Electronic Journal of Differential Equations, Vol. 2010(2010), No. 101, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

DEPENDENCE RESULTS ON ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF EVOLUTION EQUATIONS

JOËL BLOT, PHILIPPE CIEUTAT, GASTON M. N'GUÉRÉKATA

Abstract. We consider the semilinear evolution equations $x'(t) = A(t)x(t) + f(x(t), u(t), t)$ and $x'(t) = A(t)x(t) + f(x(t), \zeta, t)$ where $A(t)$ is a unbounded linear operator on a Banach space X and f is a nonlinear operator. We study the dependence of solutions x with respect to the function u in three cases: the continuous almost periodic functions, the differentiable almost periodic functions, and the almost automorphic functions. We give results on the continuous dependence and on the differentiable dependence.

1. Introduction

We consider the differential equations

$$x'(t) = A(t)x(t) + f(x(t), u(t), t), \tag{1.1}$$

$$x'(t) = A(t)x(t) + f(x(t), \zeta, t), \tag{1.2}$$

where $t \in \mathbb{R}$, $A(t)$ is a unbounded linear operator on a Banach space and f is a nonlinear operator. The function u can be seen as a perturbation or as a control term; the term ζ is a general abstract parameter. Our aim is to study the dependence of solutions x of (1.1) with respect to the function u and the dependence of the solutions x of (1.2) with respect to ζ. We consider three classes of functions: the continuous almost periodic functions, the differentiable almost periodic functions, and the almost automorphic functions. In our method we use the following linear inhomogeneous differential equation.

$$x'(t) = A(t)x(t) + b(t). \tag{1.3}$$

In the special case where A is independent of t, $A(t) = A$, the previous equations become the following equation, respectively,

$$x'(t) = Ax(t) + f(x(t), u(t), t), \tag{1.4}$$

$$x'(t) = Ax(t) + f(x(t), \zeta, t), \tag{1.5}$$

$$x'(t) = Ax(t) + b(t). \tag{1.6}$$

2000 *Mathematics Subject Classification.* 47J35, 43A60, 47D06.
Key words and phrases. Semilinear evolution equation; almost periodic function; almost automorphic function; dependence results.

Submitted May 25, 2010. Published July 21, 2010.

Now we describe the contents of this article. In Section 2 we fix our notations, the various spaces of functions considered, and assumptions used later. In Section 3 we establish preliminary results for the linear case. In Section 4 we treat the continuous dependence of the solutions of (1.1) and of (1.4) with respect to u and the continuous dependence of the solutions of (1.2) and of (1.5) with respect to ζ; we use a fixed point theorem to realize that. In Section 5 we treat the differentiable dependence of the solutions of (1.1) and of (1.4) with respect to u and the differentiable dependence of the solutions of (1.2) and of (1.5) with respect to ζ; we use an implicit function theorem to reach our goal.

2. Notation

When X is a Banach space, $AP(X)$ denotes the space of the Bohr almost periodic functions from $\mathbb{R}$ into X, [11], [12, 16, 17, 21, 23]. When n is a non negative integer number, $AP^{(n)}(X)$ denotes the space of the functions in $AP(X)$ which are of class C^n on $\mathbb{R}$ such that the derivative of order k belongs to $AP(X)$ for all k between 0 and n, [3]. $AA(X)$ denotes the space of the Bochner almost automorphic functions from $\mathbb{R}$ into X, [16]. Endowed with the norm of the uniform convergence, $\|x\|_\infty := \sup_{t\in\mathbb{R}} |x(t)|$, $AP(X)$ and $AA(X)$ are Banach spaces. Endowed with the norm

$$\|x\|_{C^n} := \|x\|_\infty + \sum_{1\le k\le n} \|\frac{d^k x}{dt^k}\|_\infty,$$

the space $AP^{(n)}(X)$ is a Banach space.

Definition 2.1 ([21, p. 5-6], [6, p. 45])**.** When X is a Banach space, a continuous mapping $f : Y \times \mathbb{R} \to X$ is so-called almost periodic in t uniformly in y when the following condition holds: for all compact $K \subset Y$ and for all $\varepsilon > 0$, there exists $\ell = \ell(K, \varepsilon) > 0$ such that, for all $r \in \mathbb{R}$, there exists $\tau \in [r, r + \ell]$ satisfying $|f(y, t + \tau) - f(y, t)| \le \varepsilon$ for all $(y, t) \in K \times \mathbb{R}$. We denote by $APU(Y \times \mathbb{R}, X)$ the space of such mappings.

Definition 2.2 ([6, p. 45])**.** A mapping $f : Y \times \mathbb{R} \to X$ is so-called almost automorphic in t uniformly in y when $f(y, .) \in AA(X)$ for all $y \in Y$ and when, for all compact $K \subset Y$ and for all $\varepsilon > 0$, there exists $\delta = \delta(K, \varepsilon) > 0$ satisfying $|f(y, t) - f(z, t)| \le \varepsilon$ for all $t \in \mathbb{R}$ and for all $y, z \in K$ such that $|y - z| \le \delta$. We denote by $AAU(Y \times \mathbb{R}, X)$ the space of such mappings.

About the continuous almost periodic functions we consider the following conditions, where U is a Banach space.

$$f \in APU((X \times U) \times \mathbb{R}, X). \tag{2.1}$$

$$\begin{gathered} f \in APU((X \times U) \times \mathbb{R}, X), \\ \forall(\xi, \zeta, t) \in X \times U \times \mathbb{R}, D_1 f(\xi, \zeta, t) \text{ and } D_2 f(\xi, \zeta, t) \text{ exist }, \\ D_1 f \in APU((X \times U) \times \mathbb{R}, \mathcal{L}(X, X)), \\ D_2 f \in APU((X \times U) \times \mathbb{R}, \mathcal{L}(U, X)), \end{gathered} \tag{2.2}$$

where $D_1 f(\xi, \zeta, t)$ is the differential of $f(., \zeta, t)$, $D_2 f(\xi, \zeta, t)$ is the differential of $f(\xi, ., t)$ and $\mathcal{L}(Y, X)$ denotes the space of the linear bounded mappings from Y into X.

About the differentiable almost periodic functions we consider the following conditions.

$$\begin{gathered} f \in APU((X \times U) \times \mathbb{R}, X) \cap C^n((X \times U) \times \mathbb{R}, X), \\ \forall k = 0, \dots, n, \; D^k f \in APU((X \times U) \times \mathbb{R}, \mathcal{L}_k(((X \times U) \times \mathbb{R})^k, X), \end{gathered} \tag{2.3}$$

where $D^k f$ denotes the differential of order k of f and where $\mathcal{L}_k(Y^k, X)$ denotes the space of the k-linear continuous mappings from Y^k into X.

$$\begin{gathered} f \in APU((X \times U) \times \mathbb{R}, X) \cap C^{n+1}((X \times U) \times \mathbb{R}, X), \\ \forall k = 0, \dots, n+1, \; D^k f \in APU((X \times U) \times \mathbb{R}, \mathcal{L}_k(((X \times U) \times \mathbb{R})^k, X))). \end{gathered} \tag{2.4}$$

About almost automorphic functions we consider the following conditions.

$$f \in AAU((X \times U) \times \mathbb{R}, X). \tag{2.5}$$

$$\begin{gathered} f \in AAU((X \times U) \times \mathbb{R}, X), \\ \forall (\xi, \zeta, t) \in X \times U \times \mathbb{R}, \; D_1 f(\xi, \zeta, t) \text{ and } D_2 f(\xi, \zeta, t) \text{ exist}, \\ D_1 f \in AAU((X \times U) \times \mathbb{R}, \mathcal{L}(X, X)), \\ D_2 f \in AAU((X \times U) \times \mathbb{R}, \mathcal{L}(U, X)). \end{gathered} \tag{2.6}$$

For a linear operator A on X, not necessarily bounded, we denote by $\mathcal{D}(A)$ its domain, by $\varrho(A)$ its resolvent set and by $R(\lambda; A)$ its resolvent operators (cf. [18, p. 8]).

Definition 2.3. A family $(F(t,s))_{t \geq s}$ of bounded linear operators on X is called an evolution family when $F(t,t) = I$ (the identity operator on X) for all $t \in \mathbb{R}$, $F(t,s)F(s,r) = F(t,r)$ for all $t \geq s \geq r$ and $(t,s) \mapsto F(t,s)x$ is continuous for all $x \in X$.

Definition 2.4. We say that the evolution family $(F(t,s))_{t \geq s}$ in $\mathcal{L}(X,X)$ is exponentially stable when there exist $c > 0$ and $\omega > 0$ such that $\|F(t,s)\| \leq c \cdot e^{-\omega(t-s)}$ for all $t \geq s$.

For all $t \in \mathbb{R}$, let $A(t) : \mathcal{D}(A(t)) \subset X \to X$ be a unbounded linear operator.

Definition 2.5 ([2], [3, p. 269])**.** *We say that $(A(t))_t$ satisfies the Acquistapace-Terrini conditions when there exist $\lambda_0 \geq 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $L \geq 0$, $K \geq 0$, $\alpha \in (0,1]$, $\beta \in (0,1]$, such that $\alpha + \beta > 1$, satisfying $\Sigma_\theta \cup \{0\} \subset \varrho(A(t) - \lambda_0 I)$ (where $\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$) for all $t \in \mathbb{R}$, $\|R(\lambda; A(t) - \lambda_0 I)\| \leq \frac{K}{1+|\lambda|}$ for all $t \in \mathbb{R}$, and $\|(A(t) - \lambda_0 I) R(\lambda; A(t) - \lambda_0 I)[R(\lambda_0; A(t)) - R(\lambda_0; A(s))]\| \leq L|t-s|^\alpha |\lambda|^{-\beta}$ for all $t, s \in \mathbb{R}$, for all $\lambda \in \Sigma_\theta$.*

Remark 2.6. Under these Acquistapace-Terrini conditions, the family $(A(t))_t$ generates a unique evolution family $(F(t,s))_{t \geq s}$ in $\mathcal{L}(X,X)$ such that, for all $s \in \mathbb{R}$ and for all $x_0 \in \overline{\mathcal{D}(A(s))}$, the function $t \mapsto F(t,s)x_0$ is continuous at $t = s$ and it is the unique solution in $C([s,\infty), X) \cap C^1((s,\infty), X)$ of the following Cauchy problem: $x'(t) = A(t)x(t)$ for $t > s$ and $x(s) = x_0$ (cf. [2]).

We consider the following condition.

$(A(t))_t$ satisfies the Acquistapace-Terrini conditions $R(\lambda_0, A(.)) \in AP(\mathcal{L}(X,X))$ for λ_0 given in Definition 2.5 and the evolution family $(F(t,s))_{t \geq s}$ is exponentially stable. (2.7)

We also consider the following condition which are the assumptions [9, (A1)-(A2)].

Definition 2.7. We say that $(A(t))_t$ satisfies the Ding-Long-N'Guérékata conditions when $(A(t))_t$ generates an evolution family $(F(t,s))_{t\geq s}$ and there exists $P \in C(\mathbb{R}, \mathcal{L}(X,X))$ such that $P(t)$ is a projection for all $t \in \mathbb{R}$, there exist $c \geq 0$, $\omega > 0$ such that $F(t,s)P(s) = P(t)F(t,s)$ for all $t \geq s$, and denoting $Q := I - P$, the restriction $F_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$, and $\|F(t,s)P(s)\| \leq c.e^{-\omega.(t-s)}$, $\|F_Q(t,s)Q(t)\| \leq c.e^{-\omega.(t-s)}$ for all $t \geq s$. Setting $\Gamma(t,s) := F(t,s)P(s)$ when $t \geq s$ and $\Gamma(t,s) := -F_Q(t,s)Q(s)$ when $t < s$, for all real sequence $(s'_m)_m$, there exists a subsequence $(s_m)_m$ of $(s'_m)_m$ such that $\Lambda(t,s)x := \lim_{m\to\infty} \Gamma(t+s_m, s+s_m)x$ is well defined for all $x \in X$ and for all $t,s \in \mathbb{R}$, and moreover $\lim_{m\to\infty} \Lambda(t-s_m, s-s_m)x = \Gamma(t,s)x$ for all $x \in X$ and for all $t,s \in \mathbb{R}$.

Note that

$$(A(t))_t \text{ satisfies the Ding-Long-N'Guérékata conditions.} \tag{2.8}$$

In the special case where $A(t) = A$ is constant with respect to t, we consider the following notion, see [16, p. 56].

Definition 2.8. We say that the linear unbounded operator $A : \mathcal{D}(A) \subset X \to X$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ in $\mathcal{L}(X,X)$ which is exponentially stable when there exist $c > 0$ and $\omega > 0$ such that $\|T(t)\| \leq c \cdot e^{-\omega t}$ for all $t \geq 0$.

Note that

$$A : \mathcal{D}(A) \subset X \to X \text{ generates an exponentially stable } C_0\text{-semigroup.} \tag{2.9}$$

Definition 2.9 ([18, pp. 106, 146, 184])**.** When $x : \mathbb{R} \to X$ is a continuous function, x is so-called a mild solution of (1.1) (respectively of (1.3) respectively of (1.2), respectively of (1.4), respectively of (1.6), respectively of (1.5)) when the following condition holds for all $t \geq s$:
$x(t) = F(t,s)x(s) + \int_s^t F(t,r)f(x(r),u(r),r)dr$ (respectively $x(t) = F(t,s)x(s) + \int_s^t F(t,r)b(r)dr$, respectively $x(t) = F(t,s)x(s) + \int_s^t F(t,r)f(x(r),\zeta,r)dr$, respectively $x(t) = T(t-s)x(s) + \int_s^t T(t-r)f(x(r),u(r),r)dr$, respectively $x(t) = T(t-s)x(s) + \int_s^t T(t-r)b(r)dr$, respectively $x(t) = T(t-s)x(s) + \int_s^t T(t-r)f(x(r),\zeta,r)dr$).

Definition 2.10 ([18, pp. 105, 146, 184])**.** A function $x : \mathbb{R} \to X$ is so-called a classical solution of (1.4) (respectively of (1.6), respectively of (1.5))) if x is continuously differentiable on $\mathbb{R}$, $x(t) \in \mathcal{D}(A)$ for all $t \in \mathbb{R}$, and (1.4) (respectively of (1.6), respectively of (1.5))) is satisfied on $\mathbb{R}$.

3. The linear case

About the linear equations, we consider the following conditions.

$$\text{For all } b \in AP(X), \text{ (1.3) has a unique mild solution in } AP(X), \tag{3.1}$$

$$\text{For all } b \in AP^{(n)}, \text{ (1.3) has a unique mild solution in } AP^{(n)}(X), \tag{3.2}$$

$$\text{For all } b \in AA(X), \text{ (1.3) has a unique mild solution in } AA(X). \tag{3.3}$$

In [3, Theorem 3.6] it is shown that (3.1) and (3.2) are fulfilled when (2.7) is satisfied. In [9, Theorem 2.2] it is shown that (3.3) is fulfilled when (2.8) is satisfied.

Theorem 3.1. *Under* (2.7) *(respectively under* (2.8)*) we define the operators* $T_{ap}: AP(X) \to AP(X)$ *and* $T_{apn}: AP^{(n)}(X) \to AP^{(n)}(X)$ *(respectively* $T_{aa}: AA(X) \to AA(X)$*) in the following way:* $T_{ap}(b)$ *(respectively* $T_{apn}(b)$*, respectively* $T_{aa}(b)$*) is the unique mild solution of* (1.3) *in* $AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*) for all* $b \in AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*). Then* T_{ap}*,* T_{apn} *and* T_{aa} *are linear bounded operators.*

Proof. The conditions (3.1)-(3.3) ensure that the operators T_{ap}, T_{apn} and T_{aa} are well defined and their linearity is clear. We prove that the graph of T_{ap}, $\mathcal{G}(T_{ap})$, is closed in $AP(X) \times AP(X)$. Let $(b_m, x_m)_m$ be a sequence in $\mathcal{G}(T_{ap})$ which (uniformly) converges to $(b, x) \in AP(X) \times AP(X)$. And so, for all $m \in \mathbb{N}$, and for $t \geq s$, the following equality holds.

$$x_m(t) = F(t,s)x_m(s) + \int_s^t F(t,r)b_m(r)dr.$$

Since the uniform convergence implies the pointwise converge, $\lim_{m\to\infty} x_m(t) = x(t)$ and $\lim_{m\to\infty} x_m(s) = x(s)$. Since $F(t,s)$ is a bounded linear operator, we have $\lim_{m\to\infty} F(t,s)x_m(s) = F(t,s)x(s)$. Note that, for all $r \in [s,t]$, we have

$$\begin{aligned}|F(t,r)b_m(r) - F(t,r)b(r)| &\leq \|F(t,r)\|.|b_m(r) - b(r)| \\ &\leq ce^{-\omega(t-s)}\|b_m - b\|_\infty \leq c\|b_m - b\|_\infty,\end{aligned}$$

and consequently we obtain that the sequence $(F(t,.)b_m)_m$ converges uniformly to $F(t,.)b$ on $[s,t]$, and then, [8], we have

$$\lim_{m\to\infty} \int_s^t F(t,r)b_m(r)dr = \int_s^t F(t,r)b(r)dr.$$

Then, when $m \to \infty$, we obtain the equality

$$x(t) = F(t,s)x(s) + \int_s^t F(t,r)b(r)dr$$

for all $t \geq s$. This proves that $(b,x) \in \mathcal{G}(T_{ap})$. Since T_{ap} is closed and since $\mathcal{D}(T_{ap}) = AP(X)$, by using the Closed Graph Theorem (Theorem II.1.9 in [10, p. 45]), we deduce that T_{ap} is bounded. The reasoning is similar for T_{apn} and T_{aa}. □

Now we treat the autonomous case. We need some lemmas. We consider the following conditions.

$$\text{For all } b \in AP(X),\ (1.6) \text{ has a unique mild solution in } AP(X), \tag{3.4}$$

$$\text{For all } b \in AP^{(n)},\ (1.6) \text{ has a unique mild solution in } AP^{(n)}(X), \tag{3.5}$$

$$\text{For all } b \in AA(X),\ (1.6) \text{ has a unique mild solution in } AA(X). \tag{3.6}$$

Lemma 3.2 ([22, p. 332])**.** *Under assumption* (2.9)*, if* $b : \mathbb{R} \to X$ *is bounded and continuous, if* $x_1 : \mathbb{R} \to X$ *and* $x_2 : \mathbb{R} \to X$ *are bounded continuous mild solutions of* (1.6) *then we have* $x_1 = x_2$*.*

Lemma 3.3. *Under assumption* (2.9)*, for all* $b \in AP^{(n)}(X)$ *there exists a unique mild solution of* (1.6) *in* $AP^{(n)}(X)$*. And moreover, for* $n \geq 1$*, the mild solution is a classical solution.*

Proof. The uniqueness is a consequence of Lemma 3.2. To prove the existence, we consider the function $x : \mathbb{R} \to X$ defined by $x(t) := \int_{-\infty}^{t} T(t-r)b(r)dr$ for all $t \in \mathbb{R}$. First, we note that $|T(t-r)b(r) - T(t-s)b(s)| \leq |T(t-r)(b(r)-b(s))| + |(T(t-r) - T(t-s))b(s)| \leq c \cdot e^{-\omega.(t-r)}|b(r)-b(s)| + |(T(t-r)-T(t-s))b(s)|$; when $r \to s$ the first term converges to zero by using the continuity of b, and the second term converges to zero by using [18, Corollary 2.3]. And so the function $r \mapsto T(t-r)b(r)$ is continuous on $(-\infty, t]$, and consequently it is Lebesgue measurable.

We note that $|T(t-r)b(r)| \leq c.e^{-\omega.(t-r)}\|b\|_\infty$ for all $r \in (-\infty, t]$. It is well-known that the function $r \mapsto ce^{-\omega.(t-r)}$ is Lebesgue integrable on $(-\infty, t]$ and consequently $r \mapsto T(t-r)b(r)$ is Lebesgue integrable on $(-\infty, t]$, see [14, Proposition 2.4.8]. And so the function x is well defined on $\mathbb{R}$. By using the change of variable formula, [14, Proposition 8.4.10], since $r \mapsto t-r$ is a C^1-diffeomorphism from $(-\infty, t)$ on $(0, \infty)$, we obtain $x(t) = \int_0^\infty T(s)b(t-s)ds$.

Reasoning as at the beginning of this proof we verify that the function $s \mapsto T(s)b(t-s)$ is continuous on $\mathbb{R}_+$, and therefore it is Lebesgue measurable on $\mathbb{R}_+$. Since it is well-known that the function $s \mapsto ce^{-\omega s}\|b\|_\infty$ is Lebesgue integrable on $\mathbb{R}_+$, and since the inequality $|T(s)b(t-s)| \leq c \cdot e^{-\omega s}\|b\|_\infty$ holds when $s \in \mathbb{R}_+$, we can use the first part of [14, Proposition 2.4.10] that permits us to say that x is continuous on $\mathbb{R}$.

From the last formula of x it is easy to obtain the inequalities $|x(t+\tau) - x(t)| \leq \int_0^\infty \|T(s)\|.|b(t+\tau-s) - b(t-s)|ds \leq \int_0^\infty c \cdot e^{-\omega s}|b(t+\tau-s) - b(t-s)|ds$ from which we easily verify that $x \in AP(X)$ by using the definition of the Bohr almost periodicity or the Bochner criterion, [12, p. 4.]. When $b \in AP^{(n)}$ with $n \geq 1$, since $T(s)$ is bounded and consequently it is differentiable, and so the function $t \mapsto T(s)b(t-s)$ is of class C^1 on $\mathbb{R}$, and its derivative satisfies the inequality $|T(s)b'(t-s)| \leq ce^{-\omega s}\|b'\|_\infty$ where the function $s \mapsto ce^{-\omega s}\|b'\|_\infty$ is Lebesgue integrable on $\mathbb{R}$, that permits us to use the second part of [14, Proposition 2.4.10], and then to say that the function x is differentiable on $\mathbb{R}$, and that its derivative is $x'(t) = \int_0^\infty T(s)b'(t-s)ds$ for all $t \in \mathbb{R}$. From the inequality $|x'(t+\tau) - x'(t)| \leq \int_0^\infty ce^{-\omega s}|b'(t+\tau-s) - b'(t-s)|ds$ it is easy to see that $x' \in AP(X)$ when $b' \in AP(X)$. Iterating this reasoning we obtain that $x^{(k)} \in AP(X)$ when $b^{(k)} \in AP(X)$ for all $k = 1, \dots, n$. And so we obtain $x \in AP^{(n)}(X)$ when $b \in AP^{(n)}(X)$.

To verify that x is a mild solution of (1.6), the reasoning is similar to this one given in [22]. To prove that the mild solution is a classical solution when $n \geq 1$, it remains to prove that $x(t) \in \mathcal{D}(A)$ and x satisfies (1.6) when $t \in \mathbb{R}$. Recall that the mild solution x of (1.6) is given by $x(t) = \int_{-\infty}^{t} T(t-r)b(r)dr$. It is easy to verify the following equality, for $h > 0$:

$$\frac{T(h)x(t) - x(t)}{h} = \frac{x(t+h) - x(t)}{h} - \frac{1}{h}\int_t^{t+h} T(t+h-r)b(r)dr. \tag{3.7}$$

From the continuity of b it is clear that the second term of the right-hand of (3.7) has the limit $b(t)$ when $h \to 0$. Since x is differentiable on $\mathbb{R}$, it follows from (3.7) that $x(t) \in \mathcal{D}(A)$ and $Ax(t) = x'(t) - b(t)$ for all $t \in \mathbb{R}$; consequently x is a classical solution of (1.6). ◻

Remark 3.4. The proof of Lemma 3.3 is an extension at the cases $n \geq 1$ of the proof of a theorem in [22] done when $n = 0$. This proof in [22] is itself an extension of the proof of the Neugebauer-Bohr theorem, for the finite-dimensional systems, for instance given in [19] p. 206-207. In [22] or in [19], the authors use the Riemann improper integral; in the previous proof we only use the Lebesgue integral.

Remark 3.5. When $A(t) = A$ is constant with respect to t, the condition (2.7) is reduced to the following condition:

$$\begin{gathered}\exists\lambda_0 \in \mathbb{R}_+, \exists\theta \in (\frac{\pi}{2}, \pi), \exists K \in \mathbb{R}_+,\\ \Sigma_\theta \cup \{0\} \subset \rho(A - \lambda_0 I),\\ \forall\lambda \in \Sigma_\theta, \|R(\lambda + \lambda_0; A)\| \le \frac{K}{1 + |\lambda|}.\end{gathered} \tag{3.8}$$

If an infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfies this last condition, then $(T(t))_{t\geq 0}$ is differentiable (and even it can be extended to an analytic semigroup), see [18, Theorem 5.2]; therefore condition (3.8) is not a consequence of (2.9) and it is not necessary to obtain the conclusion of Lemma 3.3.

Lemma 3.6 ([16, Theorem 2.17], [15, Theorem 3.1])**.** *Under assumption* (2.9)*, for all* $b \in AA(X)$ *there exists a unique mild solution of* (1.6)*.*

Theorem 3.7. *Under assumption* (2.9) *we can define the operators* T^c_{ap}*,* T^c_{apn}*, and* T^c_{aa} *as follows: for all* $b \in AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*)* $T^c_{ap}(b)$ *(respectively* $T^c_{apn}(b)$*, respectively* $T^c_{aa}(b)$*) is the unique mild solution of* (1.6) *in* $AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*). Then* $T^c_{ap} : AP(X) \to AP(X)$*,* $T^c_{apn} : AP^{(n)}(X) \to AP^{(n)}(X)$ *and* $T^c_{aa} : AA(X) \to AA(X)$ *are linear and bounded.*

Proof. Theorem in [22] ensures that (2.9) implies (3.4). Lemma 3.3 ensures that (2.9) implies (3.5). Lemma 3.6 ensures that (2.9) implies (3.6). And so the three operators T^c_{ap}, T^c_{apn}, and T^c_{aa} are well defined. The rest of the proof is similar to this one of Theorem 3.1. □

Notation. $\|T_{ap}\|_{\mathcal{L}}$ (respectively $\|T_{apn}\|_{\mathcal{L}}$, respectively $\|T_{aa}\|_{\mathcal{L}}$, respectively $\|T^c_{ap}\|_{\mathcal{L}}$, respectively $\|T^c_{apn}\|_{\mathcal{L}}$, respectively $\|T^c_{aa}\|_{\mathcal{L}}$) denotes the norm of the linear bounded operator T_{ap} (respectively T_{apn}, respectively T_{aa}, respectively T^c_{ap}, respectively T^c_{apn}, respectively T^c_{aa}).

4. The continuous dependence

4.1. **Solutions of equations** (1.1) **and** (1.2)**.** First we formulate the following conditions:

$$\begin{gathered}\exists c_{ap} \in (0, \|T_{ap}\|_{\mathcal{L}^{-1}}), \forall t \in \mathbb{R}, \forall\xi, \xi_1 \in X, \forall\zeta \in U,\\ |f(\xi, \zeta, t) - f(\xi_1, \zeta, t)| \le c_{ap}|\xi - \xi_1|.\end{gathered} \tag{4.1}$$

$$\begin{gathered}\exists c_{apn} \in (0, \|T_{apn}\|_{\mathcal{L}^{-1}}), \forall t \in \mathbb{R}, \forall\xi, \xi_1 \in X, \forall\zeta \in U,\\ |f(\xi, \zeta, t) - f(\xi_1, \zeta, t)| \le c_{apn}|\xi - \xi_1|.\end{gathered} \tag{4.2}$$

$$\begin{gathered}\exists c_{aa} \in (0, \|T_{aa}\|_{\mathcal{L}^{-1}}), \forall t \in \mathbb{R}, \forall\xi, \xi_1 \in X, \forall\zeta \in U,\\ |f(\xi, \zeta, t) - f(\xi_1, \zeta, t)| \le c_{aa}|\xi - \xi_1|.\end{gathered} \tag{4.3}$$

We recall the following parametrized fixed point theorem.

Theorem 4.1 ([20, Théorème 46-bis]). *Let (Z, d) be a complete metric space and let W be a topological space. Let $\Phi : Z \times W \to Z$ be a mapping such that the partial mappings $w \mapsto \Phi(z, w)$ are continuous for all $z \in Z$, and such that there exists $c \in [0, 1)$ satisfying $d(\Phi(z, w), \Phi(z_1, w)) \le c.d(z, z_1)$ for all $z, z_1 \in Z$ and for all $w \in W$. Then, for all $w \in W$, there exists a unique $\underline{z}(w) \in Z$ such that $\Phi(\underline{z}(w), w) = \underline{z}(w)$, and moreover the mapping $w \mapsto \underline{z}(w)$ is continuous from W into Z.*

Now we state the result on the continuous dependence for (1.1).

Theorem 4.2. *Under assumptions* (2.1), (2.7) *and* (4.1) *(respectively* (2.3), (2.7) *and* (4.2), *respectively* (2.5), (2.8) *and* (4.3)*), for all $u \in AP(U)$ (respectively $AP^{(n)}(U)$, respectively $AA(U)$) there exists a unique $\underline{x}(u) \in AP(X)$ (respectively $AP^{(n)}(X)$, respectively $AA(X)$) which is a mild solution of* (1.1)*. Moreover the mapping $u \mapsto \underline{x}(u)$ is continuous from $AP(U)$ (respectively $AP^{(n)}(U)$, respectively $AA(U)$) into $AP(X)$ (respectively $AP^{(n)}(X)$, respectively $AA(X)$).*

Proof. First we treat the case of the continuous almost periodic functions. When $u \in AP(U)$, note that $x \in AP(X)$ is a mild solution of (1.1) if and only if we have $x = T_{ap} \circ \mathcal{N}_f(x, u)$, where $\mathcal{N}_f : AP(X) \times AP(U) \to AP(X)$ is the superposition operator (or the Nemytskii operator) built on f; i.e., $\mathcal{N}_f(x, u) := [t \mapsto f(x(t), u(t), t)]$. By using [6, Lemma 3.4] we know that $\mathcal{N}_f$ is well defined. From (4.1) it is easy to verify that we have $\|\mathcal{N}_f(x, u) - \mathcal{N}_f(x_1, u)\|_\infty \le c_{ap}\|x - x_1\|_\infty$ for all $x, x_1 \in AP(X)$ and for all $u \in AP(U)$.

We set $\Phi_{ap} := T_{ap} \circ \mathcal{N}_f : AP(X) \times AP(U) \to AP(X)$. For all $x, x_1 \in AP(X)$ and for all $u \in AP(U)$, we have

$$\|\Phi_{ap}(x, u) - \Phi_{ap}(x_1, u)\|_\infty \le \|T_{ap}\|_{\mathcal{L}} c_{ap} \|x - x_1\|_\infty = d_{ap}\|x - x_1\|_\infty,$$

where $d_{ap} := \|T_{ap}\|_{\mathcal{L}} c_{ap} \in [0, 1)$. Moreover, by using Theorem 3.5 in [6] we know that $\mathcal{N}_f$ is continuous and consequently the partial mapping $u \mapsto \Phi_{ap}(x, u)$ is continuous (as a composition of continuous mappings) on $AP(U)$ for all $x \in AP(X)$. And so we can use Theorem 4.1 and we obtain the announced result for the continuous almost periodic case. For the mild solution in $AP^{(n)}(X)$ (respectively $AA(X)$), the reasoning is similar by using Theorem 7.2 (respectively [6, Lemma 9.4 and Theorem 9.6]) instead of [6, Lemma 3.4 and Theorem 3.5]. □

Now we establish the theorem on the continuous dependence for (1.2).

Theorem 4.3. *Under assumptions* (2.1), (2.7) *and* (4.1) *(respectively* (2.3), (2.7) *and* (4.2), *respectively* (2.5), (2.8) *and* (4.3)*), for all $\zeta \in U$ there exists a unique $\underline{x}(\zeta) \in AP(X)$ (respectively $AP^{(n)}(X)$, respectively $AA(X)$) which is a mild solution of* (1.2)*. Moreover the mapping $\zeta \mapsto \underline{x}(\zeta)$ is continuous from U into $AP(X)$ (respectively $AP^{(n)}(X)$, respectively $AA(X)$).*

Proof. Let ϕ be the operator from $AP(U)$ into $AP(X)$ defined as follows: $\phi(u)$ is the unique mild solution of (1.1) in $AP(X)$ provided by Theorem 4.2. By using Theorem 4.2 we obtain that ϕ is well defined and continuous. We consider U as the Banach subspace of the constant functions in $AP(U)$. And so we define the operator $\psi : U \to AP(X)$ as the restriction of ϕ at U. Then $\psi(\zeta)$ is the unique mild solution of (1.2) in $AP(X)$ and ψ is continuous. The reasoning is similar for the other cases. □

4.2. **Solutions of equations** (1.4) **and** (1.5). When $A(t) = A$ does not depend on t, we consider the following conditions.

$$\begin{gathered}\exists c^1_{ap} \in (0, \|T^c_{ap}\|_{\mathcal{L}^{-1}}), \forall t \in \mathbb{R}, \forall \xi, \xi_1 \in X, \forall \zeta \in U, \\ |f(\xi,\zeta,t) - f(\xi_1,\zeta,t)| \le c^1_{ap}|\xi - \xi_1|.\end{gathered} \tag{4.4}$$

$$\begin{gathered}\exists c^1_{apn} \in (0, \|T^c_{apn}\|_{\mathcal{L}^{-1}}), \forall t \in \mathbb{R}, \forall \xi, \xi_1 \in X, \forall \zeta \in U, \\ |f(\xi,\zeta,t) - f(\xi_1,\zeta,t)| \le c^1_{apn}|\xi - \xi_1|.\end{gathered} \tag{4.5}$$

$$\begin{gathered}\exists c^1_{aa} \in (0, \|T^c_{aa}\|_{\mathcal{L}^{-1}}), \forall t \in \mathbb{R}, \forall \xi, \xi_1 \in X, \forall \zeta \in U, \\ |f(\xi,\zeta,t) - f(\xi_1,\zeta,t)| \le c^1_{aa}|\xi - \xi_1|.\end{gathered} \tag{4.6}$$

Theorem 4.4. *We assume* (2.9) *fulfilled. Under* (2.1) *and* (4.4) *(respectively* (2.3) *and* (4.5)*, respectively* (2.5) *and* (4.6)*), for all* $u \in AP(U)$ *(respectively* $AP^{(n)}(U)$*, respectively* $AA(U)$*) there exists a unique* $\underline{x}(u) \in AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*) which is a mild solution of* (1.4)*. Moreover the mapping* $u \mapsto \underline{x}(u)$ *is continuous from* $AP(U)$ *(respectively* $AP^{(n)}(U)$*, respectively* $AA(U)$*) into* $AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*). Moreover, for* $n \ge 1$*, the mild solution* $\underline{x}(u) \in AP^{(n)}(X)$ *is a classical solution.*

Proof. For the mild solution the proof is similar to this one of Theorem 4.2. Remark that $\underline{x}(u)$ is a mild solution of (1.6) with $b(t) := f(\underline{x}(u)(t), u(t), t)$. If f satisfies (2.3) and if $u \in AP^{(n)}(X)$, then we have $\underline{x}(u) \in AP^{(n)}(X)$ and by using [6, Theorem 7.2], we obtain that $b \in AP^{(n)}(X)$. In this case, by help of Lemma 3.3, we deduce that $\underline{x}(u)$ is a classical solution. □

Theorem 4.5. *We assume* (2.9) *fulfilled. Under* (2.1) *and* (4.4) *(respectively* (2.3) *and* (4.5)*, respectively* (2.5) *and* (4.6)*), for all* $\zeta \in U$ *there exists a unique* $\underline{x}(\zeta) \in AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*) which is a mild solution of* (1.5)*. Moreover the mapping* $\zeta \mapsto \underline{x}(\zeta)$ *is continuous from* U *into* $AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*). Moreover, for* $n \ge 1$*, the mild solution* $\underline{x}(u) \in AP^{(n)}(X)$ *is a classical solution.*

The proof of Theorem 4.5 is similar to the proof of Theorem 4.3 and it is omitted.

5. The differentiable dependence

5.1. **Solutions of equations** (1.1) **and** (1.2). In this subsection, first we provide conditions to ensure the differentiability of the dependence of the solution x with respect to u for (1.1).

Theorem 5.1. *Under assumption* (2.2) *and* (2.7) *(respectively* (2.4) *and* (2.7)*, respectively* (2.6) *and* (2.8)*) we assume that there exist* $u_0 \in AP(U)$ *(respectively* $AP^{(n)}(U)$*,respectively* $AA(U)$*) and* $x_0 \in AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*) which is a mild solution of* (E, u_0)*. We also assume that the following inequality hold:*

$$\begin{gathered}\sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), u_0(t), t)\| < \|T_{ap}\|_{\mathcal{L}^{-1}} \\ \textit{(respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), u_0(t), t)\| < \|T_{apn}\|_{\mathcal{L}^{-1}}, \\ \textit{respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), u_0(t), t)\| < \|T_{aa}\|_{\mathcal{L}^{-1}}).\end{gathered}$$

Then there exist an open neighborhood $\mathcal{U}$ of u_0 in $AP(U)$ (respectively $AP^{(n)}(U)$, respectively $AA(U)$), an open neighborhood $\mathcal{X}$ of x_0 in $AP(X)$ (respectively $AP^{(n)}(X)$, respectively $AA(X)$), and a C^1-mapping $u \mapsto \underline{x}(u)$ from $\mathcal{U}$ into $\mathcal{X}$ such that, for all $u \in \mathcal{U}$, $\underline{x}(u)$ is a mild solution of (1.1). *Moreover $\underline{x}(u)$ is the unique mild solution of* (1.1) *in $\mathcal{X}$; notably we have $\underline{x}(u_0) = x_0$.*

Proof. We do the proof for the almost periodic case. The proofs of the other cases are similar. In the proof of Theorem 3.7, we have seen that, when $u \in AP(U)$, $x \in AP(X)$ is a mild solution of (1.1) if and only if we have $x = T_{ap} \circ \mathcal{N}_f(x, u)$. We denote by $\pi_1 : AP(X) \times AP(U) \to AP(X)$ the first projection, $\pi_1(x, u) := x$. Clearly π_1 is a linear bounded operator.

We introduce the nonlinear operator $\Psi_{ap} : AP(X) \times AP(U) \to AP(X)$ by setting $\Psi_{ap}(x, u) := \pi_1(x, u) - T_{ap} \circ \mathcal{N}_f(x, u)$. And so, $x \in AP(X)$ is a mild solution of (1.1) if and only if we have $\Psi_{ap}(x, u) = 0$. By using (2.2) and [6, Theorem 5.1], we know that $\mathcal{N}_f$ is of class C^1 from $AP(X) \times AP(U)$ into $AP(X)$. Since T_{ap} and π_1 are linear bounded, they are of class C^1. Consequently Ψ_{ap} is of class C^1 as a composition of operators of class C^1. Since x_0 is a mild solution of (E, u_0) we have $\Psi_{ap}(x_0, u_0) = 0$. By using the chain rule, the partial differential of Ψ_{ap} with respect to the first variable at (x_0, u_0) is $D_x\Psi_{ap}(x_0, u_0) = I - T_{ap} \circ D_x\mathcal{N}_f(x_0, u_0)$ where I is the identity operator of $\mathcal{L}(AP(X), AP(X))$. After Theorem 5.1 in [6] we know that, for all $h \in AP(X)$, $D_x\mathcal{N}_f(x_0, u_0).h = [t \mapsto D_1 f(x_0(t), u_0(t), t).h(t)]$, and then by using the assumption on $D_1 f$ we obtain

$$\|D_x\mathcal{N}_f(x_0, u_0)\|_{\mathcal{L}} \le \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), u_0(t), t)\| < \|T_{ap}\|_{\mathcal{L}^{-1}}.$$

Consequently we have $\|T_{ap} \circ D_x\mathcal{N}_f(x_0, u_0)\| < 1$. Then by using a classical argument on the Neumann series (Proof of [1, Lemma 2.5.4], or [7, Théorème 1.7.2]) we know that $I - T_{ap} \circ D_x\mathcal{N}_f(x_0, u_0)$ is invertible. And so we can use the implicit function theorem ([7, Théorème 4.7.1], or [1, Théorème 2.5.7]) and we can assert that there exist a neighborhood $\mathcal{U}$ of u_0 in $AP(U)$, a neighborhood $\mathcal{X}$ of x_0 in $AP(X)$ and a C^1-mapping $u \mapsto \underline{x}(u)$, from $\mathcal{U}$ into $\mathcal{X}$, such that $\underline{x}(u_0) = x_0$, and such that $\{(x, u) \in \mathcal{X} \times \mathcal{U} : \Psi_{ap}(x, u) = 0\} = \{(\underline{x}(u), u) : u \in \mathcal{U}\}$. The conclusion of the theorem is just a translation of these properties. ☐

The following theorem treats the differentiable dependence for the equations (1.2).

Theorem 5.2. *Under assumptions* (2.2) *and* (2.7) *(respectively* (2.4) *and* (2.7)*, respectively* (2.6) *and* (2.8)*) we assume that there exist $\zeta_0 \in U$, and $x_0 \in AP(X)$ (respectively $AP^{(n)}(X)$, respectively $AA(X)$) which is a mild solution of* (1.2) *with $\zeta = \zeta_0$. We also assume that the following inequality holds:*

$$\sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), \zeta_0, t)\| < \|T_{ap}\|_{\mathcal{L}^{-1}}$$
$$\textit{(respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), \zeta_0), t)\| < \|T_{apn}\|_{\mathcal{L}^{-1}},$$
$$\textit{respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), \zeta_0, t)\| < \|T_{aa}\|_{\mathcal{L}^{-1}}).$$

Then there exist an open neighborhood Z of ζ_0 in $AP(U)$, an open neighborhood $\mathcal{X}$ of x_0 in $AP(X)$ (respectively $AP^{(n)}(X)$, respectively $AA(X)$), and a C^1-mapping $\zeta \mapsto \underline{x}(\zeta)$ from Z into $\mathcal{X}$ such that, for all $\zeta \in Z$, $\underline{x}(\zeta)$ is a mild solution of

(1.2). *Moreover* $\underline{x}(\zeta)$ *is the unique mild solution of* (1.2) *in* $\mathcal{X}$*; notably we have* $\underline{x}(\zeta_0) = x_0$.

Proof. Let Φ be the operator from $\mathcal{U}$ into $\mathcal{X}$ defined as follows: $\Phi(u)$ is the unique mild solution of (1.1) in $\mathcal{X} \cap AP(X)$ provided by Theorem 5.1. By using Theorem 5.1 we obtain Φ is of class C^1. We consider U as the Banach subspace of the constant functions in $AP(U)$. And so we define the operator $\Psi : \mathcal{U} \cap U \to \mathcal{X} \cap U$ as the restriction of Φ to U. Then $\Psi(\zeta)$ is the unique mild solution of (1.2) in $AP(X)$ and Ψ is of class C^1. The reasoning is similar for the other cases. □

5.2. **Solutions of equations** (1.4) **and** (1.5)**.** Now we establish a result of differentiability in the special case where $A(t) = A$ is constant with respect to t; i.e., for the equations (1.4).

Theorem 5.3. *We assume* (2.9) *fulfilled. Under assumption* (2.2) *(respectively* (2.4)*, respectively* (2.6)*) we assume that there exist* $u_0 \in AP(U)$ *(respectively* $AP^{(n)}(U)$*, respectively* $AA(U)$*), and* $x_0 \in AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*) which is a mild solution of* (E_c, u_0)*. We also assume that the following inequality holds*

$$\sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), u_0(t), t)\| < \|T^c_{ap}\|_{\mathcal{L}^{-1}}$$
$$\text{(respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), u_0(t), t)\| < \|T^c_{apn}\|_{\mathcal{L}^{-1}},$$
$$\text{respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), u_0(t), t)\| < \|T^c_{aa}\|_{\mathcal{L}^{-1}}).$$

Then there exist an open neighborhood $\mathcal{U}$ *of* u_0 *in* $AP(U)$ *(respectively* $AP^{(n)}(U)$*, respectively* $AA(U)$*), an open neighborhood* $\mathcal{X}$ *in* $AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*), and a* C^1*-mapping* $u \mapsto \underline{x}(u)$ *from* $\mathcal{U}$ *into* $\mathcal{X}$ *such that, for all* $u \in \mathcal{U}$*,* $\underline{x}(u)$ *is a mild solution of* (1.4)*. Moreover* $\underline{x}(u)$ *is the unique mild solution of* (1.4) *in* $\mathcal{X}$*; notably we have* $\underline{x}(u_0) = x_0$.

The proof of Theorem 5.3 is similar to the proof of Theorem 5.1 and it is omitted. One of the main tools used in the proofs of Theorem 5.1 and Theorem 5.3 is the implicit function theorem. The use of the implicit function theorem in a functional analytic framework was done in [4] for periodic solutions of ordinary differential equations, and in [5] for almost periodic solutions of ordinary differential equations.

The following theorem is a differentiability result for (1.5).

Theorem 5.4. *We assume* (2.9) *fulfilled. Under* (2.2) *(respectively* (2.4)*, respectively* (2.6)*) we assume that there exist* $\zeta_0 \in U$ *, and* $x_0 \in AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*) which is a mild solution of* (1.5) *with* $\zeta = \zeta_0$*. We also assume that the following inequality holds:*

$$\sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), \zeta_0, t)\| < \|T^c_{ap}\|_{\mathcal{L}^{-1}}$$
$$\text{(respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), \zeta_0), t)\| < \|T^c_{apn}\|_{\mathcal{L}^{-1}},$$
$$\text{respectively } \sup_{t\in\mathbb{R}} \|D_1 f(x_0(t), \zeta_0, t)\| < \|T^c_{aa}\|_{\mathcal{L}^{-1}}).$$

Then there exist an open neighborhood Z *of* ζ_0 *in* $AP(U)$*, an open neighborhood* $\mathcal{X}$ *of* x_0 *in* $AP(X)$ *(respectively* $AP^{(n)}(X)$*, respectively* $AA(X)$*), and a* C^1*-mapping* $\zeta \mapsto \underline{x}(\zeta)$ *from* Z *into* $\mathcal{X}$ *such that, for all* $\zeta \in Z$*,* $\underline{x}(\zeta)$ *is a mild solution of*

(1.5). *Moreover* $\underline{x}(\zeta)$ *is the unique mild solution of* (1.5) *in* $\mathcal{X}$*; notably we have* $\underline{x}(\zeta_0) = x_0$.

The proof of Theorem 5.4 is similar to the proof of Theorem 5.2 and it is omitted.

Remark 5.5. For reasons similar to these ones used about Theorem 4.4, the mild solution $\underline{x}(u)$ of (1.4) (respectively (1.5)) in $AP^{(n)}(X)$, for $n \geq 1$, provided by Theorem 5.3 (respectively Theorem 5.4) is a classical solution.

References

[1] R. Abraham, J. E. Marsden, and T. Ratiu; *Manifolds, tensor analysis, and applications*, Addison-Wesley Publishing Company, Inc., Reading, Massachussets, 1983.

[2] P. Acquistapace and B. Terrini; *A unified approach to abstract linear nonaotonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova **78**, 1987, 47-107.

[3] J.-B. Baillon, J. Blot, G. M. N'Guérékata, and D. Pennequin; *On* $C^{(n)}$*-almost periodic solutions to some nonautonomous differential equations in Banach spaces*, Comment. Math. Prace Mat. **XLVI**(2), 2006, 263-273.

[4] J. Blot; *Méthodes asymptotiques dans l'étude des systèmes hamiltoniens non autonomes*, in *Advances in Hamiltonian Systems*, J.-P. Aubin, A. Bensoussan, I. Ekeland (Editors), Birkhäuser, Boston, 1982, 144-170.

[5] J. Blot, P. Cieutat, and J. Mawhin; *Almost periodic oscillations on monotone second order systems*, Adv. Differential Equations **2**(5), 1997, 693-714.

[6] J. Blot, P. Cieutat, G. M. N'Guérékata, and D. Pennequin; *Superposition operators between various almost periodic function spaces and applications*, Commun. Math. Anal. **6**(1), 2009, 42-70.

[7] H. Cartan; *Cours de calcul différentiel*, Hermann, Paris, 1977.

[8] J. Dieudonné; *Éléments d'analyse; tome 1 : fondements de l'analyse moderne*, French Edition, Gauthier-Villars, Paris, 1972.

[9] H.-S. Ding, W. Long, and G. M. N'Guérékata; *Almost automorphic solutions of nonautonomous evolution equations*, Nonlinear Anal. **70** (12), 2009, 4158-4164.

[10] S. Goldberg; *Unbounded linear operators; theory and applications*, Dover Publications, Inc., N. Y., 1966.

[11] Y. Hino, T. Naito, N.V. Minh, and J. S. Shin; *Almost periodic solutions of differential equations in Banach spaces*, Taylor and Francis, London, 2002.

[12] B. M. Levitan, and V. V. Zhikov; *Almost periodic functions and differential equations*, English Edition, Cambridge University Press, Cambridge, U.K., 1982.

[13] J. H. Liu, G. M. N'Guérékata and Nguyen Van Minh; *Topics on Stability and Periodicity in Abstract Differential Equations*, World Scientific, Series on Concrete and Applicable Mathematics, Vol.6 (2008), New Jersey-Hong Kong-London-Beijing-Singapore.

[14] C.-M. Marle; *Mesures et probabilités*, Hermann, Paris, 1974.

[15] G. M. N'Guérékata; *Existence and uniqueness of almost automorphic mild solutions to some semilinear abstract differential equations*, Semigroup Forum **69**, 2004, 80-86.

[16] G. M. N'Guérékata; *Topics in almost automorphy*, Springer Science+Business Media, Inc., New York, 2003.

[17] A. A. Pankov; *Bounded and almost periodic solutions of nonlinear operator differential equations*, English Edition, Kluwer Academic Publishers, Dordrecht, 1990.

[18] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[19] M. Roseau; *Vibrations non linéaires et théorie de la stabilité*, Springer-Verlag, Berlin, 1966.

[20] L. Schwartz; *Cours d'analyse, tome 1*, Hermann, Paris, 1967.

[21] T. Yoshizawa; *Stability theory and the existence of periodic solutions and almost periodic solutions*, Springer-Verlag, New York, 1975.

[22] S. Zaidman; *A non-linear abstract differential equation with almost-periodic solution*, Riv. Mat. Univ. Parma **10**(4), 1984, 331-336.

[23] S. Zaidman; *Almost-periodic functions in abstract spaces*, Pitman Publishing Inc., Marshfield, Mass., 1985.

Joël Blot
Laboratoire SAMM EA 4543, Université Paris 1 Panthéon-Sorbonne, centre P.M.F., 90 rue de Tolbiac, 75634 Paris cedex 13, France
E-mail address: blot@univ-paris1.fr

Philippe Cieutat
Laboratoire de Mathématiques de Versailles, UMR-CNRS 8100, Université Versailles-Saint-Quentin-en-Yvelines, 45 avenue des États-Unis, 78035 Versailles cedex, France
E-mail address: Philippe.Cieutat@math.uvsq.fr

Gaston M. N'Guérékata
Department of mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21252, USA
E-mail address: gnguerek@jewel.morgan.edu