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DISSIPATIVE BOUSSINESQ EQUATIONS ON NON-CYLINDRICAL DOMAINS IN $\mathbb{R}^n$

HAROLDO R. CLARK, ALFREDO T. COUSIN, CÍCERO L. FROTA, JUAN LÍMACO

Abstract. This article concerns the initial-boundary value problem for the nonlinear Boussinesq equations on time dependent domains in $\mathbb{R}^n$ with $1 \leq n \leq 4$. Global solvability, uniqueness of solutions and the exponential decay to the energy are established provided the initial data are bounded in some sense.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary Γ. By $Q_\infty = \Omega \times (0,\infty)$ and $\Sigma_\infty = \Gamma \times (0,\infty)$ we denote the cylindrical domain and its boundary, respectively. Given $k = k(t)$ a real function defined on $[0,\infty)$, for each $t \geq 0$ we denote Ω_t the transformed sets by the number $k(t)$; that is,

$$\Omega_t = \{x \in \mathbb{R}^n \text{ such that } x = k(t)y \text{ for all } y \in \Omega\},$$

and Γ_t is the boundary of Ω_t. Then the time dependent domain

$$\widehat{\mathcal{Q}}_\infty = \cup_{t>0}\big(\Omega_t \times \{t\}\big),$$

is a subset of $\mathbb{R}^{n+1}$, with lateral boundary

$$\widehat{\Sigma}_\infty = \cup_{t>0}\big(\Gamma_t \times \{t\}\big).$$

In this article, we study the initial-boundary value problem for the dissipative Boussinesq equation

$$u_{tt}(x,t) - \Delta\big(u(x,t) + u_t(x,t) + u^2(x,t)\big) + \Delta^2 u(x,t) = 0 \quad \text{in } \widehat{\mathcal{Q}}_\infty, \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \widehat{\Sigma}_\infty, \tag{1.2}$$

$$u(x,0) = u_0(x);\ u_t(x,0) = u_1(x) \quad \text{for } x \in \Omega_0. \tag{1.3}$$

The theory of water waves for the case of shallow water and waves of small amplitude, idealized by Scott-Russell in 1834, had one of the first mathematical analysis established in 1872 by Boussinesq [3]. His work derived a nonlinear dissipative wave system which is now known as the Boussinesq equations. See also Boussinesq [4].

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A nice survey on the history of the derivation of models of Boussinesq type can be found in Miles [10].

Initial-boundary value problem in a cylindrical domain with small initial data has been considered by Varlamov [13, 14, 15] in both 1-dimensional and 2-dimensional cases. As results, classical solutions were constructed, uniqueness of solutions and the long-time asymptotic were obtained explicitly. For more information about problems associated with Boussinesq equation, see Varlamov [16] and references therein. Liu-Russell [9] studied the existence and uniqueness of solutions to initial-boundary value problems on a 1-d periodic domain. There (1.1) have an internal weak damping $k_1 u_t$ and a linear feedback $k_2(u-[u])$.

For the one-dimensional case, we mention the works of Bona-Sachs [2] and Tsutsumi-Matahshi [12]; where the authors studied the existence, uniqueness and stability of solutions for Cauchy problems. Cauchy problem related to (1.1) in an abstract framework on a Hilbert space H has also been studied by other authors; Biler [1] and Pereira [11] established results on existence, uniqueness and asymptotic stability of solutions.

This article is motivated by the article [5] where a 1-d version of (1.1)-(1.3) is investigated. Our proof is a slight modification of the one in [5]. However we had to overcome some technically difficulties when considering this problem in $\widehat{\mathcal{Q}}_\infty$.

The paper is organized as follows: In section 2, we give some assumptions to be used later, and state the main results. Subsequently, sections 3 and 4 are devoted to prove the main results: Theorems 2.1 and 2.2.

2. Assumptions and main results

For the functional spaces we use standard notation as in Lions [7] and Lions-Magenes [8]. The inner product and norm in $L^2(\Omega)$ and $H_0^1(\Omega)$ are, respectively, denoted by

$$(f,g)=\int_\Omega f(\xi)\,g(\xi)\,d\xi,\quad |f|=\Big(\int_\Omega |f(\xi)|^2\,d\xi\Big)^{1/2},$$
$$((f,g))=\sum_{i=1}^n\int_\Omega \frac{\partial f}{\partial \xi_i}(\xi)\frac{\partial g}{\partial \xi_i}(\xi)d\xi\,,\quad \|f\|=\Big(\sum_{i=1}^n\int_\Omega |\frac{\partial f}{\partial \xi_i}(\xi)|^2\,d\xi\Big)^{1/2}.$$

For the rest of this article we consider $n\le 4$, which implies that $H_0^1(\Omega)$ is continuously imbedded in $L^4(\Omega)$. Let C_0 be such that $\|\cdot\|_{L^4(\Omega)}\le C_0\|\cdot\|$. Moreover, let C_1 and C_2 be positive real constants satisfying the inequalities $\|f\|_{H^2(\Omega)}\le C_1|\Delta f|$ and $\|f\|\le C_2|\Delta f|$ for all $f\in H_0^2(\Omega)$. Since Ω is bounded there exists C_3 such that $|y_i|\le\|y\|_{\mathbb{R}^n}\le C_3$, for all $y=(y_1,\dots y_n)\in\Omega$. From Poincaré inequality $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ and we put C_4 such that $|\cdot|\le C_4\|\cdot\|$. Henceforth we take for simplicity

$$C=\max_{0\le j\le 4} C_j\,. \tag{2.1}$$

We now state some assumptions on the function k:

$$k\in C^2\big([0,\infty)\big)\quad \text{with } k(0)=1\,, \tag{2.2}$$

$$0<k_0\le k(t)\le k_1<\frac{1}{\sqrt{2}\,C}\quad \text{for all } t\ge 0\,. \tag{2.3}$$

Let ϵ_0 be a real number such that

$$\epsilon_0 > \frac{4}{1-4k_1^2C^2}\,, \tag{2.4}$$

and for each pair of functions $(u_0,u_1)\in H_0^2(\Omega)\times L^2(\Omega)$ we denote

$$\alpha(u_0,u_1)=\frac{3}{4}|u_1|^2+\frac{1+C^2k_1^2}{k_0^2}\|u_0\|^2+\frac{1}{2k_0^4}|\Delta u_0|^2\,. \tag{2.5}$$

We also introduce the following tree polynomials:

$$\begin{aligned} p(\lambda,\eta)&=\big[(2+9nC^2k_1)C^2k_1+\frac{1}{k_0}\big]\lambda+\big[\frac{9}{4}\big((2n+1)^2+n^4C^8k_1^2\big)C^4\big]\lambda^2\\ &\quad+\big[\frac{9}{2}n^2C^4\big]\lambda^4+\big[\frac{9}{8}n^2C^4k_1^2\big]\eta^2\,; \end{aligned} \tag{2.6}$$

$$q(\lambda)=\frac{3}{k_0}\lambda; \tag{2.7}$$

$$r(\lambda,\eta)=\big[\frac{2}{k_0}+C^3k_1^3(2n+n^2C^2)\big]\lambda+(2nC^4k_1^2)\lambda^2+[nC^4k_1^3]\eta\,. \tag{2.8}$$

Now that the notation and assumptions have been set, we state the main results.

Theorem 2.1 (Existence and exponential decay)**.** *Suppose $n\le 4$ and* (2.2)–(2.3) *hold. If*

$$p(|k'(t)|,|k''(t)|)<\frac{1}{4}\,,\quad q(|k'(t)|)<\frac{1}{4}\,,\quad r(|k'(t)|,|k''(t)|)<\frac{1}{4}\,, \tag{2.9}$$

for all $t\ge 0$. Then for each $(u_0,u_1)\in H_0^2(\Omega)\times L^2(\Omega)$ such that

$$2\epsilon_0C^8k_1^6\alpha(u_0,u_1)+8C^3k_1^2\sqrt{\alpha(u_0,u_1)}<\frac{1}{4}\,, \tag{2.10}$$

there exists at least one global weak solution, u, to the problem (1.1)-(1.3)*, such that*

$$u\in L^\infty_{\rm loc}(0,\infty;H_0^2(\Omega_t)),\quad u_t\in L^2_{\rm loc}(0,\infty;H_0^1(\Omega_t)), \tag{2.11}$$

and it satisfies (1.1) *in the sense of $L^2(0,T;H^{-2}(\Omega_t))$. Moreover, there exist positive real constants $\kappa_0,\kappa_1,\kappa_2$, such that the energy*

$$E(u,t)=\frac{1}{2}\big\{|u'(t)|^2_{L^2(\Omega_t)}+|\nabla u(t)|^2_{L^2(\Omega_t)}+|\Delta u(t)|^2_{L^2(\Omega_t)}\big\}$$

of system (1.1)-(1.3) *satisfies*

$$E(u,t)\le\frac{\kappa_2\alpha(u_0,u_1)}{\kappa_1}e^{-t/\kappa_0}\quad\text{for all } t\ge 0, \tag{2.12}$$

where $\kappa_0,\kappa_1,\kappa_2$, are defined in (3.40), (3.43), (3.51)*, respectively.*

Theorem 2.2 (Uniqueness of Solutions)**.** *Under the assumption of Theorem 2.1, if k' and k'' satisfy*

$$|k'|_{L^1(0,+\infty)}+|k''|_{L^1(0,+\infty)}<\min\big\{\frac{1}{4K_1},\ \frac{1}{4K_2}\big\}, \tag{2.13}$$

where K_1,K_2 are real constants defined by (4.18)*, then the global weak solution of* (1.1)-(1.3) *is unique on $[0,T]$, for all $T>0$.*

3. Proof of Theorem 2.1

The idea is to transform the non-cylindrical mixed problem (1.1)-(1.3) into to a problem on a cylindrical domain, by using a suitable change of variables. Whence let us introduce the function $F:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}^n\times[0,\infty)$ defined by

$$F(y,t)=(k(t)y,t)=(k(t)y_1,\dots,k(t)y_n,t)\,,\quad \text{for } y=(y_1,\dots,y_n)\in\mathbb{R}^n. \tag{3.1}$$

It is not difficult to see that F is a diffeomorphism of class C^2 which satisfies:

$$F(Q_\infty)=\widehat{\mathcal{Q}}_\infty,\quad F(\Omega)=\Omega_t,\quad F(\Sigma_\infty)=\widehat{\Sigma}_\infty,\quad F^{-1}(x,t)=\big(\frac{x}{k(t)},t\big)\,.$$

Given a function $u:\widehat{\mathcal{Q}}_\infty\to\mathbb{R}$, using the diffeomorphism F, we define $v=(u\circ F):Q_\infty\to\mathbb{R}$; that is, $v(y,t)=u(k(t)y,t)$. Then we get

$$\begin{gathered}u(x,t)=v(y,t)\quad\text{where } y=\frac{x}{k(t)}\,,\\ \frac{\partial u}{\partial x_i}=\frac{1}{k(t)}\frac{\partial v}{\partial y_i}\quad\text{for } i=1,\dots,n,\\ \frac{\partial u}{\partial t}=-\frac{k'(t)}{k(t)}\sum_{j=1}^n\frac{\partial v}{\partial y_j}y_j+\frac{\partial v}{\partial t}\,.\end{gathered}\tag{3.2}$$

For the second order derivatives we find

$$\begin{gathered}\frac{\partial^2 u}{\partial x_i^2}=\frac{1}{k^2(t)}\frac{\partial^2 v}{\partial y_i^2}\quad\text{for } i=1,\dots,n\,,\\ \frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 v}{\partial t^2}-2\frac{k'(t)}{k(t)}\sum_{j=1}^n\frac{\partial^2 v}{\partial t\partial y_j}y_j+\big(\frac{k'(t)}{k(t)}\big)^2\sum_{j=1}^n\sum_{l=1}^n\frac{\partial^2 v}{\partial y_l\partial y_j}\,y_ly_j\\ +\big[\frac{2(k'(t))^2-k(t)k''(t)}{k^2(t)}\big]\sum_{j=1}^n\frac{\partial v}{\partial y_j}y_j\,,\\ \frac{\partial^2 u}{\partial x_i\partial t}=-\frac{k'(t)}{k^2(t)}\sum_{j=1}^n\frac{\partial^2 v}{\partial y_i\partial y_j}\,y_j-\frac{k'(t)}{k^2(t)}\frac{\partial v}{\partial y_i}+\frac{1}{k(t)}\frac{\partial^2 v}{\partial y_i\partial t}\,,\\ \frac{\partial^2}{\partial x_i^2}\big(\frac{\partial u}{\partial t}\big)=-\frac{k'(t)}{k^3(t)}\sum_{j=1}^n\frac{\partial^3 v}{\partial y_i^2\partial y_j}y_j-2\frac{k'(t)}{k^3(t)}\frac{\partial^2 v}{\partial y_i^2}+\frac{1}{k^2(t)}\frac{\partial^3 v}{\partial y_i^2\partial t}\,.\end{gathered}\tag{3.3}$$

Taking into account these computations, we have

$$\Delta u=\frac{1}{k^2(t)}\Delta v\,,\quad \Delta^2 u=\frac{1}{k^4(t)}\Delta^2 v\,, \tag{3.4}$$

$$\Delta\big(\frac{\partial u}{\partial t}\big)=-\frac{k'(t)}{k^3(t)}\sum_{j=1}^n\Delta\big(\frac{\partial v}{\partial y_j}\big)y_j-2\frac{k'(t)}{k^3(t)}\Delta v+\frac{1}{k^2(t)}\Delta\big(\frac{\partial v}{\partial t}\big); \tag{3.5}$$

$$\Delta(u^2)=\frac{1}{k^2(t)}\Delta(v^2). \tag{3.6}$$

From (3.3)-(3.6), a function u is a solution to the problem (1.1)-(1.3) if and only if v is a solution to the problem

$$\begin{aligned} &v_{tt}(y,t) - \frac{1}{k^2(t)}\Delta(v(y,t) + v_t(y,t) + v^2(y,t))\frac{1}{k^4(t)}\Delta^2 v(y,t) \\ &+ 2\frac{k'(t)}{k^3(t)}\Delta v(y,t) + \frac{k'(t)}{k^3(t)}\sum_{j=1}^n \Delta(\frac{\partial v}{\partial y_j}(y,t))y_j - 2\frac{k'(t)}{k(t)}\sum_{j=1}^n \frac{\partial^2 v}{\partial t\partial y_j}(y,t)y_j \\ &+ (\frac{k'(t)}{k(t)})^2\sum_{j=1}^n\sum_{l=1}^n \frac{\partial^2 v}{\partial y_l\partial y_j}(y,t)y_l y_j \\ &+ [\frac{2(k'(t))^2 - k(t)k''(t)}{k^2(t)}]\sum_{j=1}^n \frac{\partial v}{\partial y_j}(y,t)y_j = 0 \quad \text{in } Q_\infty, \end{aligned} \tag{3.7}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Sigma_\infty, \tag{3.8}$$

$$v(y,0) = v_0(y) = u_0(y), \quad v_t(y,0) = v_1(y) = u_1(y) \quad \text{for } y \in \Omega. \tag{3.9}$$

According to the above statements, it suffices to prove that under the assumptions of Theorem 2.1 there exists at least a weak solution v to (3.7)-(3.9) satisfying

$$v \in L^\infty_{\rm loc}(0,\infty; H^2_0(\Omega)), \quad v_t \in L^2_{\rm loc}(0,\infty; H^1_0(\Omega)). \tag{3.10}$$

Let $(w_j)_{j\in\mathbb{N}}$ be a basis to the Sobolev space $H^2_0(\Omega)$, and let V_m be the finite dimensional subspace of $H^2_0(\Omega)$ spanned by the vectors $\{w_1, w_2, \dots, w_m\}$. The theory of ordinary differential equations yields a local solution $v_m(x,t) = \sum_{j=1}^m g_{jm}(t)w_j(x)$ in V_m, defined in $[0,T_m]$ for each $m \in \mathbb{N}$. This solution is a local solution to the approximate initial value problem

$$\begin{aligned} &(v_m''(t), w) + \frac{1}{k^2(t)}\Big(\nabla(v_m(t) + v_m'(t) + v_m^2(t)), \nabla w\Big) + \frac{1}{k^4(t)}(\Delta v_m(t), \Delta w) \\ &- 2\frac{k'(t)}{k^3(t)}\Big(\nabla v_m(t), \nabla w\Big) - \frac{k'(t)}{k^3(t)}\sum_{j=1}^n \Big(\nabla(\frac{\partial v_m}{\partial y_j}(t)), \nabla(y_j w)\Big) \\ &- 2\frac{k'(t)}{k(t)}\sum_{j=1}^n \Big(\frac{\partial v_m'}{\partial y_j}(t)y_j, w\Big) + (\frac{k'(t)}{k(t)})^2\sum_{i=1}^n\sum_{j=1}^n \Big(\frac{\partial^2 v_m}{\partial y_i\partial y_j}(t)y_i y_j\,, w\Big) \\ &+ [\frac{2(k'(t))^2 - k(t)k''(t)}{k^2(t)}]\sum_{j=1}^n (\frac{\partial v_m}{\partial y_j}(t)y_j, w\Big) = 0 \quad \text{for all } w \in V_m\,, \end{aligned} \tag{3.11}$$

$$v_m(0) = v_{0m} \to u_0 \quad \text{in } H^2_0(\Omega) \quad \text{and} \quad v_m'(0) = v_{1m} \to u_1 \quad \text{in } L^2(\Omega). \tag{3.12}$$

Now we need estimates independent of m and t which will allow us to extend the solutions v_m to the whole interval $[0,\infty)$ and take to the limit in v_m as $m \to \infty$.

A priori estimates. First we take $w = v_m'$ in (3.11) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|v_m'(t)|^2 + \frac{1}{k^2(t)}\frac{d}{dt}\|v_m(t)\|^2 + \frac{1}{k^2(t)}\|v_m'(t)\|^2 + \frac{1}{k^2(t)}(\nabla v_m^2(t), \nabla v_m'(t)) \\ &+ \frac{1}{2k^4(t)}\frac{d}{dt}|\Delta v_m(t)|^2 - \frac{2k'(t)}{k^3(t)}(\nabla v_m(t), \nabla v_m'(t)) \end{aligned}$$

$$
\begin{aligned}
&- \frac{k'(t)}{k^3(t)} \sum_{j=1}^n \Big(\nabla(\frac{\partial v_m}{\partial y_j}(t)), \nabla(y_j v_m'(t))\Big) \\
&- 2\frac{k'(t)}{k(t)} \sum_{j=1}^n \Big(\frac{\partial v_m'}{\partial y_j}(t) y_j, v_m'(t)\Big) + (\frac{k'(t)}{k(t)})^2 \sum_{i=1}^n \sum_{j=1}^n \Big(\frac{\partial^2 v_m}{\partial y_i \partial y_j}(t) y_i y_j \,, v_m'(t)\Big) \\
&+ \big[\frac{2(k'(t))^2 - k(t)k''(t)}{k^2(t)}\big] \sum_{j=1}^n \Big(\frac{\partial v_m}{\partial y_j}(t) y_j, v_m'(t)\Big) = 0.
\end{aligned} \tag{3.13}
$$

Now we study each term in (3.13):

$$
\frac{1}{k^2(t)} \frac{d}{dt} \|v_m(t)\|^2 = \frac{d}{dt}(\frac{1}{2}\frac{\|v_m(t)\|^2}{k^2(t)}) + \frac{k'(t)}{k^3(t)} \|v_m(t)\|^2; \tag{3.14}
$$

$$
\begin{aligned}
&\big|\frac{1}{k^2(t)}(\nabla v_m^2(t), \nabla v_m'(t))\big|_{\mathbb{R}} \\
&\le \frac{2}{k^2(t)} \sum_{i=1}^n \|v_m(t)\|_{L^4(\Omega)} \,\|\frac{\partial v_m}{\partial y_i}(t)\|_{L^4(\Omega)}\,|\frac{\partial v_m'}{\partial y_i}(t)| \\
&\le \frac{2C^2}{k^2(t)} \|v_m(t)\| \sum_{i=1}^n \|\frac{\partial v_m}{\partial y_i}(t)\|\,|\frac{\partial v_m'}{\partial y_i}(t)| \\
&\le \frac{2C^2}{k^2(t)} \|v_m(t)\|\,\|v_m(t)\|_{H^2(\Omega)}\,\|v_m'(t)\| \\
&\le \frac{2C^4}{k^2(t)} |\Delta v_m(t)|^2\,\|v_m'(t)\| \\
&\le \epsilon_0 C^8 \frac{|\Delta v_m(t)|^4}{k^2(t)} + \frac{1}{\epsilon_0}\frac{\|v_m'(t)\|^2}{k^2(t)}\,;
\end{aligned} \tag{3.15}
$$

here we have used the imbedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ and the constants C and ϵ_0 given in (2.1) and (2.4), respectively. We also find

$$
\frac{1}{2k^4(t)} \frac{d}{dt}|\Delta v_m(t)|^2 = \frac{d}{dt}\Big(\frac{1}{2k^4(t)}|\Delta v_m(t)|^2\Big) + \frac{2k'(t)}{k^5(t)}|\Delta v_m(t)|^2\,; \tag{3.16}
$$

$$
\big|\frac{2k'(t)}{k^3(t)}(\nabla v_m(t), \nabla v_m(t))\big|_{\mathbb{R}} \le \frac{|k'(t)|}{k_0}\frac{\|v_m(t)\|^2}{k^2(t)} + \frac{|k'(t)|}{k_0}\frac{\|v_m'(t)\|^2}{k^2(t)}. \tag{3.17}
$$

Taking $\delta_i^j = 0$ if $i = j$ and 1 if $i \neq j$, we have

$$
\begin{aligned}
&\big|\frac{k'(t)}{k^3(t)} \sum_{j=1}^n \Big(\nabla(\frac{\partial v_m}{\partial y_j}(t)), \nabla(y_j\, v_m'(t))\Big)\big|_{\mathbb{R}} \\
&= \Big|\frac{k'(t)}{k^3(t)}\Big[\sum_{i=1}^n \Big(\frac{\partial^2 v_m}{\partial y_i^2}(t), v_m'(t)\Big) + \sum_{i=1}^n \Big(\frac{\partial^2 v_m}{\partial y_i^2}(t), y_i \frac{\partial v_m'}{\partial y_i}(t)\Big) \\
&\quad + \sum_{j=1}^n \sum_{i=1}^n \delta_i^j \Big(\frac{\partial^2 v_m}{\partial y_i \partial y_j}(t), y_j \frac{\partial v_m'}{\partial y_i}(t)\Big)\Big]\Big|_{\mathbb{R}} \\
&\le \frac{|k'(t)|}{k^3(t)}(2n+1)C^2\|v_m'(t)\|\,|\Delta v_m(t)| \\
&\le \frac{9(2n+1)^2C^4}{4}|k'(t)|^2\frac{\|v_m'(t)\|^2}{k^2(t)} + \frac{1}{9}\frac{|\Delta v_m(t)|^2}{k^4(t)}\,;
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
\Big|2\frac{k'(t)}{k(t)}\sum_{j=1}^{n}\Big(\frac{\partial v'_m}{\partial y_j}(t)y_j, v'_m(t)\Big)\Big|_{\mathbb{R}} &\le 2C\frac{|k'(t)|}{k(t)}\sum_{j=1}^{n}|\frac{\partial v'_m}{\partial y_j}(t)|\,|v'_m(t)| \\
&\le 2C\frac{|k'(t)|}{k(t)}|v'_m(t)|\,\|v'_m(t)\| \\
&\le 2C^2\frac{|k'(t)|}{k(t)}\|v'_m(t)\|^2\,;
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
&\Big|(\frac{k'(t)}{k(t)})^2\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{\partial^2 v_m}{\partial y_i\partial y_j}(t)y_i\,y_j\,, v'_m(t)\Big)\Big|_{\mathbb{R}} \\
&\le C^2\frac{|k'(t)|^2}{k^2(t)}|v'_m(t)|\sum_{i=1}^{n}\sum_{j=1}^{n}|\frac{\partial^2 v_m}{\partial y_i\partial y_j}(t)| \\
&\le n^2C^4\frac{|k'(t)|^2}{k^2(t)}\|v'_m(t)\|\,|\Delta v_m(t)| \\
&\le \frac{9n^4C^8}{4}|k'(t)|^2\|v'_m(t)\|^2+\frac{1}{9}\frac{|\Delta v_m(t)|^2}{k^4(t)}\,;
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\Big|\big[\frac{2(k'(t))^2-k(t)k''(t)}{k^2(t)}\big]\sum_{j=1}^{n}\Big(\frac{\partial v_m}{\partial y_j}(t)y_j, v'_m(t)\Big)\Big|_{\mathbb{R}} \\
&\le C^2n\frac{|2(k'(t))^2-k(t)k''(t)|}{k(t)}\|v'_m(t)\|\,\frac{\|v_m(t)\|}{k(t)} \\
&\le \frac{9C^4n^2}{4}|2(k'(t))^2-k(t)k''(t)|^2\frac{\|v'_m(t)\|^2}{k^2(t)}+\frac{1}{9}\frac{\|v_m(t)\|^2}{k^2(t)}\,;
\end{aligned}
\tag{3.21}
$$

Inserting (3.14)-(3.21) in (3.13) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big[|v'_m(t)|^2+\frac{\|v_m(t)\|^2}{k^2(t)}+\frac{|\Delta v_m(t)|^2}{k^4(t)}\big]+\frac{\|v'_m\|^2}{k^2(t)} \\
&\le \big[\frac{1}{9}+\frac{2}{k_0}|k'(t)|\big]\frac{\|v_m(t)\|^2}{k^2(t)}+\Big[\frac{1}{\epsilon_0}+(2C^2k_1+\frac{1}{k_0})|k'(t)| \\
&\quad+\frac{9C^4\big((2n+1)^2+C^8n^4k_1^2\big)}{4}|k'(t)|^2+\frac{9C^4n^2}{2}|k'(t)|^4 \\
&\quad+\frac{9C^4n^2k_1^2}{8}|k''(t)|^2\Big]\frac{\|v'_m(t)\|^2}{k^2(t)}+\Big(\frac{2}{9}+\frac{2}{k_0}|k'(t)|\Big)\frac{|\Delta v_m(t)|^2}{k^4(t)} \\
&\quad+\epsilon_0C^8\frac{|\Delta v_m(t)|^4}{k^2(t)}\,.
\end{aligned}
\tag{3.22}
$$

Now we go back to (3.11) and take $w=v_m(t)$. Hence

$$
\begin{aligned}
&\frac{d}{dt}(v'_m(t),v_m(t))-|v'_m(t)|^2+\frac{\|v_m(t)\|^2}{k^2(t)}+\frac{1}{2k^2(t)}\frac{d}{dt}\|v_m(t)\|^2 \\
&+\frac{1}{k^2(t)}\Big(\nabla(v_m(t))^2,\nabla v_m(t)\Big)+\frac{|\Delta v_m(t)|^2}{k^4(t)}-2\frac{k'(t)}{k^3(t)}\|v_m(t)\|^2 \\
&-\frac{k'(t)}{k^3(t)}\sum_{j=1}^{n}\Big(\nabla(\frac{\partial v_m}{\partial y_j}(t)),\nabla(y_j\,v_m(t))\Big)-\frac{2k'(t)}{k(t)}\sum_{j=1}^{n}\Big(\frac{\partial v'_m}{\partial y_j}(t)y_j,v_m(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{k'(t)}{k(t)}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\frac{\partial^2 v_m}{\partial y_i\partial y_j}(t)y_iy_j, v_m(t)\Big)\\
&+\Big(\frac{2(k'(t))^2-k(t)k''(t)}{k^2(t)}\Big)\sum_{j=1}^{n}\Big(\frac{\partial v_m}{\partial y_j}(t)y_j, v_m(t)\Big)=0\,.
\end{aligned}\tag{3.23}
$$

We work with each term of (3.23):

$$
\frac{1}{2k^2(t)}\frac{d}{dt}\|v_m(t)\|^2=\frac{d}{dt}\Big[\frac{1}{2k^2(t)}\|v_m(t)\|^2\Big]+\frac{k'(t)}{k^3(t)}\|v_m(t)\|^2\,;\tag{3.24}
$$

$$
\begin{aligned}
&\Big|\frac{1}{k^2(t)}\Big(\nabla(v_m(t))^2,\nabla v_m(t)\Big)\Big|_{\mathbb{R}}\\
&\le\frac{2}{k^2(t)}\sum_{i=1}^{n}\int_\Omega\big|v_m(x,t)\big|_{\mathbb{R}}\big|\frac{\partial v_m}{\partial y_i}(x,t)\big|_{\mathbb{R}}\big|\frac{\partial v_m}{\partial y_i}(x,t)\big|_{\mathbb{R}}\,dy\\
&\le\frac{2}{k^2(t)}\sum_{i=1}^{n}\|v_m(t)\|_{L^4(\Omega)}\|\frac{\partial v_m}{\partial y_i}(t)\|_{L^4(\Omega)}|\frac{\partial v_m}{\partial y_i}(t)|\\
&\le\frac{2C^2n}{k^2(t)}\|v_m(t)\|^2\,\|v_m(t)\|_{H^2(\Omega)}\\
&\le\frac{2nC^3}{k^2(t)}\,\|v_m(t)\|\,|\Delta v_m(t)|^2\,;
\end{aligned}\tag{3.25}
$$

$$
\begin{aligned}
&\big|\frac{k'(t)}{k^3(t)}\sum_{j=1}^{n}\Big(\nabla(\frac{\partial v_m}{\partial y_j}(t)),\nabla(y_j\,v_m(t))\Big)\Big|_{\mathbb{R}}\\
&\le\frac{|k'(t)|}{k^3(t)}\Big[|v_m(t)|\sum_{i=1}^{n}|\frac{\partial^2 v_m}{\partial y_i^2}(t)|+C\sum_{j=1}^{n}\sum_{i=1}^{n}|\frac{\partial^2 v_m}{\partial y_i\partial y_j}(t)||\frac{\partial v_m}{\partial y_i}(t)|\Big]\\
&\le\frac{|k'(t)|}{k^3(t)}[n|v_m(t)|\|v_m(t)\|_{H^2(\Omega)}+nC\|v_m(t)\|\|v_m(t)\|_{H^2(\Omega)}]\\
&\le 2nC^3\frac{|k'(t)|}{k^3(t)}|\Delta v_m(t)|^2\,;
\end{aligned}\tag{3.26}
$$

$$
\begin{aligned}
\Big|\frac{2k'(t)}{k(t)}\sum_{j=1}^{n}\Big(\frac{\partial v_m'}{\partial y_j}(t)y_j, v_m(t)\Big)\Big|_{\mathbb{R}}&\le\frac{2C|k'(t)|}{k(t)}\sum_{j=1}^{n}|\frac{\partial v_m'}{\partial y_j}||v_m(t)|\\
&\le 2\frac{\|v_m(t)\|}{k(t)}\,nC^2|k'(t)|\|v_m'(t)\|\\
&\le 9nC^4|k'(t)|^2\|v_m'(t)\|^2+\frac{1}{9}\frac{\|v_m(t)\|^2}{k^2(t)}\,;
\end{aligned}\tag{3.27}
$$

$$
\begin{aligned}
&\Big|\Big(\frac{2(k'(t))^2-k(t)k''(t)}{k^2(t)}\Big)\sum_{j=1}^{n}\Big(\frac{\partial v_m}{\partial y_j}(t)y_j, v_m(t)\Big)\Big|_{\mathbb{R}}\\
&\le\Big(\frac{2|k'(t)|^2+k(t)|k''(t)|}{k^2(t)}\Big)C|v_m(t)|\sum_{j=1}^{n}|\frac{\partial v_m}{\partial y_j}(t)|\\
&\le\Big(\frac{2|k'(t)|^2+k(t)|k''(t)|}{k^2(t)}\Big)nC^4|\Delta v_m(t)|^2\,.
\end{aligned}\tag{3.28}
$$

Since $0 < k_0 \le k(t) \le k_1$, taking into account (3.23)-(3.28) we find

$$
\begin{aligned}
&\frac{d}{dt}\big[(v_m'(t), v_m(t)) + \frac{1}{2}\frac{\|v_m(t)\|^2}{k^2(t)}\big] + \frac{8}{9}\frac{\|v_m(t)\|^2}{k^2(t)} + \frac{|\Delta v_m(t)|^2}{k^4(t)} \\
&\le \frac{1}{k_0}|k'(t)|\frac{\|v_m(t)\|^2}{k^2(t)} + \Big(k_1^2C^2 + 9nC^4k_1^2|k'(t)|\Big)\frac{\|v_m'(t)\|^2}{k^2(t)} \\
&\quad + 2nC^3k_1^2\|v_m(t)\|\frac{|\Delta v_m(t)|^2}{k^4(t)} + \Big[(2n + n^2C^2)C^3k_1^3|k'(t)| \\
&\quad + nC^4k_1^2\Big(2|k'(t)|^2 + k_1|k''(t)|\Big)\Big]\frac{|\Delta v_m(t)|^2}{k^4(t)}\,.
\end{aligned} \tag{3.29}
$$

This inequality and (3.22) yields

$$
\begin{aligned}
&\frac{dH}{dt}(t) + \big(1 - k_1^2C^2 - \frac{1}{\epsilon_0}\big)\frac{\|v_m'(t)\|^2}{k^2(t)} + \frac{7}{9}\frac{\|v_m(t)\|^2}{k^2(t)} + \frac{7}{9}\frac{|\Delta v_m(t)|^2}{k^4(t)} \\
&\le p(|k'(t)|, |k''(t)|)\frac{\|v_m'(t)\|^2}{k^2(t)} + q(|k'(t)|)\frac{\|v_m(t)\|^2}{k^2(t)} \\
&\quad + r(|k'(t)|, |k''(t)|)\frac{|\Delta v_m(t)|^2}{k^4(t)} + \epsilon_0C^8\frac{|\Delta v_m(t)|^4}{k^2(t)} \\
&\quad + 2nC^3k_1^2\|v_m(t)\|\frac{|\Delta v_m(t)|^2}{k^4(t)}\,,
\end{aligned} \tag{3.30}
$$

where

$$
H(t) = \frac{1}{2}|v_m'(t)|^2 + \frac{\|v_m(t)\|^2}{k^2(t)} + \frac{1}{2}\frac{|\Delta v_m(t)|^2}{k^4(t)} + (v_m'(t), v_m(t)). \tag{3.31}
$$

From (2.3) and (2.4) we can see that $(1 - k_1^2C^2 - \frac{1}{\epsilon_0}) > 3/4$. Therefore, we rewrite (3.30) as

$$
\begin{aligned}
&\frac{dH}{dt}(t) + \big(\frac{3}{4} - p(|k'(t)|, |k''(t)|)\big)\frac{\|v_m'(t)\|^2}{k^2(t)} \\
&+ \big(\frac{3}{4} - q(|k'(t)|)\big)\frac{\|v_m(t)\|^2}{k^2(t)} + (\frac{3}{4} - r(|k'(t)|, |k''(t)|))\frac{|\Delta v_m(t)|^2}{k^4(t)} \\
&- \epsilon_0C^8\frac{|\Delta v_m(t)|^4}{k^2(t)} - 2nC^3k_1^2\|v_m(t)\|\frac{|\Delta v_m(t)|^2}{k^4(t)} \le 0\,.
\end{aligned} \tag{3.32}
$$

This inequality and (2.9) yield

$$
\frac{dH}{dt}(t) + \frac{1}{2}\frac{\|v_m'(t)\|^2}{k^2(t)} + \frac{1}{2}\frac{\|v_m(t)\|^2}{k^2(t)} + \frac{1}{4}\frac{|\Delta v_m(t)|^2}{k^4(t)} + \Big(\frac{1}{4} - \gamma(t)\Big)\frac{|\Delta v_m(t)|^2}{k^4(t)} \le 0\,, \tag{3.33}
$$

where

$$
\gamma(t) = \epsilon_0C^8k_1^2|\Delta v_m(t)|^2 + 2nC^3k_1\|v_m(t)\|\,.
$$

On the other hand,

$$
|(v_m'(t), v_m(t))| \le \frac{1}{4}|v_m'(t)|^2 + C^2k_1^2\frac{\|v_m(t)\|^2}{k^2(t)}\,. \tag{3.34}
$$

Taking into account the definition of $H(t)$, (3.33) and (3.34), for all $t \geq 0$ we find

$$\begin{aligned} &\frac{1}{4}|v_m'(t)|^2 + \frac{1}{2}\frac{\|v_m(t)\|^2}{k^2(t)} + \frac{1}{2}\frac{|\Delta v_m(t)|^2}{k^4(t)} \\ &\leq H(t) \\ &\leq \frac{3}{4}|v_m'(t)|^2 + \frac{(1+C^2k_1^2)}{k_0^2}\|v_m(t)\|^2 + \frac{1}{2k_0^4}|\Delta v_m(t)|^2 , \end{aligned} \tag{3.35}$$

which in particular for $t = 0$ gives $H(0) \leq \alpha(u_0, u_1)$. Simple computations then lead to

$$\gamma(t) \leq 2\epsilon_0 C^8 k_1^6 H(t) + 2nC^3 k_1^2 H^{1/2}(t) \quad \forall t \geq 0 , \tag{3.36}$$

and from (2.10), we obtain $\gamma(0) \leq 2\epsilon_0 C^8 k_1^6 \alpha(u_0, u_1) + 2nC^3 k_1^2 \sqrt{\alpha(u_0, u_1)} < 1/4$. Now we claim that

$$\gamma(t) < \frac{1}{4} \quad \text{for all } t \geq 0. \tag{3.37}$$

By contradiction let us suppose that (3.37) does not hold. The continuity gives $t^* > 0$ such that

$$\gamma(t) < \frac{1}{4} \quad \text{for all } t \in [0, t^*) \quad \text{and} \quad \gamma(t^*) = \frac{1}{4} . \tag{3.38}$$

Integrating (3.32) from 0 to t^* we come to $H(t^*) \leq H(0) \leq \alpha(u_0, u_1)$. This inequality, (3.36) and (2.10) yield $\gamma(t^*) < 1/4$, which contradicts (3.38) and our claim is proved.

Since we have (3.32), (3.35) and (3.37) one can easily gets a constant $A > 0$ such that

$$|v_m'(t)|^2 + \|v_m(t)\|^2 + |\Delta v_m(t)|^2 + \int_0^t \|v_m'(s)\|^2 \, ds \leq A . \tag{3.39}$$

Hence for all $T > 0$ we have $(v_m)_{m\in\mathbb{N}}$ bounded in $L^\infty(0, T; H_0^2(\Omega))$ and $(v_m')_{m\in\mathbb{N}}$ bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$. From standard compactness arguments we are able to get the existence of global solutions.

To complete the proof of Theorem 2.1, we must to establish a rate decay estimate to the total energy of the problem (1.1)-(1.3). In fact, from (3.33) and (3.37), we get

$$\frac{dH}{dt}(t) + \frac{1}{2}\frac{\|v_m'(t)\|^2}{k^2(t)} + \frac{1}{2}\frac{\|v_m(t)\|^2}{k^2(t)} + \frac{1}{4}\frac{|\Delta v_m(t)|^2}{k^4(t)} \leq 0 .$$

From this inequality, (2.1) and (2.3), we obtain

$$\frac{dH}{dt}(t) + \frac{1}{2C}\frac{|v_m'(t)|^2}{k_1^2} + \frac{1}{2}\frac{\|v_m(t)\|^2}{k_1^2} + \frac{1}{4}\frac{|\Delta v_m(t)|^2}{k_1^4} \leq 0 .$$

From this inequality there exists a positive real constant κ_0 such that

$$\frac{dH}{dt}(t) + \kappa_0\Big(\frac{3}{4}|v_m'(t)|^2 + \frac{(1+C^2k_1^2)}{k_0^2}\|v_m(t)\|^2 + \frac{1}{2k_0^4}|\Delta v_m(t)|^2\Big) \leq 0 ,$$

where

$$\kappa_0 = \min\big\{\frac{2}{3Ck_1^2}, \frac{k_0^2}{2k_1^2(1+C^2k_1^2)}, \frac{k_0^4}{2k_1^4}\big\}. \tag{3.40}$$

Therefore, by using $(3.35)_2$ in this inequality we get

$$\frac{dH}{dt}(t) + \frac{1}{\kappa_0}H(t) \leq 0 \quad \text{for all } t \geq 0,$$

which gives

$$H(t) \le H(0)e^{-t/\kappa_0} \quad \text{for all } t \ge 0. \tag{3.41}$$

The total energy of the approximate system (3.11)-(3.12) comes from identity (3.13); that is,

$$E(v_m,t) = \frac{1}{2}\big\{|v_m'(t)|^2 + \frac{\|v_m(t)\|^2}{k^2(t)} + \frac{|\Delta v_m(t)|^2}{k^4(t)}\big\}. \tag{3.42}$$

From (3.42) and $(3.35)_1$ there exists $0 < \kappa_1 \le 1/2$ such that $\kappa_1 E_m(t) \le H(t)$. Also we have that $H(0) \le \alpha(u_0,u_1)$. Therefore, from (3.41) we get

$$E(v_m,t) \le \frac{\alpha(u_0,u_1)}{\kappa_1} e^{-t/\kappa_0} \quad \text{for all} \quad t \ge 0. \tag{3.43}$$

The estimate (3.39) gives enough convergence to take to the limit $m \to \infty$ in E_m, via Banach-Steinhauss theorem, which implies

$$E(v,t) \le \frac{\alpha(u_0,u_1)}{\kappa_1} e^{-t/\kappa_0} \quad \text{for all } t \ge 0, \tag{3.44}$$

where

$$E(v,t) = \frac{1}{2}\big\{|v'(t)|^2 + \frac{\|v(t)\|^2}{k^2(t)} + \frac{|\Delta v(t)|^2}{k^4(t)}\big\}, \tag{3.45}$$

is the total energy associated with the system (3.7)-(3.9). Finally, we have to compare the terms of $E(u,t)$ with those of $E(v,t)$. In fact, from (3.2)-(3.4) we have the identities

$$\begin{gathered} \frac{\partial u}{\partial x_i} = \frac{1}{k(t)} \frac{\partial v}{\partial y_i} \quad \text{for } i = 1,\dots,n; \\ \frac{\partial u}{\partial t} = -\frac{k'(t)}{k(t)} \sum_{j=1}^{n} \frac{\partial v}{\partial y_j} y_j + \frac{\partial v}{\partial t}, \quad \Delta u = \frac{1}{k^2(t)} \Delta v. \end{gathered} \tag{3.46}$$

From the first identity above, we have

$$\|\nabla u(x,t)\|_{\mathbb{R}^n}^2 = \frac{1}{k^2(t)} \|\nabla v(y,t)\|_{\mathbb{R}^n}^2.$$

Integrating this over Ω_t, using $x = k(t)y$ and $dx = k^n(t)dy$ we get, thanks to hypothesis (2.3), that

$$|\nabla u(t)|_{L^2(\Omega_t)}^2 \le \frac{k_1^n}{k_0^2} |\nabla v(t)|_{L^2(\Omega)}^2. \tag{3.47}$$

From the second identity of (3.46), we obtain

$$\big|\frac{\partial u}{\partial t}(x,t)\big|_{\mathbb{R}} \le \frac{k_2}{k_0} \|\nabla v(y,t)\|_{\mathbb{R}^n} \|y\|_{\mathbb{R}^n}^2 + \big|\frac{\partial v}{\partial t}(y,t)\big|_{\mathbb{R}}.$$

Squaring both sides, yields

$$\big|\frac{\partial u}{\partial t}(x,t)\big|_{\mathbb{R}}^2 \le \frac{2k_2^2 C^2}{k_0^2} \|\nabla v(y,t)\|_{\mathbb{R}^n}^2 + 2\big|\frac{\partial v}{\partial t}(y,t)\big|_{\mathbb{R}}^2.$$

Integrating this over Ω_t and observing that $dx = k^n(t)dy$, we get

$$|u'(t)|_{L^2(\Omega_t)}^2 \le \frac{2k_2^2 C^2 k_1^n}{k_0^2} |\nabla v(t)|_{L^2(\Omega)}^2 + 2k_1^n |v'(t)|_{L^2(\Omega)}^2 . \tag{3.48}$$

Repeating the same arguments as above in the third identity of (3.46), we find

$$|\Delta u(t)|^2_{L^2(\Omega_t)} \leq \frac{k_1^n}{k_0^4}|\Delta v(t)|^2_{L^2(\Omega)}. \tag{3.49}$$

Now, to compare the function $E(u,t)$ with the function $E(v,t)$ it will be used the equivalences of the norms: $\|z\|$ and $|\nabla z|$ in $H_0^1(\Omega)$. Thus, from (3.47)-(3.49) we get

$$\begin{aligned} E(u,t) &= \frac{1}{2}\big\{|u'(t)|^2_{L^2(\Omega_t)} + |\nabla u(t)|^2_{L^2(\Omega_t)} + |\Delta u(t)|^2_{L^2(\Omega_t)}\big\} \\ &\leq 2k_1^n|v'(t)|^2_{L^2(\Omega)} + \big(\frac{k_1^n}{k_0^2} + \frac{2k_2^2C^2k_1^n}{k_0^2}\big)|\nabla v(t)|^2_{L^2(\Omega)} + \frac{k_1^n}{k_0^4}|\Delta v(t)|^2_{L^2(\Omega)}. \end{aligned} \tag{3.50}$$

Thus, choosing

$$\kappa_2 = \max\big\{4k_1^n,\, 2k_1\big(\frac{k_1^n}{k_0^2} + \frac{2k_2^2C^2k_1^n}{k_0^2}\big),\, \frac{2k_1^n}{k_0^4}\big\}, \tag{3.51}$$

we get from (3.45) and (3.50) that $E(u,t) \leq \kappa_2 E(v,t)$ for all $t \geq 0$. Therefore, from (3.44) we obtain the desired estimate (2.12) and consequently the proof of Theorem 2.1 is finished

4. Proof of Theorem 2.2

Problems (1.1)-(1.3) and (3.7)-(3.9) are equivalent, then it is sufficient to show the uniqueness of solutions to (3.7)-(3.9). Suppose v and $\widehat{v}$ two solutions of (3.7)-(3.9). Thus, $\phi = v - \widehat{v}$ satisfies

$$\begin{aligned} &\phi_{tt}(y,t) - \frac{1}{k^2(t)}\Delta(\phi(y,t) + \phi_t(y,t)) + \frac{1}{k^4(t)}\Delta^2\phi(y,t) + 2\frac{k'(t)}{k^3(t)}\Delta\phi(y,t) \\ &+ \frac{k'(t)}{k^3(t)}\sum_{j=1}^n \Delta\big(\frac{\partial\phi}{\partial y_j}(y,t)\big)\,y_j - 2\frac{k'(t)}{k(t)}\sum_{j=1}^n \frac{\partial^2\phi}{\partial t\partial y_j}(y,t)y_j \\ &+ \big(\frac{k'(t)}{k(t)}\big)^2\sum_{j,\,l=1}^n \frac{\partial^2\phi}{\partial y_l\partial y_j}(y,t)y_l y_j + \big[\frac{2(k'(t))^2 - k(t)k''(t)}{k^2(t)}\big]\sum_{j=1}^n \frac{\partial\phi}{\partial y_j}(y,t)y_j \\ &= -\frac{1}{k^2(t)}(\Delta v^2(y,t) - \Delta\widehat{v}^2(y,t)) \quad \text{in } Q_\infty, \end{aligned} \tag{4.1}$$

$$\phi = \frac{\partial\phi}{\partial\nu} = 0 \quad \text{on} \quad \Sigma_\infty, \tag{4.2}$$

$$\phi(y,0) = \phi_t(y,0) = 0 \quad \text{for} \quad y \in \Omega. \tag{4.3}$$

Equation (4.1) is given in the sense of $L^2(0,T;H^{-2}(\Omega))$ and $\phi_t \in L^2(0,T;H_0^1(\Omega))$, then the duality $\big\langle\phi_{tt}(t),\phi_t(t)\big\rangle_{H^{-2}(\Omega)\times H_0^1(\Omega)}$ does not make sense. To overcome this difficulty, the uniqueness will be obtained following the argument contained in Ladyzhenskaya-Visik [6]. In fact, for each $s \in (0,T)$ let $\psi(t)$, be a real function defined for all t, in $]0,T[$, by

$$\psi(y,t) = \begin{cases} -\int_t^s \phi(y,r)dr & \text{if } 0 < t \leq s, \\ 0 & \text{if } s < t < T, \end{cases} \tag{4.4}$$

where ϕ, is a solution of (4.1)-(4.3). Since ϕ is in $L^\infty(0,T;H_0^2(\Omega))$ and ψ in $L^\infty(0,T;H_0^2(\Omega))$, the duality $\langle \phi_{tt}(t),\psi(t)\rangle_{H^{-2}(\Omega)\times H_0^2(\Omega)}$ makes sense. Moreover,

$$\psi_t(y,t)=\phi(y,t) \quad \text{and} \quad \psi(y,s)=0. \tag{4.5}$$

Setting $\psi_1(y,t)=\int_0^t\phi(y,r)dr$, we have

$$\psi(y,t)=\psi_1(y,t)-\psi_1(y,s) \quad \text{and} \quad \psi(y,0)=-\psi_1(y,s). \tag{4.6}$$

Taking the scalar product on $L^2(\Omega)$ of ψ with both sides of (4.1) and integrating from 0, to s, yields

$$\begin{aligned}
&\int_0^s\langle\phi_{tt}(t),\psi(t)\rangle dt-\int_0^s\frac{1}{k^2(t)}\langle\Delta(\phi(t)+\phi_t(t)),\psi(t)\rangle dt+\int_0^s\frac{1}{k^4(t)}\langle\Delta^2\phi(t),\psi(t)\rangle dt\\
&+2\int_0^s\frac{k'(t)}{k^3(t)}\langle\Delta\phi(t),\psi(t)\rangle dt+\int_0^s\frac{k'(t)}{k^3(t)}\Big\langle\sum_{j=1}^n\Delta\big(\frac{\partial\phi}{\partial y_j}(t)\big)\,y_j,\psi(t)\Big\rangle dt\\
&-2\int_0^s\frac{k'(t)}{k(t)}\Big\langle\sum_{j=1}^n\frac{\partial^2\phi}{\partial t\partial y_j}(t)y_j,\psi(t)\Big\rangle dt+\int_0^s\big(\frac{k'(t)}{k(t)}\big)^2\Big\langle\sum_{j,\,l=1}^n\frac{\partial^2\phi}{\partial y_l\partial y_j}(t)y_ly_j,\psi(t)\Big\rangle dt\\
&+\int_0^s\Big[\frac{2(k'(t))^2-k(t)k''(t)}{k^2(t)}\Big]\Big\langle\sum_{j=1}^n\frac{\partial\phi}{\partial y_j}(t)y_j,\psi(t)\Big\rangle dt \qquad (4.7)\\
&=-\int_0^s\frac{1}{k^2(t)}\langle(\Delta v^2(t)-\Delta\widehat{v}^{\,2}(t)),\psi(t)\rangle dt.
\end{aligned}$$

Now, we modify each terms of (4.7) by using several times integration by parts, the Green formula, the identities (4.5), (4.6), the null initial condition (4.3), the hypotheses (2.2), (2.3), (2.12) and usual inequalities like: Cauchy-Schwartz, Young so on. In fact, The first term can be changed as

$$\int_0^s\langle\phi_{tt}(t),\psi(t)\rangle dt=(\phi_t(t),\psi(t))\Big|_0^s-\int_0^s(\phi_t(t),\phi(t))dt=-\frac12|\phi(s)|^2. \tag{4.8}$$

The second term of (4.7) is modified as

$$\begin{aligned}
-\int_0^s\frac{1}{k^2(t)}\langle\Delta\phi(t),\psi(t)\rangle dt&=\int_0^s\frac{1}{k^2(t)}(\nabla\psi_t(t),\nabla\psi(t))dt\\
&=\frac12\int_0^s\Big\{\frac{d}{dt}\Big[\frac{1}{k^2(t)}|\nabla\psi(t)|^2\Big]-2\frac{k'(t)}{k^3(t)}|\nabla\psi(t)|^2\Big\}dt\\
&=-\frac12\frac{1}{k^2(0)}|\nabla\psi_1(s)|^2-\int_0^s\frac{k'(t)}{k^3(t)}|\nabla\psi_1(s)|^2dt.
\end{aligned}$$

The last integral above is bounded from above as follows:

$$\begin{aligned}
\Big|-\int_0^s\frac{k'(t)}{k^3(t)}|\nabla\psi_1(s)|^2dt\Big|_{\mathbb{R}}&\le2\int_0^s\frac{|k'(t)|}{k^3(t)}|\nabla\psi_1(t)|^2dt+\frac{2}{k_0^3}|\nabla\psi_1(s)|^2\int_0^s|k'(t)|dt\\
&=2\int_0^s\frac{|k'(t)|}{k^3(t)}|\nabla\psi_1(t)|^2dt+\frac{2}{k_0^3}|\nabla\psi_1(s)|^2|k'|_{L^1(0,\infty)}.
\end{aligned}$$

Therefore, as $k(0)=1$, see (2.2), we obtain

$$
\begin{aligned}
&-\int_0^s \frac{1}{k^2(t)}\langle \Delta\phi(t),\psi(t)\rangle dt\\
&\geq -\frac{1}{2}|\nabla\psi_1(s)|^2 - 2\int_0^s \frac{|k'(t)|}{k^3(t)}|\nabla\psi_1(t)|^2dt - \frac{2}{k_0^3}|k'|_{L^1(0,\infty)}|\nabla\psi_1(s)|^2
\end{aligned}
\tag{4.9}
$$

The third term of (4.7) is modified as

$$
\begin{aligned}
-\int_0^s \frac{1}{k^2(t)}\langle \Delta\phi_t(t),\psi(t)\rangle dt &= -\int_0^s (\nabla\phi(t),\big[\frac{1}{k^2(t)}\nabla\psi(t)\big]')\\
&= -\int_0^s \frac{1}{k^2(t)}|\nabla\phi(t)|^2dt - 2\int_0^s \frac{k'(t)}{k^3(t)}(\phi(t),\Delta\psi(t))dt.
\end{aligned}
$$

The last integral above is the same that the fifth term of (4.7) with positive sign. Thus, we have

$$
-\int_0^s \frac{1}{k^2(t)}\langle \Delta\phi_t(t),\psi(t)\rangle dt + 2\int_0^s \frac{k'(t)}{k^3(t)}\langle \Delta\phi(t),\psi(t)\rangle dt = -\int_0^s \frac{1}{k^2(t)}|\nabla\phi(t)|^2dt.
\tag{4.10}
$$

Now, we estimate the fourth term of (4.7):

$$
\begin{aligned}
\int_0^s \frac{1}{k^4(t)}\langle \Delta^2\phi(t),\psi(t)\rangle dt &= -\frac{1}{2}\left|\Delta\psi_1(s)\right|^2 + 2\int_0^s \frac{k'(t)}{k^5(t)}|\Delta\psi(t)|^2dt\\
&\geq -\frac{1}{2}\left|\Delta\psi_1(s)\right|^2 - \int_0^s \frac{|k'(t)|}{k_0^5}|\Delta\psi_1(t)|^2dt\\
&\quad - \frac{1}{k_0^5}|\Delta\psi_1(s)|^2|k'|_{L^1(0,\infty)}.
\end{aligned}
\tag{4.11}
$$

The sixth term of (4.7) is estimated as

$$
\begin{aligned}
&\Big|\int_0^s \frac{k'(t)}{k^3(t)}\langle\sum_{j=1}^n \Delta(\frac{\partial\phi}{\partial y_j}(t))\,y_j,\psi(t)\rangle dt\Big|_{\mathbb{R}}\\
&= \Big|\int_0^s \frac{k'(t)}{k^3(t)}(\frac{\partial\phi}{\partial y_j}(t), 2\frac{\partial\psi}{\partial y_j}(t) + y_j\Delta\psi(t))dt\Big|_{\mathbb{R}}\\
&\leq 2\int_0^s \frac{|k'(t)|}{k^3(t)}|\nabla\phi(t)||\nabla\psi_1(s)|dt + \int_0^s \frac{|k'(t)|}{k^3(t)}|\nabla\phi(t)|\|y\|_{\mathbb{R}^n}|\Delta\psi(t)|dt\\
&\leq \int_0^s \big[\frac{3\epsilon_1}{2k^2(t)}|\nabla\phi(t)|^2 + \frac{k_2^2}{\epsilon_1 k_0^4}|\nabla\psi_1(t)|^2\big]dt + \frac{k_2}{\epsilon_1 k_0^4}|\nabla\psi_1(s)|^2|k'|_{L^1(0,\infty)}\\
&\quad + \frac{k_2^2C^2}{2\epsilon_1 k_0^4}\int_0^s |\Delta\psi_1(t)|^2dt + \frac{C^2k_2}{2\epsilon_1 k_0^4}|\Delta\psi_1(s)|^2|k'|_{L^1(0,\infty)},
\end{aligned}
\tag{4.12}
$$

where k_2 is a constant that comes from the hypothesis (2.2). That is, $|k'(t)|\leq k_2$.

The seventh term of (4.7) is estimated as

$$
\begin{aligned}
&\big| -2\int_0^s \frac{k'(t)}{k(t)}\big\langle \sum_{j=1}^n \frac{\partial^2\phi}{\partial t\partial y_j}(t)y_j, \psi(t)\big\rangle dt\big|_{\mathbb{R}} \\
&= \big|2\int_0^s \frac{k'(t)}{k(t)}\Big(\sum_{j=1}^n \frac{\partial\phi}{\partial y_j}(t)y_j, \phi(t)\Big)dt\big|_{\mathbb{R}} \\
&= \big|\int_0^s \frac{k'(t)}{k(t)}(\sum_{j=1}^n \frac{\partial}{\partial y_j}[\phi(t)]^2, y_j)dt\big|_{\mathbb{R}} \\
&= \big|\int_0^s \frac{nk'(t)}{k(t)}|\phi(t)|^2 dt\big|_{\mathbb{R}} \\
&\le \frac{nk_2}{k_0}\int_0^s |\phi(t)|^2 dt.
\end{aligned} \tag{4.13}
$$

The eighth term of (4.7) is estimated as

$$
\begin{aligned}
&\Big|\int_0^s (\frac{k'(t)}{k(t)})^2\big\langle \sum_{j,\,l=1}^n \frac{\partial^2\phi}{\partial y_l\partial y_j}(t)y_l y_j, \psi(t)\big\rangle dt\Big|_{\mathbb{R}} \\
&= \Big| -\int_0^s (\frac{k'(t)}{k(t)})^2 \sum_{j,\,l=1}^n \Big(\frac{\partial\phi}{\partial y_l}(t), [\delta_{l\,j}y_j + y_l]\psi(t) + y_l y_j\frac{\partial\psi}{\partial y_j}(t)\Big)dt\Big|_{\mathbb{R}} \\
&\le \int_0^s (\frac{k'(t)}{k(t)})^2 \sum_{j,\,l=1}^n \big[2\big|\frac{\partial\phi}{\partial y_l}(t)\big|d(\Omega)|\psi(t)| + \big|\frac{\partial\phi}{\partial y_l}(t)\big|[d(\Omega)]^2\big|\frac{\partial\psi}{\partial y_j}(t)\big|\big]dt \\
&\le \int_0^s (\frac{k'(t)}{k(t)})^2\big[2n\sqrt{n}C|\nabla\phi(t)||\psi(t)| + nC^2|\nabla\phi(t)||\nabla\psi(t)|\big]dt \\
&\le \epsilon_2\int_0^s \frac{1}{k^2(t)}|\nabla\phi(t)|^2 dt + \frac{2n^3C^2k_2^4}{\epsilon_2}\int_0^s \frac{1}{k^2(t)}|\psi_1(t)|^2 dt \\
&\quad + \frac{2n^2C^2k_2^3}{\epsilon_2 k_0^2}|\psi_1(s)|^2|k'|_{L^1(0,\infty)} + \frac{n^2C^4k_2^4}{\epsilon_2}\int_0^s \frac{1}{k^2(t)}|\nabla\psi_1(t)|^2 dt \\
&\quad + \frac{n^2C^4k_2^3}{2\epsilon_2 k_0^2}|\nabla\psi_1(s)|^2|k'|_{L^1(0,\infty)}.
\end{aligned} \tag{4.14}
$$

The ninth term of (4.7) is estimated as

$$
\begin{aligned}
&\Big|\int_0^s [\frac{2(k'(t))^2 - k(t)k''(t)}{k^2(t)}]\big\langle \sum_{j=1}^n \frac{\partial\phi}{\partial y_j}(t)y_j, \psi(t)\big\rangle dt\Big|_{\mathbb{R}} \\
&= \Big|\int_0^s [-\frac{2(k'(t))^2 - k(t)k''(t)}{k^2(t)}]\sum_{j=1}^n \Big(\phi(t), \psi(t) + y_j\frac{\partial\psi}{\partial y_j}(t)\Big)dt\Big|_{\mathbb{R}} \\
&\le \int_0^s \big[\frac{(2k_2+k_1)}{k^2(t)}(|k'(t)| + |k''(t)|)\big][n|\phi(t)||\psi(t)| + d(\Omega)|\phi(t)||\nabla\psi(t)|]dt \\
&\le \big[\frac{n^2(2k_2+k_1)^2}{k_0^4} + \frac{C^2(2k_2+k_1)^2}{4k_0^4}\big]\int_0^s |\phi(t)|^2 dt \\
&\quad + 2(C^2+1)\int_0^s (|k'(t)| + |k''(t)|)^2|\nabla\psi_1(t)|^2 dt
\end{aligned} \tag{4.15}
$$

$$+ 4(C^2+1)(k_2+k_3)|\nabla\psi_1(s)|^2(|k'|_{L^1(0,\infty)}+|k''|_{L^1(0,\infty)}).$$

where k_3 is a constant due to (2.2) defined by $|k''(t)|\le k_3$.

The last term of (4.7) is estimated as

$$\begin{aligned}
&\Big|-\int_0^s \frac{1}{k^2(t)}\langle(\Delta v^2(t)-\Delta\widehat{v}^2(t)),\psi(t)\rangle dt\Big|_{\mathbb{R}}\\
&=\Big|-\int_0^s \frac{1}{k^2(t)}([v(t)+\widehat{v}(t)]\phi(t),\Delta\psi(t))dt\Big|_{\mathbb{R}}\\
&\le C^2\Big(\|v\|_{L^\infty(0,T;H_0^1(\Omega))}+\|\widehat{v}\|_{L^\infty(0,T;H_0^1(\Omega))}\Big)\int_0^s\frac{1}{k^2(t)}|\nabla\phi(t)||\Delta\psi(t)|dt\\
&\le \epsilon_3\int_0^s\frac{1}{k^2(t)}|\nabla\phi(t)|^2dt\\
&\quad+\frac{C^4\Big(\|v\|_{L^\infty(0,T;H_0^1(\Omega))}+\|\widehat{v}\|_{L^\infty(0,T;H_0^1(\Omega))}\Big)^2}{2\epsilon_3k_0^2}\int_0^s|\Delta\psi_1(t)|^2dt\\
&\quad+\frac{C^4\Big(\|v\|_{L^\infty(0,T;H_0^1(\Omega))}+\|\widehat{v}\|_{L^\infty(0,T;H_0^1(\Omega))}\Big)^2s}{2\epsilon_3k_0^2}|\Delta\psi_1(s)|^2.
\end{aligned}\tag{4.16}$$

Inserting (4.8)-(4.16) in (4.7), we have

$$\begin{aligned}
&\frac{1}{2}|\phi(s)|^2+\big[\frac{1}{2k_1^2}-K_1[|k'|_{L^1(0,\infty)}+|k''|_{L^1(0,\infty)}]\big]|\nabla\psi_1(s)|^2\\
&+[\frac{1}{2k_1^4}-K_2|k'|_{L^1(0,\infty)}-K_3s]|\Delta\psi_1(s)|^2\\
&+[1-(2\epsilon_1+2\epsilon_2+\epsilon_3)]\int_0^s\frac{1}{k^2(t)}|\nabla\phi(t)|^2dt\\
&\le K_4\int_0^s[|\phi(t)|^2+|\nabla\psi_1(t)|^2+|\Delta\psi_1(t)|^2]dt,
\end{aligned}\tag{4.17}$$

where

$$\begin{gathered}
K_1=\frac{2}{k_0^3}+\frac{k_2}{\epsilon_1k_0^4}+\frac{2n^2C^3k_2^3}{\epsilon_2k_0^2}+\frac{n^2C^4k_2^3}{2\epsilon_2k_0^2}+4(C^2+1)(k_2+k_3);\\
K_2=\frac{1}{k_0^5}+\frac{C^2k_2}{2\epsilon_1k_0^4};\\
K_3=\frac{C^4}{2\epsilon_3k_0^2}\Big(\|v\|_{L^\infty(0,T;H_0^1(\Omega))}+\|\widehat{v}\|_{L^\infty(0,T;H_0^1(\Omega))}\Big)^2;\\
K_4=\frac{4k_2}{k_0^3}+\frac{k_2}{k_0^5}+\frac{3\epsilon_1}{2k_0^2}+\frac{k_2^2}{\epsilon_1k_0^4}+\frac{k_2^2C^2}{2\epsilon_1k_0^4}+\frac{nk_2}{k_0}+\frac{\epsilon_2}{k_0^2}+\frac{2n^3C^3k_2^4}{\epsilon_2}+\frac{n^2C^4k_2^4}{\epsilon_2k_0^2}\\
+2(C^2+1)+\frac{n^2(2k_2+k_1)^2}{k_0^4}+\frac{C^2(2k_2+k_1)^2}{4k_0^4}+\frac{\epsilon_3}{k_0^2}+K_3.
\end{gathered}\tag{4.18}$$

Thus, if $\epsilon_1,\epsilon_2,\epsilon_3$, are taking such that $2\epsilon_1+2\epsilon_2+\epsilon_3\le 1/2$, and k',k'', satisfying the hypothesis (2.13), we get

$$\frac{1}{2}|\phi(s)|^2+\frac{1}{4}|\nabla\psi_1(s)|^2+(\frac{1}{4}-K_3s)|\Delta\psi_1(s)|^2+\frac{1}{2}\int_0^s\frac{1}{k^2(t)}|\nabla\phi(t)|^2dt$$

$$\leq K_4 \int_0^s \Big[|\phi(s)|^2 + |\nabla\psi_1(s)|^2 + |\Delta\psi_1(s)|^2\Big] dt.$$

Now, if $s \leq T_0 = 1/8K_3$, then

$$|\phi(s)|^2 + |\nabla\psi_1(s)|^2 + |\Delta\psi_1(s)|^2 \leq 8K_4 \int_0^s \Big[|\phi(s)|^2 + |\nabla\psi_1(s)|^2 + |\Delta\psi_1(s)|^2\Big] dt.$$

From this and Gronwall's inequality we find $\phi(x, s) = 0$ a. e. for all $s \in [0, T_0]$. This gives the uniqueness of solutions over the interval $[0, T_0]$.

The task now is to show the uniqueness of the solutions over the interval $[T_0, 2T_0]$. In fact, being the solutions unique on $[0, T_0]$ we obtain from (4.3) that

$$\phi(y, T_0) = \phi_t(y, T_0) = 0 \quad \text{for } y \in \Omega.$$

Now, we consider in (4.4) the variable $s \in (T_0, T)$ and take $\psi_1(y, t) = \int_{T_0}^t \phi(y, r) dr$. Thus, we get

$$\psi(y, t) = \psi_1(y, t) - \psi_1(y, s) \quad \text{and} \quad \psi(y, T_0) = -\psi_1(y, s).$$

Repeating the steps (4.7)-(4.17) on the whole interval $(T_0, T) \times \Omega$ we get

$$\frac{1}{2}|\phi(s)|^2 + \frac{1}{4}|\nabla\psi_1(s)|^2 + \Big(\frac{1}{4} - K_3(s - T_0)\Big)|\Delta\psi_1(s)|^2 + \frac{1}{2}\int_{T_0}^s \frac{1}{k^2(t)}|\nabla\phi(t)|^2 dt$$
$$\leq K_4 \int_{T_0}^s \Big[|\phi(s)|^2 + |\nabla\psi_1(s)|^2 + |\Delta\psi_1(s)|^2\Big] dt.$$

Now, if $s \leq 2T_0 = 1/4K_3$ then we will have

$$|\phi(s)|^2 + |\nabla\psi_1(s)|^2 + |\Delta\psi_1(s)|^2 \leq 8K_4 \int_0^s \Big[|\phi(s)|^2 + |\nabla\psi_1(s)|^2 + |\Delta\psi_1(s)|^2\Big] dt.$$

From this and Gronwall's inequality we have $\phi(x, s) = 0$ a. e. for all $s \in [T_0, 2T_0]$. This gives the uniqueness of solutions over the interval $[0, 2T_0]$. Repeating the precedent process N times until that $NT_0 \geq T$ we will obtain the uniqueness of solutions over the interval $[0, T]$, and thus the proof of the Theorem 2.2 is complete

References

[1] Biler, P.; *Time decay of solutions of semilinear strongly damped generalized wave equations*, Math. Methods Appl. Sci., 14 (1991), pp. 427-443.

[2] Bona, J.; Sachs, R.; *Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation*, Commun. Math. Phys., 118 (1988), pp. 15-29.

[3] Boussinesq, M. J.; *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses senciblement pareilles de la surface au fond*, Journal de Mathématiques Pures et Appliqueés, Vol. 17 (1872), pp. 55-108.

[4] Boussinesq, M. J.; *Essai sur la théorie des eaux courantes*, Mém. Présentés par divers savants à l'Académie des Sciences Inst. France (séries 2), Vol. 17 (1877), pp. 1-680.

[5] Clark, H. R.; Cousin, A. T.; Frota, C. L.; Limaco, J.; *On the dissipative Boussinesq equation in a non-cylindrical domain*, Nonlinear Analysis TMA, 67 (2007), pp. 2321-2334.

[6] Ladyzhenskaia, O. A.; Visik, M. I.; *Boundary value problems for partial differential equations and certain classes of operator equations*, AMS Translations Series 2, 10 (1958), pp. 223-281.

[7] Lions, J, L.; *Quelques methodes de resolution des problèmes aux limites non lineaires*, Dunod, Paris (1969).

[8] Lions, J. L.; Magenes, E.; *Problèmes aux limites non homogènes at applications*, Vol. 1, Dunod, Paris (1968).

[9] Liu, F. L.; Russell, D.; *Solutions of the Boussinesq equation on a periodic domain*, J. Math. Anal. Appl., 192 (1995), pp. 194-219.

[10] Miles, J. W.; *Solitary waves*, Ann. Rev. Fluid Mech., 12 (1980), pp. 11-43.

[11] Pereira, D. C.; *Existence, uniqueness and asymptotic behavior for solutions of the nonlinear beam equation*, Nonlinear Analysis TMA, 8, (1990), pp. 613-623.
[12] Tsutsumi M.; Matahshi, T.; *On the Cauchy problem for the Boussinesq-type equation*, Math. Japonica, 36 (1991), pp. 371-379.
[13] Varlamov, V. V.; *On the initial boundary value problem for the damped Boussinesq equation*, Discrete and Continuous Dynamical Systems, Vol. 4, N$^{\underline{0}}$ 3 (1998), pp. 431-444.
[14] Varlamov, V. V.; *Asymptotic behavior of solutions of the damped Boussinesq equation in two space dimensions*, Internat. J. Math. Sciences, 22 (1) (1999), pp. 131-145.
[15] Varlamov, V. V.; *On the damped Boussinesq equation in a circle*, Nonlinear Analysis TMA, 38 (1999), pp. 447-470.
[16] Varlamov, V. V.; *On the spatially two-dimensional Boussinesq equation in a circular domain*, Nonlinear Analysis TMA, 46 (2001), pp. 699-725.

Haroldo R. Clark
Universidade Federal Fluminense, IM-GAN, RJ, Brasil
E-mail address: hclark@vm.uff.br

Alfredo T. Cousin
Universidade Estadual de Maringá, DMA, PR, Brasil
E-mail address: atcousin@uem.br

Cícero Lopes Frota
Universidade Estadual de Maringá, DMA, PR, Brasil
E-mail address: clfrota@uem.br

Juan Límaco
Universidade Federal Fluminense, IM-GMA, RJ, Brasil
E-mail address: jlimaco@vm.uff.br