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# SLOW AND FAST SYSTEMS WITH HAMILTONIAN REDUCED PROBLEMS

MAAMAR BENBACHIR, KARIM YADI, RACHID BEBBOUCHI

Abstract. Slow and fast systems are characterized by having some of the derivatives multiplied by a small parameter $\epsilon$. We study systems of reduced problems which are Hamiltonian equations, with or without a slowly varying parameter. Tikhonov's theorem gives approximate solutions for times of order 1. Using the stroboscopic method, we give approximations for time of order $1/\epsilon$. More precisely, the variation of the total energy of the problem, and the eventual slow parameter, are approximated by a certain averaged differential equation. The results are illustrated by some numerical simulations. The results are formulated in classical mathematics and proved within internal set theory which is an axiomatic approach to nonstandard analysis.

## 1. Introduction

A slow and fast system (or two-time scale system) is a perturbed system of the form

$$\begin{aligned} \frac{dx}{dt} &= F(x,z,\epsilon), \quad x(0)=\alpha_\epsilon, \\ \epsilon\frac{dz}{dt} &= G(x,z,\epsilon), \quad z(0)=\beta_\epsilon, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ are the slow and fast components and $\epsilon$ is a positive real number small enough. The functions $F$ and $G$ are continuous and defined on an open subset of $\mathbb{R}^{n+m}$. The initial conditions depend continuously on $\epsilon$. The system (1.1) is more exactly a family of problems depending on the parameter $\epsilon$ varying in a small interval $]0,\epsilon_0]$. The fact that $\epsilon$ multiplies the derivative makes non valid the theory of continuous dependence of the solutions with respect to parameters. We are in presence of a singularly perturbed problem. The purpose of *Singular Perturbation Theory* is to investigate the behavior of solutions of (1.1) as $\epsilon \to 0$ on a bounded, or eventually, unbounded time interval. A recommended reference is the tenth chapter of the book of Wasow [21]. The change of the time scale $s = t/\epsilon$

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transforms (1.1) into

$$\begin{aligned} \frac{dx}{ds} &= \epsilon F(x,z,\epsilon), \quad x(0)=\alpha_\epsilon, \\ \frac{dz}{ds} &= G(x,z,\epsilon), \quad z(0)=\beta_\epsilon, \end{aligned}$$

which is a one parameter deformation of the unperturbed system

$$\begin{aligned} \frac{dx}{ds} &= 0, \quad x(0)=\alpha_0, \\ \frac{dz}{ds} &= G(x,z,0), \quad z(0)=\beta_0. \end{aligned} \tag{1.2}$$

The following system, where $x$ is considered as a parameter, is called the *fast equation*

$$\frac{dz}{ds} = G(x,z,0). \tag{1.3}$$

Hence, the $z$-component of the solution of (1.1) varies very quickly according to the so-called *boundary layer equation*

$$\frac{dz}{ds} = G(\alpha_0,z,0), \quad z(0)=\beta_0, \tag{1.4}$$

where $x$ has been frozen at its initial value. The set of the zeros of the right member of (1.3) is called the *slow manifold* of the problem (1.1). It is formed by the equilibrium points of the fast dynamics described by (1.3). A solution of (1.3) may be unbounded when $s \to \infty$, or may tend to an equilibrium point or may approach a more complex attractor and this asymptotic behavior depends on the initial condition. Assume for instance that the solutions of the fast equation (1.3) tend toward an equilibrium point $\xi(x)$ on the slow manifold. The equation $z=\xi(x)$ defines a component $\mathcal{L}$ of the slow manifold. A fast transition brings a solution of (1.1) close to the slow manifold. Then a slow motion takes place near the slow manifold and is approximated by the *reduced problem*

$$\frac{dx}{dt} = F(x,\xi(x),0), \quad x(0)=\alpha_0. \tag{1.5}$$

When the solution of (1.1) is unique, the results of Tikhonov's Theorem [20] (see [7]), under suitable conditions (among the others, the asymptotic stability of the equilibrium $\xi(x)$ uniformly with respect to $x$ in a compact domain), are mainly as follows:

> Let $\tilde{z}(s)$ and $\bar{x}(t)$ be the solutions, supposed to be unique, of the boundary layer equation (1.4) and the reduced problem (1.5). Suppose that $\bar{x}(t)$ is defined on $[0,T]$. Then for $\epsilon$ sufficiently small, the solution $(x(t,\epsilon), z(t,\epsilon))$ of the perturbed problem (1.1) is defined at least on $[0,T]$ and satisfies
>
> $$\begin{aligned} \lim_{\epsilon\to 0^+} x(t,\epsilon) &= \bar{x}(t) \quad \text{for all } t\in[0,T], \\ \lim_{\epsilon\to 0^+} z(t,\epsilon) &= \xi(\bar{x}(t)) \quad \text{for all } t\in]0,T], \\ \lim_{\epsilon\to 0^+} z(\epsilon s,\epsilon) &= \tilde{z}(s \quad \text{for all } s\geq 0\,. \end{aligned}$$

Another tool, namely the *stroboscopic method*, is needed in this work to give approximations of the solutions for larger time of order $1/\epsilon$. It is a method of the nonstandard perturbation theory of differential equations proposed by Callot and Reeb and improved by Lutz and Sari (see [2, 8, 15, 16, 18]). The principle is as follows (one should at this stage admit the intuitive meaning of the symbol "$\simeq$"; i.e., "infinitely close to"): Let $\phi(t)$ be a function. Suppose we are able to define a sequence $(t_n, \phi_n = \phi(t_n))$ such that $t_{n+1} \simeq t_n$, $\phi(t) \simeq \phi_n$ on $[t_n, t_{n+1}]$ and

$$\frac{\phi_{n+1} - \phi_n}{t_{n+1} - t_n} \simeq F(t_n, \phi_n),$$

where $F$ is a continuous function. We can conclude that the function $\phi(t)$ is infinitely close to a solution of the differential equation $\frac{d\phi}{dt} = F(t, \phi)$. Actually, we use in this paper an improved version established by T. Sari.

After the generalizations of the famous results of Tikhonov and Pontryagin-Rodygin for slow and fast systems obtained in [7, 19], it was quite natural to think about the case where the trajectories approach an oscillating motion on the slow manifold. As far as we know, this case has been described in [12]. Unfortunately, this reference has not been diffused. The author considered among others the scalar slow-fast system

$$\frac{dx}{dt} = f(x, y, z), \quad \frac{dy}{dt} = g(x, y, z), \quad \epsilon\frac{dz}{dt} = h(x, y) - z,$$

the slow equation of which presents periodic orbits and admits a first integral. With the use of the first return map of Poincaré, he showed that, after a fast transition, the considered trajectory fills the region of oscillations lying on the slow manifold. Note that this study was entirely qualitative. The starting point of the present work was the study of the singular perturbation of the harmonic oscillator [1]

$$\epsilon\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + x = 0,$$

or more exactly of the associated system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = z, \quad \epsilon\frac{dz}{dt} = -x - z.$$

By the change of variable $\epsilon z_1 = x + z$, one has

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \epsilon z_1, \quad \epsilon\frac{dz_1}{dt} = y - z_1.$$

This system is a particular case of the general slow-fast problem

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y}(x, y) + \epsilon f(x, y, z, \epsilon), \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x, y) + \epsilon g(x, y, z, \epsilon), \\ \epsilon\frac{dz}{dt} &= h(x, y, z, \epsilon), \end{aligned} \tag{1.6}$$

the slow equation of which is a Hamiltonian system. We also consider the more general case where the Hamiltonian depends on a slowly varying parameter, more exactly

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{\partial H}{\partial y}(x,y,\lambda) + \epsilon f(x,y,z,\lambda,\epsilon),\\
\frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x,y,\lambda) + \epsilon g(x,y,z,\lambda,\epsilon),\\
\epsilon\frac{dz}{dt} &= h(x,y,z,\lambda,\epsilon),\\
\frac{d\lambda}{dt} &= \epsilon\alpha(x,y,z,\lambda,\epsilon).
\end{aligned}
\tag{1.7}
$$

We define an averaged system

$$
E' = M_1(E,\lambda), \quad \lambda' = M_2(E,\lambda), \tag{1.8}
$$

where the prime denotes the derivative with respect to $\tau = \epsilon t$ and the functions $M_1$ and $M_2$ are the averages of the functions

$$
\Omega(x,y,\lambda) = \omega(x,y,\xi(x,y,\lambda),\lambda), \quad A(x,y,\lambda) = \alpha(x,y,\xi(x,y,\lambda),\lambda,0),
$$

on the closed orbits of the Hamiltonian system

$$
\frac{dx}{dt} = \frac{\partial H}{\partial y}(x,y,\lambda), \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}(x,y,\lambda), \tag{1.9}
$$

where $\lambda$ is considered as a constant parameter. The function $\omega$ is given by

$$
\begin{aligned}
\omega(x,y,z,\lambda) = {}& \frac{\partial H}{\partial x}(x,y,\lambda)f(x,y,z,\lambda,0) + \frac{\partial H}{\partial y}(x,y,\lambda)g(x,y,z,\lambda,0)\\
&+ \frac{\partial H}{\partial \lambda}(x,y,\lambda)\alpha(x,y,z,\lambda,0),
\end{aligned}
$$

and the function $\xi$ defines the slow manifold $z = \xi(x,y,\lambda)$ of (1.7). We prove in Theorem 4.1 that for any solution $(x(\tau,\epsilon), y(\tau,\epsilon), z(\tau,\epsilon), \lambda(\tau,\epsilon))$ of (1.7), written in the time scale $\tau$, the functions

$$
E(\tau) = H(x(\tau,\epsilon), y(\tau,\epsilon))
$$

and $\lambda(\tau,\epsilon)$ are approximated by a solution of the averaged system (1.8). We must attract the attention that this result could be deduced from a result published by T. Sari in [17]. Actually, this author considers the non Hamiltonian perturbation of a Hamiltonian system with slowly varying parameters

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{\partial H}{\partial y}(x,y,\lambda) + \epsilon f_1(x,y,\lambda,\epsilon),\\
\frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x,y,\lambda) + \epsilon g_1(x,y,\lambda,\epsilon),\\
\frac{d\lambda}{dt} &= \epsilon\alpha_1(x,y,\lambda,\epsilon),
\end{aligned}
\tag{1.10}
$$

to show how to use the stroboscopic method to obtain adiabatic invariants. He proves (see Theorem 2 in [17] and references therein for classical results) that its solutions are approximated by the solutions of the averaged system(1.8) where the functions $M_1$ and $M_2$ are the averages of the functions

$$
\Omega_1(x,y,\lambda) = \omega_1(x,y,\lambda), \quad A_1(x,y,\lambda) = \alpha_1(x,y,\lambda,0),
$$

on the closed orbits of the Hamiltonian system (1.9). Here, the function $\omega_1$ is given by

$$\omega_1(x,y,\lambda) = \frac{\partial H}{\partial x}(x,y,\lambda) f_1(x,y,\lambda,0) + \frac{\partial H}{\partial y}(x,y,\lambda) g_1(x,y,\lambda,0) + \frac{\partial H}{\partial \lambda}(x,y,\lambda)\alpha_1(x,y,\lambda,0).$$

To show how the last result in [17] can be used to prove our result despite the fact that the slow-fast system we consider contains also the fast variable $z$, we need to say something about the Geometric Singular Perturbation Theory, namely the Fenichel invariant manifold Theorem (for details and definitions one can see [5]). This last statement concerns systems of the form

$$y' = \epsilon u(y,z,\epsilon), \quad z' = v(y,z,\epsilon), \tag{1.11}$$

where $u$ and $v$ are $C^\infty$ in an open subset $U \times I$ of $\mathbb{R}^{m+n+1}$, $0 \in I$. Suppose that the set $\mathcal{N}_0 = \{(y,z), v(y,z,0) = 0\}$ is a normally hyperbolic manifold given by the graph of a $C^\infty$ function $z = \xi(y)$ defined on a compact subset $Y$. Under these assumptions, Fenichel's Theorem ensures that "$\mathcal{N}_0$ *persists for small values of* $\epsilon$", more precisely, *for any positive integer* $r$ *and for any* $\epsilon > 0$ *small enough, there exists a* $C^r$ *function* $z = \mathcal{Z}(y,\epsilon)$ *defined for* $y$ in $Y$ *such that the manifold* $\mathcal{N}_\epsilon = \{(y,z),\ z = \mathcal{Z}(y,\epsilon)\}$ is locally invariant under (1.11). Moreover $\mathcal{N}_\epsilon \to \mathcal{N}_0$ *when* $\epsilon \to 0$. Hence, on the invariant Fenichel slow manifold $\mathcal{N}_\epsilon$, the system (1.11) is reduced to

$$y' = u(y, \mathcal{Z}(y,\epsilon), \epsilon).$$

Let us start by saying that the assumptions we make in our work do not require strong conditions of differentiability of functions appearing in the problem. Hence the slow manifold is not supposed to be differentiable nor normally hyperbolic, that is for what Fenichel's theory is not completely satisfactory in this case. However, if we suppose that the problem (1.7) admits an invariant manifold given by $z = \mathcal{Z}(x,y,\lambda,\epsilon)$, it becomes simply

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial H}{\partial y}(x,y,\lambda) + \epsilon f(x,y,\mathcal{Z}(x,y,\lambda,\epsilon),\lambda,\epsilon),\\
\frac{dy}{dt} &= -\frac{\partial H}{\partial x}(x,y,\lambda) + \epsilon g(x,y,\mathcal{Z}(x,y,\lambda,\epsilon),\lambda,\epsilon),\\
\frac{d\lambda}{dt} &= \epsilon \alpha(x,y,\mathcal{Z}(x,y,\lambda,\epsilon),\lambda,\epsilon),
\end{aligned}$$

which is a perturbed Hamiltonian system with slowly varying parameters of the form (1.10) where

$$\begin{aligned}
f_1(x,y,\lambda,\epsilon) &= f(x,y,\mathcal{Z}(x,y,\lambda,\epsilon),\lambda,\epsilon),\\
g_1(x,y,\lambda,\epsilon) &= g(x,y,\mathcal{Z}(x,y,\lambda,\epsilon),\lambda,\epsilon),\\
\alpha_1(x,y,\lambda,\epsilon) &= \alpha(x,y,\mathcal{Z}(x,y,\lambda,\epsilon),\lambda,\epsilon).
\end{aligned}$$

Moreover, since $\mathcal{Z}(x,y,\lambda,0) = \xi(x,y,\lambda)$, one has

$$\Omega_1(x,y,\lambda) = \Omega(x,y,\lambda), \quad A_1(x,y,\lambda) = A(x,y,\lambda).$$

According to [17, Theorem 2], the solutions of (1.7) are approximated by the solutions of the averaged system (1.8), provided that the conditions of Fenichel's

Theorem are fulfilled. It is worth noting that our contribution consists of a direct proof based on both Tikhonov's theory and the stroboscopic method.

For convenience, we prefer to detail our approach for the analysis of the particular case of system (1.6). In Section 2, we state two theorems, the first being just an application of Tikhonov's Theorem giving the behavior of the solutions of (1.6) over time 1. Theorem 2.2 provides an approximation of the total energy of (1.6) over time $1/\epsilon$. Section 3 is devoted to the "non standard version" of the first results and the proof is given in Section 5. Internal Set Theory is an extension of ordinary mathematics due to E. Nelson [11]. It axiomatizes Robinson's nonstandard analysis (NSA) [13]. For a short tutorial in NSA, one can consult for instance [7] or [19]. Historically, the *nonstandard perturbation theory of differential equations*, which is today a well-established tool in asymptotic theory, has its roots in the seventies, when the Reebian school (see [4, 9, 10, 14]) introduced the use of nonstandard analysis into the field of perturbed differential equations. For more information on the subject, the interested reader is referred to texts such as [3] and to papers such as [6, 7, 15, 19] among many others. In Section 4, we consider the system (1.7) where the Hamiltonian system depends on a slowly varying vectorial parameter. We choose to present this last result (Theorem 4.1) only in a non standard form. In the last section, we give some examples of application of Theorems 2.2 and 4.1.

## 2. Averaging on the slow manifold

Consider the Hamiltonian system

$$\begin{aligned}\frac{dq}{dt} &= \frac{\partial H}{\partial p}(p,q),\\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q}(p,q).\end{aligned} \tag{2.1}$$

The level curves $H(p,q) = E$, where $E$ is constant (energy), are integral curves of (2.1). We call *region of oscillations* of the Hamiltonian function $H(p,q)$ an interval $I$ of $\mathbb{R}$ such that, for all $E$ in $I$, $H(p,q) = E$ defines a non trivial closed curve $C(E)$ in the $p,q$-plane which does not contain any singular point where both $\frac{\partial H}{\partial p}$ and $\frac{\partial H}{\partial q}$ vanish. A periodic solution of (2.1) corresponding to the closed orbit $C(E)$ is denoted by $(q(t,E),p(t,E))$ with period $P(E)$ and is defined for all $t$. Consider the system (1.6) with the initial conditions

$$x(0) = q_0, \quad y(0) = p_0, \quad z(0) = z_0, \tag{2.2}$$

where $(x,y,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n$. First, we describe the solutions of the system (1.6) on a finite time interval when $\epsilon \to 0$, by the use of the theory of singular perturbations. We use the following assumptions:

- (H1) The functions $f$, $g$, $h$ and the partial derivatives of $H$ are continuous with respect to their arguments.

We assume that the fast equation

$$\frac{dz}{ds} = h(x,y,z,0), \tag{2.3}$$

where $s = t/\epsilon$, has an asymptotically stable equilibrium point $z = \xi(x,y)$. More exactly

(H2) There exists a compact domain $K$ in $\mathbb{R}^2$ and a continuous function $\xi$ such that for all $(x,y)\in K$, $z=\xi(x,y)$ is an isolated root of $h(x,y,z,0)=0$. The point $z=\xi(x,y)$ is an asymptotically stable equilibrium of (2.3) uniformly over $K$.

The graph of $z=\xi(x,y)$ is an attractive component of the slow manifold $h(x,y,z,0)=0$. The slow equation is the Hamiltonian system (2.1). We refer to the boundary layer equation as

$$\frac{dz}{ds}=h(x,y,z,0),\ z(0)=z_0, \tag{2.4}$$

and to the reduced equation as

$$\begin{gathered}\frac{dq}{dt}=\frac{\partial H}{\partial q}(p,q),\ q(0,E_0)=q_0,\\ \frac{dp}{dt}=-\frac{\partial H}{\partial p}(p,q),\ p(0,E_0)=p_0,\end{gathered} \tag{2.5}$$

where $(q,p)$ is in the interior $\operatorname{int}K$ of $K$ and $E_0$ is the energy level such that $H(p_0,q_0)=E_0$. We make the last assumptions:

(H3) The fast equation (2.3) and the slow equation (2.1) have the property of uniqueness of solutions with prescribed initial conditions, $(q_0,p_0)\in intK$ and $z_0$ is in the basin of attraction of $\xi(q_0,p_0)$.

The theorem below is just an application of Tikhonov's theorem for slow and fast systems and gives an approximation of the solutions of (1.6), (2.2) over time 1 for $\epsilon$ sufficiently small [20, 7]. We do not give its proof.

**Theorem 2.1.** *Suppose that* (H1), (H3) *are satisfied. Let $\tilde z(s)$ be the solution of the boundary layer equation* (2.4) *and $(q(t,E_0),p(t,E_0))$ the solution of the reduced equation* (2.5). *Let $T>0$ be in the positive interval of definition of $(q(t,E_0),p(t,E_0))$. For every $\eta>0$, there exists $\epsilon^*>0$ such that, for all $0<\epsilon<\epsilon^*$, any solution $\gamma(t,\epsilon)=((x(t,\epsilon),y(t,\epsilon),z(t,\epsilon))$ of* (1.6), (2.2) *is defined at least on $[0,T]$ and there exists $\omega>0$ such that*

$$\begin{gathered}\epsilon\omega<\eta,\\ |z(\epsilon s)-\tilde z(s)|<\eta\quad\textit{for all } 0\le s\le\omega,\\ |x(t,\epsilon)-q(t,E_0)|<\eta,\quad |y(t,\epsilon)-p(t,E_0)|<\eta\quad\textit{for all } 0\le t\le T,\\ |z(t,\epsilon)-\xi(q(t,E_0),p(t,E_0))|<\eta\quad\textit{for all } \epsilon\omega\le t\le T.\end{gathered}$$

According to what precedes, for $\epsilon$ small enough, a phase trajectory $\gamma(t,\epsilon)$ starting at the point $(q_0,p_0,z_0)$ jumps, after a small time $t_0=\epsilon\omega$, to the neighborhood of the slow manifold $z=\xi(x,y)$, $(x,y)\in\operatorname{int}K$. Then it remains close to the curve $C(E_0)$ defined by $H(x,y)=E_0$ over time 1. Now, the total energy $E(t,\epsilon)=H(x(t,\epsilon),y(t,\epsilon))$ of the system (1.6) is slowly varying since its derivative is given by

$$\frac{dE}{dt}=\epsilon\omega(x,y,z,\epsilon), \tag{2.6}$$

where

$$\omega(x,y,z,\epsilon)=\frac{\partial H}{\partial x}(x,y).f(x,y,z,\epsilon)+\frac{\partial H}{\partial y}(x,y).g(x,y,z,\epsilon). \tag{2.7}$$

Over time 1, the quantity $E(t,\epsilon)$ remains nearly constant and the problem is to describe what happens over time $1/\epsilon$. It is more natural to consider system (1.6) and equation (2.6) at the time scale $\tau = \epsilon t$. Let $' = d/d\tau$ be the derivative with respect to $\tau$. System (1.6), and equation (2.6) becomes

$$\begin{aligned} x' &= \frac{1}{\epsilon}\frac{\partial H}{\partial y}(x,y) + f(x,y,z,\epsilon), \\ y' &= -\frac{1}{\epsilon}\frac{\partial H}{\partial x}(x,y) + g(x,y,z,\epsilon), \\ z' &= \frac{1}{\epsilon^2}h(x,y,z,\epsilon), \end{aligned} \tag{2.8}$$

and

$$E' = \omega(x,y,z,\epsilon). \tag{2.9}$$

Let us denote $\Omega(x,y) := \omega(x,y,\xi(x,y),0)$, and make another assumption to avoid boundary problems:

- (H4) The region of oscillations $I$ of (2.1) is non empty and there exists a compact subinterval $J$ of $I$ such that $K = \cup_{E\in J} C(E)$.

Consider the averaged equation

$$\bar{E}' = \frac{M(\bar{E})}{P(\bar{E})} := \frac{1}{P(\bar{E})}\int_0^{P(\bar{E})} \Omega(q(v,\bar{E}),p(v,\bar{E}))dv, \tag{2.10}$$

defined in int$J$. We recall that $(q(t,E),p(t,E))$ is the periodic solution of (2.1) of energy $E$ and period $P(E)$.

- (H5) Equation (2.10) has the property of uniqueness of solutions with prescribed initial conditions.

**Theorem 2.2.** *Suppose that assumptions $H1$ to* (H5) *are satisfied. Let $\gamma(\tau,\epsilon) = ((x(\tau,\epsilon),y(\tau,\epsilon),z(\tau,\epsilon))$ be a solution of* (2.8) *with initial condition* (2.2)*. Suppose that $E_0 = H(q_0,p_0) \in \operatorname{int}J$. Let $E(\tau) = H(x(\tau,\epsilon),y(\tau,\epsilon))$ be the total energy of $\gamma(\tau,\epsilon)$. Let $\bar{E}(\tau)$ be the solution of the averaged equation* (2.10) *with initial condition $E_0$ and let $L$ be in its positive interval of definition. Then, for every $\eta > 0$, there exists $\epsilon^* > 0$ such that for all $0 < \epsilon < \epsilon^*$ the function $E(\tau)$ satisfies $|E(\tau) - \bar{E}(\tau)| < \eta$ for all $\tau$ in $[0,L]$.*

## 3. External results

Theorems 3.1 and 3.2 below are external results which respectively imply Theorems 2.1 and 2.2.

**Theorem 3.1.** *Let $f$, $g$, $H$, $h$, $\xi$, $p_0$, $q_0$, $z_0$ and $E_0$ be standard. Suppose that* (H1)–(H3) *are satisfied. Let $\tilde{z}(s)$ be the solution of the boundary layer equation* (2.4) *and $(q(t,E_0),p(t,E_0))$ the solution of the reduced equation* (2.5)*. Let $\epsilon > 0$ be infinitesimal and $T$ be a standard real number in the positive interval of definition of $(q(t,E_0),p(t,E_0))$. Then, any solution $\gamma(t) = (x(t),y(t),z(t))$ of* (1.6) *is defined at least on $[0,T]$ and there exists $\omega$ such that $\epsilon\omega \simeq 0$ and*

$$\begin{gathered} z(\epsilon s) \simeq \tilde{z}(s), \quad \text{for all } 0 \le s \le \omega, \\ x(t) \simeq q(t,E_0), \quad y(t) \simeq p(t,E_0) \quad \text{for all } 0 \le t \le T, \\ z(t) \simeq \xi(q(t,E_0),p(t,E_0)), \quad \text{for all } \epsilon\omega \le t \le T. \end{gathered}$$

**Theorem 3.2.** *Suppose that* (H1)–(H5) *are satisfied. Let* $f$*,* $g$*,* $H$*,* $h$*,* $\xi$*,* $p_0$*,* $q_0$*,* $z_0$ *be standard. Let* $\epsilon$ *be positive infinitesimal. Let* $\gamma(\tau) = ((x(\tau), y(\tau), z(\tau))$ *be a solution of* (2.8) *with initial condition* (2.2)*. Suppose that* $E_0 = H(q_0, p_0) \in \mathrm{int}J$*. Let* $E(\tau) = H(x(\tau), y(\tau))$ *be the total energy of* $\gamma(\tau)$*. Let* $\bar{E}(\tau)$ *be the solution of the averaged equation* (2.10) *with initial condition* $E_0$ *and let* $L$ *standard be in its positive interval of definition. Then, the function* $E(\tau)$ *satisfies* $E(\tau) \simeq \bar{E}(\tau)$ *for all* $\tau \in [0, L]$*.*

The proof is postponed to Section 5. Let us first show that Theorem 3.2 reduces to Theorem 2.2. We will need the following frequent reduction formula due to Nelson [11]

$$\forall x\ (\forall^{st} y\ A \Rightarrow \forall^{st} z\ B) \equiv \forall z\ \exists^{fin} y'\ \forall x\ (\forall y \in y'\ A \Rightarrow B), \tag{3.1}$$

where $A$ (respectively $B$) is an internal formula with free variable $y$ (respectively $z$) and standard parameters. The notation $\forall^{st}$ means "for all standard" and $\exists^{fin}$ means "there is a finite".

*Proof of Theorem 2.2.* Let $B$ be the formula occurring in Theorem 2.2: "the function $E(\tau)$ satisfies $|E(\tau) - \bar{E}(\tau)| < \eta$ for all $\tau$ in $[0, L]$". To say that "the function $E(\tau)$ satisfies $E(\tau) \simeq \bar{E}(\tau)$ for all $\tau$ in $[0, L]$" is the same as to say $\forall^{st}\eta\ B$. To say that "$\epsilon > 0$ is infinitesimal" is the same as to say $\forall^{st}\epsilon^*\ 0 < \epsilon < \epsilon^*$. Hence, Theorem 3.2 asserts that

$$\forall \epsilon\ (\forall^{st}\epsilon^*\ 0 < \epsilon < \epsilon^* \Rightarrow \forall^{st}\eta\ B).$$

In this formula, $f$, $g$, $H$, $h$, $p_0$, $q_0$, $E_0$ and $L$ are standard parameters, $\epsilon$ and $\eta$ range over the strictly positive real numbers. By (3.1), the last formula is equivalent to

$$\forall \eta\ \exists^{fin}\epsilon^{*'}\ \forall \epsilon\ (\forall \epsilon^* \in \epsilon^{*'}\ 0 < \epsilon < \epsilon^* \Rightarrow B).$$

But for $\epsilon^{*'}$ finite set, to say $\forall \epsilon^* \in \epsilon^{*'}\ 0 < \epsilon < \epsilon^*$ is the same as to say $0 < \epsilon < \epsilon^*$ for $\epsilon^* = \min \epsilon^{*'}$. Hence, the formula is equivalent to

$$\forall \eta\ \exists \epsilon^*\ \forall \epsilon\ (0 < \epsilon < \epsilon^* \Rightarrow B).$$

This means that for any standard $f$, $g$, $H$, $h$, $p_0$, $q_0$, $E_0$ and $L$, the statement of Theorem 2.2 holds, thus by transfer, it holds for all $f$, $g$, $H$, $h$, $p_0$, $q_0$, $E_0$ and $L > 0$. □

## 4. Case of a slowly varying parameter

As explained in the introduction, we present the case where the slow motion is described by a Hamiltonian system depending on a slowly varying parameter $\lambda \in D$ where $D$ is a compact of $\mathbb{R}^k$ such that $\mathrm{int}D \neq \emptyset$. More exactly, at the time scale $\tau = \epsilon t$ , we examine the problem[1]

$$\begin{gathered} x' = \frac{1}{\epsilon}\frac{\partial H}{\partial y}(x, y, \lambda) + f(x, y, z, \lambda), \\ y' = -\frac{1}{\epsilon}\frac{\partial H}{\partial x}(x, y, \lambda) + g(x, y, z, \lambda), \\ z' = \frac{1}{\epsilon^2}h(x, y, z, \lambda), \\ \lambda' = \alpha(x, y, z, \lambda), \end{gathered} \tag{4.1}$$

[1] Contrarily to system (1.7) given in the introduction, we dropped the parameter $\epsilon$ in the expressions of the functions without lost of generality.

with initial condition

$$x(0) = q_0, \quad y(0) = p_0, \quad z(0) = z_0, \quad \lambda(0) = \lambda_0. \tag{4.2}$$

We still denote by $J$ a compact region of oscillations of the Hamiltonian function $H(p, q, \lambda)$. The total energy of a solution $\gamma(\tau) = ((x(\tau), y(\tau), z(\tau), \lambda(\tau))$ verifies

$$E' = \omega(x, y, z, \lambda), \tag{4.3}$$

where

$$\omega(x, y, z, \lambda) = \frac{\partial H}{\partial x}.f + \frac{\partial H}{\partial y}.g + \frac{\partial H}{\partial \lambda}\alpha. \tag{4.4}$$

Under Tikhonov's Theorem conditions, the trajectory is supposed to jump to the neighborhood of the slow attractive manifold $\{z = \xi(x, y, \lambda), \lambda = \lambda_0\}$ and is first approximated by the solution of the reduced equation

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p}(p, q, \lambda_0), \quad q(0, E_0, \lambda_0) = q_0, \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q}(p, q, \lambda_0), \quad p(0, E_0, \lambda_0) = p_0. \end{aligned}$$

We want to give an approximation of the very slow drift of $E$ and $\lambda$. Let us denote by

$$\begin{aligned} \Omega(x, y, \lambda) &:= \omega(x, y, \xi(x, y, \lambda), \lambda), \\ A(x, y, \lambda) &:= \alpha(x, y, \xi(x, y, \lambda), \lambda), \end{aligned} \tag{4.5}$$

and define the equations

$$\bar{E}' = \frac{M(\bar{E}, \bar{\lambda})}{P(\bar{E}, \bar{\lambda})} := \frac{1}{P(\bar{E}, \bar{\lambda})} \int_0^{P(\bar{E}, \bar{\lambda})} \Omega(q(\nu, \bar{E}, \bar{\lambda}), p(\nu, \bar{E}, \bar{\lambda}), \bar{\lambda}) d\nu, \tag{4.6}$$

and

$$\bar{\lambda}' = \frac{N(\bar{E}, \bar{\lambda})}{P(\bar{E}, \bar{\lambda})} := \frac{1}{P(\bar{E}, \bar{\lambda})} \int_0^{P(\bar{E}, \bar{\lambda})} A(q(\nu, \bar{E}, \bar{\lambda}), p(\nu, \bar{E}, \bar{\lambda}), \bar{\lambda}) d\nu, \tag{4.7}$$

where $(q(t, E, \lambda), p(t, E, \lambda))$ is the periodic solution of

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}(p, q, \lambda), \ \frac{dp}{dt} = -\frac{\partial H}{\partial q}(p, q, \lambda),$$

of energy $E$ and period $P(E, \lambda)$. We claim what follows.

**Theorem 4.1.** *Let $f$, $g$, $H$, $h$, $\alpha$, $\xi$, $p_0$, $q_0$, $z_0$, $\lambda_0$ be standard. Suppose that Tikhonov's conditions are satisfied and that (4.6) and (4.7) have the property of uniqueness. Let $\epsilon > 0$ be infinitesimal. Let $\gamma(\tau) = ((x(\tau), y(\tau), z(\tau), \lambda(\tau))$ be a solution of (4.1) with initial condition (4.2). Suppose that $E_0 = H(q_0, p_0, \lambda_0) \in \operatorname{int} J$ and $\lambda_0 \in \operatorname{int} D$. Let $E(\tau) = H(x(\tau), y(\tau), \lambda(\tau))$ be the total energy of $\gamma(\tau)$. Let $\bar{E}(\tau)$ and $\bar{\lambda}(\tau)$ be the solutions of the equations (4.6) and (4.7) with initial conditions $E_0$ and $\lambda_0$ and let $L$ standard be in their positive interval of definition. Then, the functions $E(\tau)$ and $\lambda(\tau)$ satisfy $E(\tau) \simeq \bar{E}(\tau)$ and $\lambda(\tau) \simeq \bar{\lambda}(\tau)$ for all $\tau \in [0, L]$.*

The proof is given in the next Section.

## 5. Proof of the main results

The key for proving the main results is the so called *Stroboscopy Lemma* (see [16, 18]) which is an extension and an improvement of the stroboscopic method outlined in the introduction. Let $\mathcal{O}$ be a standard open subset of $\mathbb{R}^n$, $F:\mathcal{O}\to\mathbb{R}^n$ a standard continuous function. Let $\mathcal{I}$ be an interval of $\mathbb{R}$ containing 0 and $\phi:\mathcal{I}\to\mathbb{R}^n$ a function such that $\phi(0)$ is nearstandard in $\mathcal{O}$. Let $\mathcal{J}$ be a connected subset of $\mathcal{I}$, eventually an external collection, such that $0\in\mathcal{J}$.

**Definition 5.1** (Stroboscopic property)**.** Let $t$ and $t'$ be in $\mathcal{J}$. The function $\phi$ is said to satisfy the stroboscopic property $\mathcal{S}(t,t')$ if $[t,t']\subset\mathcal{J}$, $t'\simeq t$, $\phi(s)\simeq\phi(t)$ for all $s$ in $[t,t']$ and

$$\frac{\phi(t)-\phi(t')}{t-t'}\simeq F(\phi(t)).$$

Under suitable conditions, the Stroboscopy Lemma asserts that the function $\phi$ is approximated by the solution of the initial value problem

$$\frac{dx}{dt}=F(x),\quad x(0)={}^\circ(\phi(0)), \tag{5.1}$$

where ${}^\circ(\phi(0))$ denotes the standard part of $\phi(0)$.

**Theorem 5.2** (Stroboscopy Lemma)**.** *Suppose that*

(i) *There exists $\mu>0$ such that, whenever $t\in\mathcal{J}$ is limited and $\phi(t)$ is nearstandard in $\mathcal{O}$, there is $t'\in\mathcal{J}$ such that $t'-t\geq\mu$ and the function $\phi$ satisfies the stroboscopic property $\mathcal{S}(t,t')$.*

(ii) *The initial value problem (5.1) has a unique solution $x(t)$.*

*Then, for any standard $L$ in the maximal positive interval of definition of $x(t)$, we have $[0,L]\subset\mathcal{J}$ and $\phi(t)\simeq x(t)$ for all $t\in[0,L]$.*

**Proof of Theorem 3.2.** Consider $\tau_1\geq 0$ such that $[0,\tau_1]\subset[0,L]$ and $E(\tau)$ is nearstandard in $\mathrm{int}J$ for all $\tau\in[0,\tau_1]$. Let us consider the external collection

$$\mathcal{J}=\{\tau\geq 0: E(s)\text{ nearstandard in }\mathrm{int}J\text{ for all }s\in[0,\tau]\}$$

which contains the interval $[0,\tau_1]$. Let us show that $E(\tau)$ satisfies the hypothesis $(i)$ of the Stroboscopy Lemma (Theorem 5.2).

Let $\mu=\epsilon\min_{E\in J}P(E)$. Since $P$ does not vanish and is continuous and $J$ is a compact subset, $\mu$ is positive. Let $\tau'$ limited in $\mathcal{J}$, thus $E(\tau')$ is nearstandard in $\mathrm{int}J$. Let us make the change of variables

$$r=\frac{\tau-\tau'}{\epsilon},\quad F(r)=\frac{E(\tau'+\epsilon r)-E(\tau')}{\epsilon}, \tag{5.2}$$

which transforms the system formed by (2.8) and (2.9) with initial condition $(x(\tau'),y(\tau'),z(\tau'),E(\tau'))$ into

$$\begin{gathered}\frac{dx}{dr}=\frac{\partial H}{\partial y}(x,y)+\epsilon f(x,y,z,\epsilon),\\ \frac{dy}{dr}=-\frac{\partial H}{\partial x}(x,y)+\epsilon g(x,y,z,\epsilon),\\ \epsilon\frac{dz}{dr}=h(x,y,z,\epsilon),\quad \frac{dF}{dr}=\omega(x,y,z,\epsilon),\end{gathered} \tag{5.3}$$

with initial condition $(x(\tau'), y(\tau'), z(\tau'), 0)$. Moreover, according to Tikhonov's Theorem, the components $x(r)$, $y(r)$ and $F(r)$ of (5.3) are infinitely close, for all limited $r$, to the solution of the standard system

$$\begin{aligned} \frac{dx}{dr} &= \frac{\partial H}{\partial y}(x, y), \\ \frac{dy}{dr} &= -\frac{\partial H}{\partial x}(x, y), \\ \frac{dF}{dr} &= \Omega(x, y), \end{aligned}$$

with initial condition $({}^o x(\tau'{}^o y(\tau'), 0)$, where ${}^o x(\tau')$ and ${}^o y(\tau')$ are the standard parts of $x(\tau')$ and $y(\tau')$. Hence, for all limited $r$,

$$\begin{aligned} x(r) &\simeq q(r, E(\tau')) \simeq q(r, E'), \\ y(r) &\simeq p(r, E(\tau')) \simeq p(r, E'), \end{aligned}$$

where $E'$ is the standard part of $E(\tau')$ and

$$F(r) \simeq \int_0^r \Omega(q(\nu, E'), p(\nu, E'))d\nu.$$

In particular, by periodicity, we obtain

$$F(P(E')) \simeq \int_0^{P(E')} \Omega(q(\nu, E'), p(\nu, E'))d\nu. \tag{5.4}$$

We define now the successive instant of observation by $\tau'' = \tau' + \epsilon P(E')$. We claim that $\tau'' \in \mathcal{J}$. Indeed, since $\tau'$ is in $\mathcal{J}$, $E(s)$ is nearstandard in $\text{int}J$ for all $s \in [0, \tau']$. On the other hand, let $s \in [\tau', \tau'']$. Let $s = \tau' + \epsilon r$. By (5.2) we have that $E(s) = E(\tau') + \epsilon F(r) \simeq E(\tau') \simeq E'$ for all $r$ in $[0, P(E')]$. Thus, $E(s)$ is nearstandard in $\text{int}J$. We proved that, for any $\tau'$ limited in $\mathcal{J}$ and $E(\tau')$ nearstandard in $\text{int}J$, there exists $\tau''$ such that $0 \simeq \tau'' - \tau' \geq \mu$, $[\tau', \tau''] \subset \mathcal{J}$, $E(s) \simeq E(\tau')$ for all $s$ in $[\tau', \tau'']$. Moreover, by (5.4),

$$\frac{E(\tau'') - E(\tau')}{\tau'' - \tau'} = \frac{F(P(E'))}{P(E')} \simeq \frac{M(E')}{P(E')} \simeq \frac{M(E(\tau'))}{P(E(\tau'))}.$$

By the Stroboscopy Lemma 5.2,

$$[0, L] \subset \mathcal{J} \quad \text{and} \quad E(\tau) \simeq \bar{E}(\tau) \text{ for all } \tau \in [0, L]. \tag{5.5}$$

**Proof of Theorem 4.1.** Consider $\tau_1 \geq 0$ such that $[0, \tau_1] \subset [0, L]$ and $E(\tau)$, (resp. $\lambda(\tau)$) nearstandard in $\text{int}J$ (in $\text{int}D$ ) for all $\tau \in [0, \tau_1]$. Let us consider the external collection

$$\mathcal{J} = \{\tau \geq 0 : E(s), \text{ (resp. } \lambda(s)) \text{ nearstandard in int}J \text{ (in int}D \text{ ) for all } s \in [0, \tau]\}.$$

Let $\mu = \epsilon \min\{P(E, \lambda),\ E \in J,\ \lambda \in D\}$. Let $\tau'$ limited in $\mathcal{J}$. The change of variables

$$r = \frac{\tau - \tau'}{\epsilon}, \quad F(r) = \frac{E(\tau' + \epsilon r) - E(\tau')}{\epsilon}, \quad \Lambda(r) = \frac{\lambda(\tau' + \epsilon r) - \lambda(\tau')}{\epsilon}, \tag{5.6}$$

transforms the system formed by (4.1) and (4.3) with initial condition

$$(x(\tau'), y(\tau'), z(\tau'), \lambda(\tau'), E(\tau'))$$

into

$$
\begin{aligned}
\frac{dx}{dr} &= \frac{\partial H}{\partial y}(x,y,\lambda(\tau')+\epsilon\Lambda(r)) + \epsilon f(x,y,z,\lambda(\tau')+\epsilon\Lambda(r)),\\
\frac{dy}{dr} &= -\frac{\partial H}{\partial x}(x,y,\lambda(\tau')+\epsilon\Lambda(r)) + \epsilon g(x,y,z,\lambda(\tau')+\epsilon\Lambda(r)),\\
\epsilon\frac{dz}{dr} &= h(x,y,z,\lambda(\tau')+\epsilon\Lambda(r)),\\
\frac{d\Lambda}{dr} &= \alpha(x,y,z,\lambda(\tau')+\epsilon\Lambda(r)),\\
\frac{dF}{dr} &= \omega(x,y,z,\lambda(\tau')+\epsilon\Lambda(r)),
\end{aligned} \tag{5.7}
$$

with initial condition $(x(\tau'), y(\tau'), z(\tau'), 0, 0)$. Let $\lambda'$ be the standard part of $\lambda(\tau')$. According to Tikhonov's theorem one can state that for all limited $r$, the components $x(r)$, $y(r)$, $\lambda(r)$ and $F(r)$ of (5.7) are infinitely close to the solution of the standard system

$$
\begin{aligned}
\frac{dx}{dr} &= \frac{\partial H}{\partial y}(x,y,\lambda'),\\
\frac{dy}{dr} &= -\frac{\partial H}{\partial x}(x,y,\lambda'),\\
\frac{d\Lambda}{dr} &= A(x,y,\lambda'),\\
\frac{dF}{dr} &= \Omega(x,y,\lambda'),
\end{aligned}
$$

with initial condition $({}^o x(\tau' {}^o y(\tau'), 0, 0)$, where ${}^o x(\tau')$ and ${}^o y(\tau')$ are the standard parts of $x(\tau')$ and $y(\tau')$. That is, if $E'$ is the standard part of $E(\tau')$, then for all limited $r$,

$$
\begin{aligned}
x(r) &\simeq q(r,E',\lambda'),\\
y(r) &\simeq p(r,E',\lambda')),\\
F(r) &\simeq \int_0^r \Omega(q(\nu,E',\lambda'),p(\nu,E',\lambda'),\lambda')d\nu,\\
\Lambda(r) &\simeq \int_0^r A(q(\nu,E',\lambda'),p(\nu,E',\lambda'),\lambda')d\nu.
\end{aligned}
$$

By periodicity, we also have

$$
\begin{aligned}
F(P(E',\lambda')) &\simeq \int_0^{P(E',\lambda')} \Omega(q(\nu,E',\lambda'),p(\nu,E',\lambda'),\lambda')d\nu,\\
\Lambda(P(E',\lambda')) &\simeq \int_0^{P(E',\lambda')} A(q(\nu,E',\lambda'),p(\nu,E',\lambda'),\lambda')d\nu.
\end{aligned}
$$

Let $\tau'' = \tau' + \epsilon P(E',\lambda') \in \mathcal{J}$ be the successive instant. By (5.6),

$$
\begin{aligned}
\frac{E(\tau'')-E(\tau')}{\tau''-\tau'} &= \frac{F(P(E',\lambda'))}{P(E',\lambda')} \simeq \frac{M(E',\lambda')}{P(E',\lambda')} \simeq \frac{M(E(\tau'),\lambda(\tau'))}{P(E(\tau'),\lambda(\tau'))},\\
\frac{\lambda(\tau'')-\lambda(\tau')}{\tau''-\tau'} &= \frac{A(P(E',\lambda'))}{P(E',\lambda')} \simeq \frac{N(E',\lambda')}{P(E',\lambda')} \simeq \frac{N(E(\tau'),\lambda(\tau'))}{P(E(\tau'),\lambda(\tau'))}
\end{aligned}
$$

By the Stroboscopy Lemma, $[0,L] \subset \mathcal{J}$, $E(\tau) \simeq \bar{E}(\tau)$ and $\lambda(\tau) \simeq \bar{\lambda}(\tau)$ for all $\tau$ in $[0,L]$.

## 6. Applications

The following examples should be viewed more as didactic examples to illustrate the results, than as arising from practical problems.

**Example 1.** The system associated to the following third order differential equation

$$\epsilon \dddot{x} = h(x, \ddot{x}),$$

where $\epsilon > 0$ is a small real parameter and $h$ a sufficiently smooth function, is given by

$$\dot{x} = y, \quad \dot{y} = z_1, \quad \epsilon \dot{z}_1 = h(x, z_1), \tag{6.1}$$

where the dot denotes the derivative with respect to $t$. We suppose that $z_1 = u(x)$ is an isolated root of $h(x, z_1) = 0$ and that Tikhonov's Theorem conditions are satisfied; in particular, $\frac{\partial h}{\partial z_1}(x, z_1) < 0$, which makes the slow manifold $z_1 = u(x)$ attractive. To obtain the general form (1.6), we apply the change of variable

$$\epsilon z = z_1 - u(x),$$

which transforms (6.1) into the system

$$\dot{x} = y, \quad \dot{y} = u(x) + \epsilon z, \quad \epsilon \dot{z} = \tilde{h}(x, y, z_1, \epsilon), \tag{6.2}$$

where

$$\tilde{h}(x, y, z, \epsilon) = \frac{\partial h}{\partial z_1}(x, u(x)).z - u'(x)y + o(\epsilon).$$

The slow equation

$$\dot{q} = p, \quad \dot{p} = u(q), \tag{6.3}$$

is a Hamiltonian system with the Hamiltonian function

$$H(p, q) = -U(q) + \frac{p^2}{2},$$

where $U' = u$. The formula (2.7) becomes

$$\omega(x, y, z, \epsilon) = \epsilon yz,$$

and the averaged equation (2.10), where $\tau = \epsilon t$, takes the form

$$\frac{d\bar{E}}{d\tau} = \frac{M(\bar{E})}{P(\bar{E})} = \frac{1}{P(\bar{E})} \int_0^{P(\bar{E})} \Omega(q(v, \bar{E}), p(v, \bar{E}))dv, \tag{6.4}$$

where

$$\Omega(q, p) = u'^2 \big(\frac{\partial h}{\partial z_1}(q, u(q))\big)^{-1}, \tag{6.5}$$

and $(q(v, \bar{E}), p(v, \bar{E}))$ is a $P(\bar{E})$-periodic solution of (6.3). One can see that

$$P(\bar{E}) = 2 \int_{q_1(\bar{E})}^{q_2(\bar{E})} \frac{dq}{\sqrt{2(\bar{E} + U(q))}}, \tag{6.6}$$

where $q_1(\bar{E})$ and $q_2(\bar{E})$ are respectively the minimum and the maximum of an oscillation on the closed orbit $C(\bar{E})$. According to Theorem 2.1, the solution $(x(t, \epsilon), y(t, \epsilon), z(t, \epsilon))$ of (6.2) with initial condition $(p_0, q_0, z_0)$ mainly satisfies

$$\lim_{\epsilon \to 0}(x(t, \epsilon), y(t, \epsilon)) = (q(t, E_0), p(t, E_0)) \quad \text{for all } t \in [0, kP(E_0)],\ k \in \mathbb{N},$$

$$\lim_{\epsilon \to 0} z(t, \epsilon) = u(q(t, E_0)) \quad \text{for all } t \in ]0, kP(E_0)],$$

where $E_0 = \frac{p_0^2}{2} - U(q_0)$. Moreover, according to Theorem 2.2, the total energy $E(t,\epsilon) = H(x(t,\epsilon), y(t,\epsilon))$ of the system satisfies

$$\lim_{\epsilon\to 0} E(t,\epsilon) = \bar{E}(\epsilon t) \quad \text{for all } t \in [0, L/\epsilon],$$

where $\bar{E}(\tau)$ is the solution of (6.4) with initial condition $E_0$ and defined on $[0, L]$.

To illustrate the effectiveness of the stroboscopy method, we present a numerical simulation of the example above where we chose $h(x, \ddot{x}) = -\ddot{x} - x$, which gives a singularly perturbed harmonic oscillator. Hence, $u(x) = -x$ and system (6.2) corresponds to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \epsilon z, \\ \epsilon \dot{z} &= y - z. \end{aligned} \tag{6.7}$$

The Hamiltonian function of the corresponding slow Hamiltonian equation is

$$H(q,p) = \frac{1}{2}q^2 + \frac{1}{2}p^2,$$

and the period is exactly

$$P(\bar{E}) = 2\int_{-\sqrt{2\bar{E}}}^{\sqrt{2\bar{E}}} \frac{dq}{\sqrt{2(\bar{E} - \frac{1}{2}q^2)}} = 2\pi.$$

Moreover, according to (6.5) and to the first equation of (6.7),

$$\begin{aligned} M(\bar{E}) &= \oint_{C(\bar{E})} p^2(\nu, \bar{E}) d\nu = \oint_{C(\bar{E})} \sqrt{2\bar{E} - q^2} dq \\ &= 2\sqrt{2\bar{E}} \int_{-\sqrt{2\bar{E}}}^{\sqrt{2\bar{E}}} \sqrt{1 - (\frac{q}{\sqrt{2\bar{E}}})^2} dq. \end{aligned}$$

By the change of variable $X = q/\sqrt{2\bar{E}}$, one has

$$M(\bar{E}) = 4\bar{E}\int_{-1}^{1} (\sqrt{1 - X^2}) dX = 2\pi\bar{E}.$$

If we fix the initial conditions $(p_0, q_0, z_0) = (1, 2, 1)$, the averaged equation (6.4) is simply

$$\frac{d\bar{E}}{d\tau} = \bar{E}, \quad \bar{E}(0) = \frac{5}{2}, \tag{6.8}$$

with exact solution

$$\bar{E}(\tau) = \frac{5}{2} e^{\tau}.$$

Figure 1 shows how the considered trajectory jumps to the neighborhood of the slow manifold $z = y$ of the system (6.7) before it evolves along the closed orbits of the slow equation drawn on this slow manifold. Figure 2 is a comparison between the exact variation of the total energy $E(\tau)$ and the solution $\bar{E}(\tau)$ of the averaged equation (6.8) with the mentioned initial conditions. Note that the oscillating curve corresponds to $E(\tau)$.

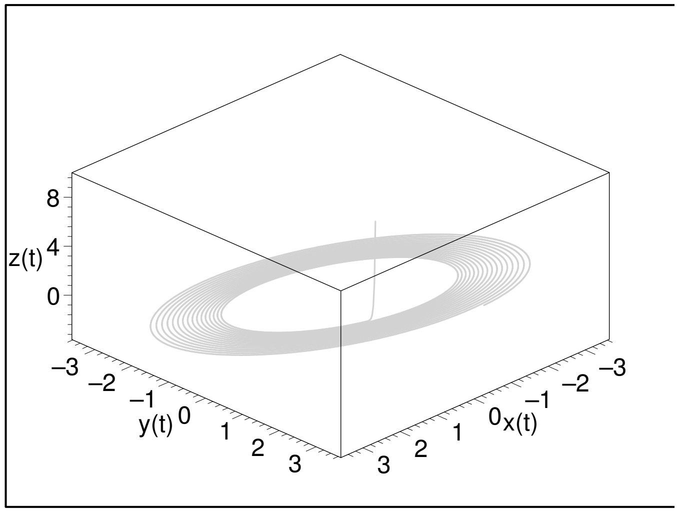


FIGURE 1. Numerical simulation of the trajectory of (6.7) with initial condition $(1, 2, 10)$, $\epsilon = .01$, $t = 0..100$

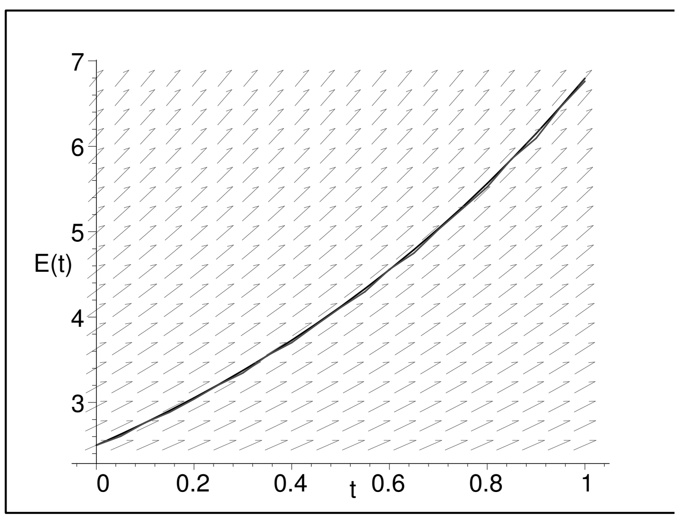


FIGURE 2. Comparison between $E(\tau)$ and $\bar{E}(\tau)$ for the system (6.7) with $\epsilon = .01$

**Example 2.** Consider the system

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -\lambda x + \epsilon z, \\
\epsilon \dot{z} &= -z + \lambda y, \\
\dot{\lambda} &= \epsilon \lambda x y,
\end{aligned}
\tag{6.9}
$$

In the same way as above, one can obtain

$$P(\bar{E},\bar{\lambda}) = 2\int_{-\sqrt{2\bar{E}/\bar{\lambda}}}^{\sqrt{2\bar{E}/\bar{\lambda}}} \frac{dq}{\sqrt{2(\bar{E}-\frac{1}{2}\bar{\lambda}q^2)}} = \frac{2\pi}{\sqrt{\bar{\lambda}}}.$$

According to (4.4), (4.5), (4.6) and the first equation of (6.9), we get

$$\begin{aligned} M(\bar{E},\bar{\lambda}) &= \oint_{C(\bar{E},\bar{\lambda})} [\bar{\lambda}p^2(\nu,\bar{E},\bar{\lambda}) + \frac{\bar{\lambda}}{2}q^3(\nu,\bar{E},\bar{\lambda})p(\nu,\bar{E},\bar{\lambda}]d\nu \\ &= \oint_{C(\bar{E},\bar{\lambda})} \bar{\lambda}pdq + \oint_{C(\bar{E},\bar{\lambda})} \frac{\bar{\lambda}}{2}q^3dq \\ &= 2\sqrt{2\bar{E}}\int_{-\sqrt{2\bar{E}/\bar{\lambda}}}^{\sqrt{2\bar{E}/\bar{\lambda}}} \bar{\lambda}\sqrt{2\bar{E}-\bar{\lambda}q^2}dq + 0 \\ &= \frac{2\pi\bar{\lambda}\bar{E}}{\sqrt{\bar{\lambda}}}. \end{aligned}$$

According to (4.5), (4.7) and the first equation of (6.9), we also get

$$N(\bar{E},\bar{\lambda}) = \oint_{C(\bar{E},\bar{\lambda})} \bar{\lambda}^2 qpd\nu = 2\int_{-\sqrt{2\bar{E}/\bar{\lambda}}}^{\sqrt{2\bar{E}/\bar{\lambda}}} qdq = 0.$$

Hence, the averaged system describing the drift of $E$ and $\lambda$ is given by the simple system

$$\begin{aligned} \bar{E}' &= \frac{M(\bar{E},\bar{\lambda})}{P(\bar{E},\bar{\lambda})} := \lambda\bar{E}, \\ \bar{\lambda}' &= \frac{N(\bar{E},\bar{\lambda})}{P(\bar{E},\bar{\lambda})} := 0. \end{aligned} \tag{6.10}$$

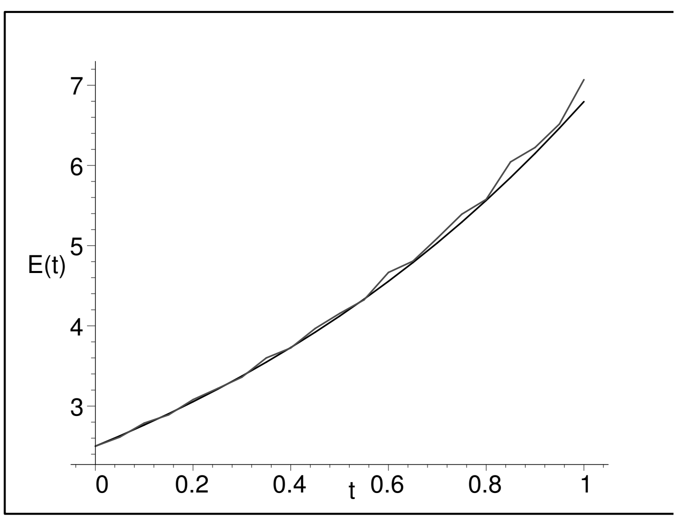


FIGURE 3. Comparison between $E(\tau)$ and $\bar{E}(\tau)$ for the system (6.9) with $\epsilon = .01$

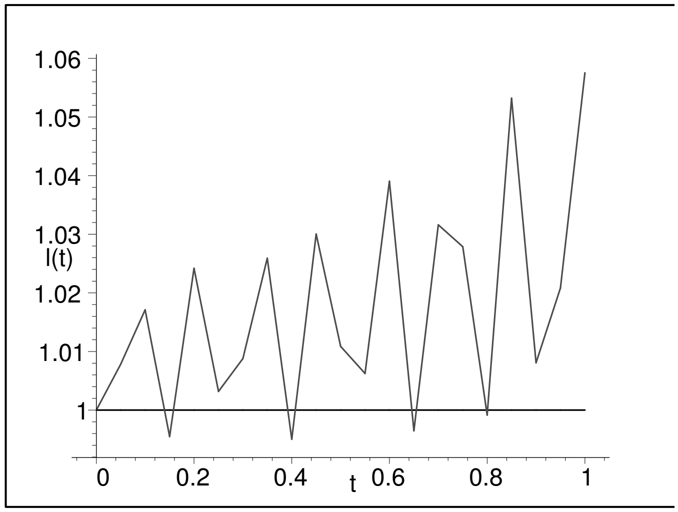


FIGURE 4. Comparison between $\lambda(\tau)$ and $\bar{\lambda}(\tau)$ for the system (6.9) with $\epsilon = .01$

Figures 3 and 4 compare the exact solutions $E(\tau)$ and $\lambda(\tau)$ of (6.9) with initial condition $E_0 = 5/2$ and $\lambda_0 = 1$ at time scale $\tau = \epsilon t$, and the solutions $\bar{E}(\tau) = \frac{5}{2}e^{\tau}$ and $\bar{\lambda}(\tau) = 1$ of (6.10). It is worth noting that in Figure 4 the difference between the oscillating curve and the averaged one does not exceed 0.06 for $0 \leq \tau \leq 1$, that is for $0 \leq t \leq 100$.

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Maamar Benbachir
Université de Béchar, BP 417, Route de Kenadsa, 08000 Béchar Algérie
*E-mail address*: `mbenbachir2001@yahoo.fr`

Karim Yadi
Laboratoire des Systémes Dynamiques et Applications, Université Aboubekr Belkaïd, BP 119, 13000 Tlemcen Algérie
*E-mail address*: `yadikdz@yahoo.fr, k_yadi@mail.univ-tlemcen.dz`

Rachid Bebbouchi
Faculté de Mathématiques, Université des Sciences et de la Technologie
Houari Boumediéne, Alger Bab ezzouar, Algérie
*E-mail address*: `rbebbouchi@hotmail.com`