Big\| r_0^j \widehat{\mu}_{01} \Big(\int_{\mathbb{R}_+} e^{2\operatorname{Re} r_0 x}\,dx\Big)^{1/2}\Big\|_{L^2(\mathbb{R})} \\ &\le c\||\lambda|^{j/3-1/6}\widehat{\mu}_{01}\|_{L^2(\mathbb{R})} \\ &\le c_1\|\mu_0\|_{H^{j/3-1/6}(\mathbb{R})}, \end{aligned}$$

whence by interpolation it follows that for $s \ge 0$

$$\|J_0\|_{L^2(\mathbb{R};H^s(\mathbb{R}))} \le c\|\mu_0\|_{H^{s/3-1/6}(\mathbb{R})}.$$

Similar arguments can be applied to the function J_1 and so the well-known embedding $H^{1/2+\varepsilon} \subset L^\infty$ provides the property $\psi_x \in L^2(0,T;L^\infty)$. $\square$

5. Results for local solutions

In this section local well-posedness for the original nonlinear problem is proved under natural assumptions on initial and boundary data.

Proof of Theorem 2.8. For $t_0 \in (0,T]$ introduce a set of functions

$$\widetilde{X}_k(Q_{t_0}) = \{v(t,x) \in X_k(Q_{t_0}) : \partial_t^n v\big|_{t=0} = \Phi_n,\ n = 0,\dots,k-1\}$$

and define on this set a map Λ in such a way: $u = \Lambda v \in \widetilde{X}_k(Q_{t_0})$ is a solution in Q_{t_0} to an initial boundary value problem for the equation

$$u_t + Pu = f - vv_x \tag{5.1}$$

with initial and boundary conditions (1.2)–(1.4). Making use of Lemma 4.3, we have to estimate $\|vv_x\|_{M_k(Q_{t_0})}$. Let $k=0$, then

$$\begin{aligned}\|vv_x\|_{L^2(0,t_0;H^{-l})} &\le \frac{1}{2}\|v\|^2_{L^4(Q_{t_0})}\\ &\le \frac{1}{2}\sup_{t\in[0,t_0]}\|v\|_{L^2}\Big(\int_0^{t_0}\sup_{x\in[0,1]}v^2\,dt\Big)^{1/2}\\ &\le c\sup_{t\in[0,t_0]}\|v\|_{L^2}\Big(\int_0^{t_0}\left(\|v_x\|_{L^2}\|v\|_{L^2}+\|v\|^2_{L^2}\right)\,dt\Big)^{1/2}\\ &\le c_1 t_0^{1/4}\|v\|^2_{X_0(Q_{t_0})}.\end{aligned}\tag{5.2}$$

Let $k=1$, then

$$\begin{aligned}&\|vv_x\|_{L^2(0,t_0;H^l)}\\ &\le c\sum_{j=0}^{l+1}\|\partial_x^j v\|^2_{L^4(Q_{t_0})}\\ &\le c_1\sum_{j=0}^{l+1}\sup_{t\in[0,t_0]}\|\partial_x^j v\|_{L^2}\Big(\int_0^{t_0}\left(\|\partial_x^{j+1}v\|_{L^2}\|\partial_x^j v\|_{L^2}+\|\partial_x^j v\|^2_{L^2}\right)\,dt\Big)^{1/2}\\ &\le c_2 t_0^{1/2}\|v\|^2_{X_1(Q_{t_0})};\end{aligned}$$

similarly to (5.2),

$$\begin{aligned}\|(vv_x)_t\|_{L^2(0,t_0;H^{-l})} &\le \|vv_t\|_{L^2(Q_{t_0})}\\ &\le \sup_{t\in[0,t_0]}\|v_t\|_{L^2}\Big(\int_0^{t_0}\sup_{x\in[0,1]}v^2\,dt\Big)^{1/2}\\ &\le ct_0^{1/4}\|v_t\|_{X_0(Q_{t_0})}\|v\|_{X_0(Q_{t_0})}\\ &\le ct_0^{1/4}\|v\|^2_{X_1(Q_{t_0})}\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}\|vv_x\|_{C([0,t_0];L^2)} &\le \|u_0u_0'\|_{L_2}+\|(vv_x)_t\|_{L^1(0,t_0;L^2)}\\ &\le c\|u_0\|^2_{H^{2l+1}}+ct_0^{1/2}\|(vv_x)_t\|_{L^2(Q_{t_0})}\\ &\le c\|u_0\|^2_{H^{2l+1}}+ct_0^{1/2}\|v\|_{C([0,t_0];C^1)}\|v_t\|_{L^2(0,t_0;H^1)}\\ &\le c\|u_0\|^2_{H^{2l+1}}+c_1t_0^{1/2}\|v\|^2_{X_1(Q_{t_0})}.\end{aligned}$$

The cases $k\ge 2$ can be handled in the same manner as the case $k=1$.

As a result, the map Λ exists and it follows from (4.2) that

$$\|\Lambda v\|_{X_k(Q_{t_0})}\le c\left(1+t_0^{1/4}\|v\|^2_{X_k(Q_{t_0})}\right).\tag{5.4}$$

By standard arguments, see [32], it can be derived from (5.4) that for small t_0 the map Λ transforms a certain large ball in $\widetilde{X}_k(Q_{t_0})$ into itself. Similarly to (5.4), one can obtain for two functions $v,\widetilde{v}\in\widetilde{X}_k(Q_{t_0})$:

$$\|\Lambda v-\Lambda\widetilde{v}\|_{X_k(Q_{t_0})}\le c\left(\|v\|_{X_k(Q_{t_0})},\|\widetilde{v}\|_{X_k(Q_{t_0})}\right)t_0^{1/4}\|v-\widetilde{v}\|_{X_k(Q_{t_0})},$$

and t_0 can be reduced in such a way that Λ becomes a contraction on this ball in $\widetilde{X}_k(Q_{t_0})$, therefore, there exists a unique solution $u \in X_k(Q_{t_0})$ to problem (1.1)–(1.4). Continuous dependence can be established by analogous arguments. $\square$

Remark 5.1. The idea to use the contraction principle to establish local well-posedness for KdV-like equations goes back to [32].

6. Results for global solutions

Theorem 2.9 succeeds from the already established local well-posedness and the following global a priori estimates.

Lemma 6.1. *Let the hypothesis of Theorem 2.9 be satisfied and* $u(t,x) \in X_k(Q_{T'})$ *be a solution to* (1.1)–(1.4) *for certain* $T' \in (0,T]$. *Then*

$$\|u\|_{C([0,T'];H^{(2l+1)k})} \le c, \tag{6.1}$$

where the constant c depends upon T, l, k; properties of the operator P_0 and the norms of initial data, boundary data and the right-hand side of (1.1) *in the spaces from the hypothesis of the theorem hold, but do not depend on T'.*

Proof. First put $k=0$. Let for $l=1$ a function ψ is given by formula (4.14) and for $l \ge 2$ a function $\psi \in X_0(Q_T)$ be a solution to (4.1), (1.2)–(1.4) for the same u_0, μ_j, ν_j and f. Then

$$\psi \in L^2(0,T;H^l) \cap C([0,T];L^2).$$

Define

$$U(t,x) \equiv u(t,x) - \psi(t,x),$$

$$F \equiv f - uu_x - \psi_t - P(\partial_x)\psi \quad \text{for } l=1 \quad \text{and} \quad F \equiv -uu_x \quad \text{for } l \ge 2.$$

Then the function U satisfies

$$U_t + PU = F \in L^1(0,T;L^2) \cap L^2(0,T;H^{-l}),$$

$$U\big|_{x=0} = U\big|_{x=1} = 0, \quad U_x\big|_{x=1} = \nu_1 - \psi_x\big|_{x=1} \in L^2(0,T) \quad \text{for } l=1,$$

zero boundary conditions of (1.3), (1.4) type for $l \ge 2$ and the initial condition

$$U(0,x) = U_0(x),$$

where $U_0 \equiv u_0 - \psi|_{t=0} \in L^2$ for $l=1$ and $U_0 \equiv 0$ for $l \ge 2$. Write down inequality (4.11) for the function U. Since

$$2\int_0^1 uu_x U\,dx = \int_0^1 \psi_x(U^2 + 2U\psi)\,dx,$$

where $\psi_x \in L^2(0,T;L^\infty)$, estimate (6.1) for $k=0$ (and, consequently, desired global well-posedness) follows.

Next let $k=1$, then a function $u_1 \equiv u_t \in X_0(Q_{T'})$ is a solution to an initial boundary value problem for the equation

$$u_{1t} + Pu_1 = (f - uu_x)_t$$

with initial data $\Phi_1 = f\big|_{t=0} - Pu_0 - u_0u_0' \in L^2$ and boundary data $(\mu_0', \dots, \mu_{l-1}') \in \mathcal{B}_0^{l-1}(0,T)$, $(\nu_0', \dots, \nu_l') \in \mathcal{B}_0^l(0,T)$. We use for the function u_1 the inequality (4.2) in the case $k=0$. Note that, by virtue of (5.3) (where v is substituted by u), for any $t_0 \in (0,T']$

$$\|(uu_x)_t\|_{L^2(0,t_0;H^{-l})} \le ct_0^{1/4}\|u\|_{X_0(Q_{T'})}\|u_t\|_{X_0(Q_{t_0})}.$$

Since for $k = 0$ global well-posedness is already established and, consequently, $\|u\|_{X_0(Q_{T'})} \le c$, we first derive from inequality (4.2) that

$$\|u_t\|_{X_0(Q_{t_0})} \le c\Big(1 + \|u_t\big|_{t=0}\|_{L^2} + t_0^{1/4}\|u_t\|_{X_0(Q_{t_0})}\Big)$$

and, finally, by standard arguments the estimate

$$\|u_t\|_{C([0,T'];L^2)} \le c. \tag{6.2}$$

Using the equality

$$\partial_x^{2l+1} u = (-1)^{l+1}(f - u_t - P_0 u - uu_x),$$

the Ehrling inequality, (6.1) for $k = 0$, and (6.2), one can obtain estimate (6.1) for $k = 1$. The cases $k \ge 2$ are handled similarly to the case $k = 1$. $\square$

7. Decay of small solutions

Consider (1.1)–(1.4) with zero boundary data, $f \equiv 0$ and small initial data u_0. Define

$$A_j = (-1)^{j+1}(2j+1)a_{2j+1} + (-1)^j \sigma a_{2j}, \quad j = 0, \dots, l, \tag{7.1}$$

where $\sigma = 2$ if $(-1)^j a_{2j} \ge 0$, $\sigma = 4$ if $(-1)^j a_{2j} < 0$; $(-1)^{l+1} a_{2l+1} = 1$.

Theorem 7.1. *Let $u_0 \in L^2$, $\mu_j = \nu_j = \nu_l \equiv 0$ for $j = 0, \dots, l-1$, $f \equiv 0$ and Assumption A is satisfied. Let*

$$A_l + \sum_{j:A_j<0} 2^{3(j-l)} A_j = 2K > 0 \tag{7.2}$$

$$\|(1+x)^{\frac{1}{2}} u_0\|_{L^2} < 3 \times 2^{3(l-1)} K. \tag{7.3}$$

Then a unique solution $u(t,x)$ to (1.1)–(1.4), such that $u \in X_0(Q_T)$ for any $T > 0$, satisfies the following inequality, for all $t \ge 0$,

$$\|u(t,\cdot)\|_{L^2}^2 \le 2e^{-\kappa t}\|u_0\|_{L^2}^2, \tag{7.4}$$

where

$$\kappa = 2^{3l} K + \sum_{j:A_j \ge 0} 2^{3j} A_j.$$

Proof. First of all note that the hypothesis of Theorem 2.9 are satisfied, hence such a unique solution exists. By Assumption A, $(-1)^l a_{2l} \ge 0$, hence $A_l \ge 2l + 1 > 0$. Multiplying (1.1) by $2(1+x)u(t,x)$ and integrating, we find

$$\frac{d}{dt}\int_0^1 (1+x)u^2(t,x)\,dx + \sum_{j=0}^{l} \int_0^1 \big[(-1)^{j+1}(2j+1)a_{2j+1} + (-1)^j 2a_{2j}(1+x)\big](\partial_x^j u)^2\,dx - \frac{2}{3}\int_0^1 u^3\,dx = 0. \tag{7.5}$$

(In fact, such a calculation must be first performed for smooth solutions and the general case can be obtained via closure). We use the Friedrichs inequality as follows: for any $\varphi \in H_0^l$

$$\|\varphi\|_{L^\infty} \le 2^{1-3l/2}\|\varphi^{(l)}\|_{L^2}, \quad \|\varphi\|_{L^2} \le 2^{-3l/2}\|\varphi^{(l)}\|_{L^2}. \tag{7.6}$$

Then

$$\Big|\int_0^1 u^3\,dx\Big| \le \|u\|_{L^\infty}^2 \|u\|_{L^2} \le 2^{2-3l}\|u(t,\cdot)\|_{L^2}\|\partial_x^l u\|_{L^2}^2.$$

This and (7.1), (7.2) allow us to rewrite (7.5) as

$$\frac{d}{dt}\int_0^1 (1+x)u^2(t,x)\,dx + \int_0^1 \big[2K - \frac{1}{3}2^{3(1-l)}\|(1+x)^{\frac12}u(t,\cdot)\|_{L^2}\big](\partial_x^l u)^2\,dx \le 0.$$

Taking into account (7.3) and exploiting standard arguments, one can prove that

$$\|(1+x)^{\frac12}u(t,\cdot)\|_{L^2} < 3\times 2^{3(l-1)}K \text{ for all } t\ge 0.$$

Returning to (7.5), we rewrite it as

$$\frac{d}{dt}\int_0^1 (1+x)u^2(t,x)\,dx + \int_0^1 2^{3l}Ku^2\,dx + \int_0^1 \sum_{j:A_j\ge 0} 2^{3j}A_j u^2\,dx \le 0,$$

whence

$$\frac{d}{dt}\int_0^1 (1+x)u^2(t,x)\,dx + \kappa\int_0^1 (1+x)u^2\,dx \le 0.$$

From here follows (7.4). □

Remark 7.2. Zero boundary data are chosen here for simplicity in order to show the idea of the method. In the general case similar results can be established for a difference between the solution to problem (1.1)–(1.4) and a solution to a certain linear problem with the same boundary data under suitable assumptions on behavior of the solution of this linear problem as $t\to+\infty$.

Remark 7.3. In [28] a non-trivial stationary solution to the initial-boundary value problem for the homogeneous KdV equation under zero boundary data (1.3), (1.4) (here $l=1$) was constructed. Therefore certain assumptions on the initial data u_0 are necessary for the decay of the corresponding solution as $t\to+\infty$.

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Andrei V. Faminskii
Department of Mathematics, Peoples Friendship University of Russia, Miklukho-Maklai str. 6, Moscow, 117198, Russia
E-mail address: `andrei_faminskii@mail.ru`

Nikolai A. Larkin
Departamento de Matemática, Universidade Estadual de Maringá, Av. Colombo 5790: Agência UEM, 87020-900, Maringá, PR, Brazil
E-mail address: `nlarkine@uem.br`