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TRAVELLING WAVES IN THE LATTICE EPIDEMIC MODEL

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Abstract. In this article, we establish the existence and nonexistence of travelling waves for a lattice non-monotone integral equation which is an epidemic model. Moreover, the wave is either convergent to the positive equilibrium or oscillating on the positive equilibrium at positive infinity, and has the exponential asymptotic behavior at negative infinity. For the non-monotone case, the asymptotic speed of propagation also coincides with the minimal wave speed.

1. Introduction

One of the epidemic models on the lattice $\mathbb{Z}$ is

$$u_n(t) = \int_0^t \sum_{j\in\mathbb{Z}} A_{n-j}(t-\tau)g(u_j(\tau))d\tau + f_n(t), \quad t \geq 0,\ n \in \mathbb{Z} = \{0, \pm 1, \dots\}. \tag{1.1}$$

The existence of nonnegative solution and the asymptotic speed of propagation of (1.1) have been studied in [24] under some suitable assumptions on the functions A_n, g, and f_n.

Equation (1.1) corresponds to an initial value problem (the history up to $t = 0$ is prescribed; in fact it is incorporated in the function f_n). On the other hand, if one wants to describe an epidemic which has been evolving from the beginning of time then one arrives at the time-translation invariant homogeneous equation

$$u_n(t) = \int_{-\infty}^t \sum_{j\in\mathbb{Z}} A_{n-j}(t-\tau)g(u_j(\tau))d\tau, \quad t \in \mathbb{R} := (-\infty, \infty),\ n \in \mathbb{Z}, \tag{1.2}$$

which is investigated in [24] about the existence and nonexistence of the travelling wave (i.e. solution of the form $u_n(t) = u(n + ct)$, where $c > 0$) when g is a nondecreasing function. Furthermore, authors in [24] showed that the asymptotic spread speed coincides with the minimal wave speed in the case that g is a nondecreasing function.

For a continuous analogue of (1.1), one may consider the integral equation

$$u(t,x) = \int_0^t A(t-\tau) \int_{R^n} g(u(\tau,\xi))V(x-\xi)d\xi d\tau + f(t,x), \quad t \geq 0,\ x \in \mathbb{R}. \tag{1.3}$$

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The behaviors of (1.3) and its corresponding homogeneous equation were investigated in [4, 5, 6], such as the existence of travelling waves, the minimal wave speed, the asymptotic speed of propagation, etc. There have been extensive investigations on travelling waves for reaction-diffusion equations [7, 8, 12, 13, 15, 16, 17, 21, 20, 25, 26, 27, 28, 29], integral equations [4, 5, 6, 18, 19], and lattice equations [3, 9, 10, 14, 22]. The concept of the asymptotic speed of spread was first introduced by Aronson & Weinberger [1, 2], (see also [9, 10, 11, 14, 18, 19, 22, 23, 24] and the references therein).

Note that authors in [24] established the asymptotic speed of propagation for (1.1) when g may be a non-monotone function, but the existence of travelling waves for (1.2) was admitted only when g is nondecreasing. To our knowledge, the existence of travelling wave solutions for (1.2) with a non-monotone function g is still an open problem, let alone the relation between the spreading speed and the minimal wave speed for (1.2) with a non-monotone function g. In this paper, we will give an affirmative answer. More precisely, we will be mainly concerned with the existence and nonexistence of travelling waves, and the relation between the spreading speed and the minimal wave speed for (1.2) when g is a non-monotone function. Moreover, the wave is either convergent to the positive equilibrium or oscillating on the positive equilibrium at positive infinity.

We use the notation: $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{N}_+ = \{1, 2, \dots\}$, $\mathbb{R}_+ = [0, +\infty)$. We shall assume that the following hold through this article.

- (A1) For any $j \in \mathbb{Z}$, $A_j \in C(\mathbb{R}_+)$ and $A_j(t) = A_{-j}(t) \geq 0$; $\sum_{j\in\mathbb{Z}} A_j \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ with $\int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau)d\tau = 1$; there exists a $\bar{\lambda} : 0 < \bar{\lambda} \leq \infty$ such that
$$\int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau)e^{-\lambda j}d\tau < \infty$$
for $\lambda \in [0, \bar{\lambda})$.
- (G1) $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $|g(u) - g(v)| \leq g'(0)|u - v|$ for $u, v \in \mathbb{R}_+$;
- (G2) $g(0) = 0$; there exists $K > 0$ such that $g(K) = K$, $g(u) > u$ for $0 < u < K$ and $g(u) < u$ for $u > K$;
- (G3) $g'(0)u \geq g(u)$ for $u \in \mathbb{R}_+$ and there exists $K^* \geq K$ such that $g(u) \leq K^*$ for all $u \in [0, K^*]$.
- (G4) There exists $\sigma \in (0, 1]$ such that
$$\limsup_{u\to 0^+}[g'(0) - g(u)/u]u^{-\sigma} < \infty.$$
- (G5) $g(u) < 2K - u$ for $u \in [0, K)$ and $g(u) > 2K - u$ for $u \in (K, K^*]$.

2. Main results

In this section, we first establish the existence of travelling waves for the system (1.2) by using the Schauder's fixed point theorem. The key idea is to construct two monotone functions to squeeze g. This approach was initially used in [13]. We will further establish the nonexistence of travelling wave solutions.

A travelling wave of (1.2) is a special translation invariant solution of (1.2) with the form $u_n(t) = \phi(n + ct)$, where $c > 0$ is the wave speed. Letting $\xi = n + ct$, it follows that ϕ must be a solution of the following wave profile equation

$$\phi(\xi) = \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g(\phi(\xi - j - c\tau))d\tau,\ \xi \in \mathbb{R}. \tag{2.1}$$

Under the assumptions (G1)–(G2), it is easily seen that 0 and K are the only two equilibria of (2.1). We will find a solution ϕ of (2.1) with the boundary conditions

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \quad \liminf_{\xi\to\infty} \phi(\xi) > 0. \tag{2.2}$$

Define

$$g^-(u) = \inf_{u\le v\le K^*}\{g(v)\}, \quad g^+(u) = \min\{g'(0)u, K^*\} \quad \text{for } u \in [0, K^*].$$

Similar to the proof of [13, Lemma 3.1], it is easily seen that the following lemma holds.

Lemma 2.1. *Assume that* (A1), (G1)–(G4) *hold. Then the following assertions hold.*

- (i) *$g^-(u)$ and $g^+(u)$ are nondecreasing on $[0, K^*]$ and Lispchitz continuous on $[0, K^*]$; that is, $|g^\pm(u) - g^\pm(v)| \le g'(0)|u - v|$ for $u, v \in [0, K^*]$;*
- (ii) *$g^-(u) \le g(u) \le g^+(u)$, for $u \in [0, K^*]$;*
- (iii) *$g'(0)u \ge g^-(u) > 0$ and $g'(0)u \ge g^+(u) > 0$ for all $u \in (0, K^*]$;*
- (iv) *There exists $0 < K_* \le K$ such that $g^-(K_*) = K_*$. Moreover, $g^-(0) = g^-(K_*) - K_* = 0$ and $g^+(0) = g^+(K^*) - K^* = 0$;*
- (v) *$\limsup_{u\to0^+}[g'(0) - g^-(u)/u]u^{-\sigma} < \infty$.*

Let

$$\Delta_c(\lambda) = g'(0) \int_0^\infty e^{-\lambda c\tau} \sum_{j\in\mathbb{Z}} A_j(\tau)e^{-\lambda j} d\tau.$$

According to Assumption (A1), for any $c > 0$, $\Delta_c(\lambda)$ is well defined on $[0, \bar{\lambda})$. We have the following lemma.

Lemma 2.2 (Lemma 3.1, [24])**.** *Assume that* (A1) *holds and $g'(0) > 1$. Then, there exist $c_* > 0$ and $\lambda_* > 0$ such that the following assertions hold.*

- (i) *$\Delta_{c_*}(\lambda_*) = 1$, $\frac{\partial\Delta_c(\lambda)}{\partial\lambda}|_{c=c_*,\lambda=\lambda_*} = 0$; i.e, λ_* is the minimal zero point of $\Delta_{c_*}(\lambda) = 1$;*
- (ii) *For any $c \in (0, c_*)$ and $\lambda \in [0, \bar{\lambda})$, $\Delta_c(\lambda) > 1$;*
- (iii) *For any $c > c_*$, the equation $\Delta_c(\lambda) = 1$ has two real roots λ_1 and λ_2; that is, $\Delta_c(\lambda_1) = \Delta_c(\lambda_2) = 1$: $0 < \lambda_1 < \lambda_* < \lambda_2 < \bar{\lambda}$. Moreover, $\Delta_c(\lambda) < 1$ for any $\lambda \in (\lambda_1, \lambda_2)$.*

Now we are in a position to state our main results about the existence and nonexistence of travelling waves.

Theorem 2.3 (Existence)**.** *Assume that* (A1), (G1)–(G4) *hold, and that $g'(0) > 1$. Then we have*

- (i) *For $c > c_*$, the system* (1.2) *has a travelling wave $\phi \in C(\mathbb{R}, [0, K^*])$ satisfying*

$$\lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1\xi} = 1,$$
$$K_* \le \alpha := \liminf_{\xi\to\infty} \phi(\xi) \le K \le \beta := \limsup_{\xi\to\infty} \phi(\xi) \le K^*$$

Moreover, either $\alpha = \beta = K$ or $K_* < \alpha < K < \beta$; that is, the wave is either convergent to the positive equilibrium or oscillating on the positive equilibrium at positive infinity.

(ii) *If, in addition, either $g(u)$ is nondecreasing in $u \in [0, K]$ or* (G5) *holds, then $\alpha = \beta = K$; that is, the wave is convergent to the positive equilibrium.*

(iii) *For $c = c_*$, the system* (1.2) *has a travelling wave $\phi(n+c_*t) \in C(\mathbb{R}, [0, K^*]) \setminus \{0, K\}$. Moreover, if $g(u)$ is nondecreasing in $u \in [0, K]$, then the travelling wave $\phi(n + c_*t)$ satisfies the asymptotic behavior*

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \quad \lim_{\xi\to\infty} \phi(\xi) = K.$$

Theorem 2.4 (Non-existence)**.** *Assume that* (A1), (G1)–(G4) *hold. Then we have the following assertions:*

(i) *If $g'(0) < 1$, for any $c > 0$, the system* (1.2) *has no nonnegative bounded travelling wave solution satisfying* (2.2)*;*

(ii) *If $g'(0) > 1$, for $0 < c < c_*$, the system* (1.2) *has also no nonnegative bounded travelling wave satisfying* (2.2).

Remark 2.5. If $g(u)$ is nondecreasing in $u \in [0, K]$, letting $K^* = K$, then Theorem 2.3 in this present paper reduces to [24, Theorem 4.1]. The nonexistence, (ii) of Theorem 2.4, is a consequence of the result that c^* is the spreading speed which is established in [24, Theorem 3.2]. Note that the non-existence result of travelling waves still has not been reported in [24] when $g'(0) < 1$.

Remark 2.6. By Theorems 2.3 and 2.4, it is easily seen that c_* is the minimal wave speed. According to the results in [24], c_* is also the asymptotic speed of propagation for the model (1.1) without presupposing that the function g be monotone. Thus we can also conclude that the minimal wave speed for the model (1.2) coincides with the asymptotic speed of propagation for the model (1.1) without the monotonicity of g. We can further obtain some good properties of travelling waves at positive infinity for the model (1.2) with non-monotone function g.

According to the above remarks, we need to prove only Theorem 2.3 with non-monotonicity of g and Theorem 2.4(i). To complete our main results, we need to make some preparations. Define an operator $T : C(\mathbb{R}, [0, K^*]) \to C(\mathbb{R}, \mathbb{R}_+)$ by

$$T(\phi)(\xi) = \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g(\phi(\xi - j - c\tau)) d\tau, \quad \xi \in \mathbb{R}, \ \phi \in C(\mathbb{R}, [0, K^*]). \tag{2.3}$$

It is obvious that T is well defined and a fixed point of T is a solution of (2.1), which is a travelling wave of (1.2). Let $T^\pm$ be as in (2.3) with g replaced by $g^\pm$. By Lemma 2.1, it is easily seen that $T^\pm$ are nondecreasing on $C(\mathbb{R}, [0, K^*])$ and

$$T^-(\phi) \le T(\phi) \le T^+(\phi), \quad \text{for } \phi \in C(\mathbb{R}, [0, K^*]). \tag{2.4}$$

Lemma 2.7. *Assume* (A1), (G1)–(G4)*, and that $g'(0) > 1$. Then $T^+(\bar\phi^+)(\xi) \le \bar\phi^+(\xi)$, where*

$$\bar\phi^+(\xi) =: \min\{K^*, e^{\lambda_1\xi}\}, \quad \textit{for } \xi \in \mathbb{R}. \tag{2.5}$$

Proof. Since $g^+(u)$ is nondecreasing in u and $\bar\phi^+(\xi) \le K^*$, for $\xi \in \mathbb{R}$, we can obtain

$$T^+(\bar\phi^+)(\xi) \le \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g^+(K^*) d\tau = g^+(K^*) = K^*, \quad \xi \in \mathbb{R}. \tag{2.6}$$

Note that $g^+(u) \le g'(0)u$, for $u \in [0, K^*]$, and $\bar\phi^+(\xi) \le e^{\lambda_1 \xi}$ for $\xi \in \mathbb{R}$. It follows that

$$\begin{aligned} T^+(\bar\phi^+)(\xi) &\le \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g'(0)\bar\phi^+(\xi - c\tau - j)d\tau \\ &\le \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g'(0) e^{\lambda_1(\xi - c\tau - j)}d\tau \\ &= e^{\lambda_1\xi}\Delta_c(\lambda_1) = e^{\lambda_1\xi}. \end{aligned} \tag{2.7}$$

According to the definition of $\bar\phi^+$ and (2.6)-(2.7), we have $T^+(\bar\phi^+)(\xi) \le \bar\phi^+(\xi)$ for $\xi \in \mathbb{R}$. This completes the proof. □

Similar to the proof of Lemma 2.7, it is easily seen that the following lemma holds.

Lemma 2.8. *Assume* (A1), (G1)–(G4), *and that* $g'(0) > 1$. *Then* $T^-(\bar\phi^-)(\xi) \le \bar\phi^-(\xi)$, *where*

$$\bar\phi^-(\xi) =: \min\{K_*, e^{\lambda_1\xi}\}, \quad \textit{for } \xi \in \mathbb{R}.$$

Lemma 2.9. *Assume that* (A1), (G1)–(G4) *hold, and* $g'(0) > 1$. *Then for every* $\gamma \in (1, \min\{1+\sigma, \frac{\lambda_2}{\lambda_1}\})$, *there exists large enough* $Q(\gamma) > 1$ *such that for any* $q \ge Q(\gamma)$, *we have* $T^-(\underline\phi^-)(\xi) \ge \underline\phi^-(\xi)$, *where*

$$\underline\phi^-(\xi) =: \max\{0, e^{\lambda_1\xi} - qe^{\gamma\lambda_1\xi}\}, \quad \textit{for } \xi \in \mathbb{R}. \tag{2.8}$$

Proof. Let $\xi_0 = -\frac{\ln q}{\lambda_1(\gamma-1)}$. Then we have

$$\begin{gathered} \underline\phi^-(\xi) = 0 \quad \text{for } \xi \ge \xi_0, \\ \underline\phi^-(\xi) = e^{\lambda_1\xi} - qe^{\gamma\lambda_1\xi} \quad \text{for } \xi \le \xi_0. \end{gathered}$$

Obviously, it follows that

$$T^-(\underline\phi^-)(\xi) \ge 0 \quad \text{for } \xi \in \mathbb{R}. \tag{2.9}$$

By the definition of $g^-(u)$ and Lemma 2.1(v), there exist $\delta > 0$ and $0 < M < \infty$ such that

$$g(u) \ge g^-(u) \ge g'(0)u - Mu^{1+\sigma} \quad \text{for } u \in [0, \delta]. \tag{2.10}$$

It is easily seen that there exists $Q_1(\gamma) > 1$ such that $e^{\lambda_1\xi} - qe^{\gamma\lambda_1\xi} \le \delta$ for $q \ge Q_1(\gamma)$. Therefore,

$$0 \le \underline\phi^-(\xi) \le \delta \quad \text{for } \xi \in \mathbb{R}. \tag{2.11}$$

Since $\xi_0 < 0$ and $1 + \sigma > \gamma$, it is easy to see that

$$\bar\phi^+(\xi) \ge \underline\phi^-(\xi) \ge e^{\lambda_1\xi} - qe^{\gamma\lambda_1\xi} \quad \text{for } \xi \in \mathbb{R}, \tag{2.12}$$

$$[\underline\phi^-(\xi)]^{1+\sigma} \le e^{\gamma\lambda_1\xi} \quad \text{for } \xi \in \mathbb{R}. \tag{2.13}$$

Thus, according to (2.10)-(2.13), we have

$$g'(0)\underline\phi^-(\xi) - M[\underline\phi^-(\xi)]^{1+\sigma} \ge g'(0)e^{\lambda_1\xi} - g'(0)qe^{\gamma\lambda_1\xi} - Me^{\gamma\lambda_1\xi}$$

and

$$\begin{aligned}
&T^-(\underline{\phi}^-)(\xi)\\
&\geq \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau)[g'(0)\underline{\phi}^-(\xi-c\tau-j)-M[\underline{\phi}^-(\xi-c\tau-j)]^{1+\sigma}d\tau\\
&\geq \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau)\Big[g'(0)e^{\lambda_1(\xi-c\tau-j)}-g'(0)qe^{\gamma\lambda_1(\xi-c\tau-j)}-Me^{\gamma\lambda_1(\xi-c\tau-j)}\Big]d\tau\\
&= e^{\lambda_1\xi}\Delta_c(\lambda_1)-qe^{\gamma\lambda_1\xi}\Big[\Delta_c(\gamma\lambda_1)+\frac{M}{q}\int_0^\infty\sum_{j\in\mathbb{Z}}A_j(\tau)e^{-\gamma\lambda_1(j+c\tau)}\Big]d\tau.
\end{aligned} \tag{2.14}$$

Since $\lambda_1<\gamma\lambda_1<\lambda_2$, we have

$$\Delta_c(\gamma\lambda_1)<1.$$

Therefore, there exists

$$Q(\gamma)\geq \max\{1,Q_1(\gamma),M\int_0^\infty\sum_{j\in\mathbb{Z}}A_j(\tau)e^{-\gamma\lambda_1(j+c\tau)}d\tau\},$$

large enough, such that for $q\geq Q(\gamma)$, it follows

$$\Delta_c(\gamma\lambda_1)+\frac{M}{q}\int_0^\infty\sum_{j\in\mathbb{Z}}A_j(\tau)e^{-\gamma\lambda_1(j+c\tau)}d\tau\leq 1. \tag{2.15}$$

Hence, by (2.14)-(2.15), we have

$$T^-(\underline{\phi}^-)(\xi)\geq e^{\lambda_1\xi}-qe^{\gamma\lambda_1\xi}\quad \text{for } \xi\in\mathbb{R}. \tag{2.16}$$

According to the definition of $\underline{\phi}^-(\xi)$, (2.9) and (2.16), it is easy to see that

$$T^-(\underline{\phi}^-)(\xi)\geq \underline{\phi}^-(\xi)\quad \text{for } \xi\in\mathbb{R}.$$

This completes the proof. □

Remark 2.10. The construction of $\bar{\phi}^+$, $\bar{\phi}^-$ and $\underline{\phi}^-$ in (2.5) and (2.8) is due to that for monotone case in [5] and [22].

Applying the iteration monotone scheme

$$\begin{gathered}
\phi_{n+1}(\xi)=T^-(\phi_n)(\xi)\\
\phi_0(\xi)=\bar{\phi}^-(\xi),
\end{gathered}$$

by Lemmas 2.8 and 2.9, we can easily obtain the following result.

Proposition 2.11. *Assume that* (A1), (G1)–(G4) *hold, and* $g'(0)>1$. *Then, for any* $c>c^*$, *there exists a nondecreasing fixed point* ϕ^- *of* T^- *such that* $T^-(\phi^-)(\xi)=\phi^-(\xi)$ *and* $\phi^-(\xi)\leq\bar{\phi}^-(\xi)$ *for all* $\xi\in\mathbb{R}$. *Moreover,*

$$\lim_{\xi\to-\infty}\phi^-(\xi)e^{-\lambda_1\xi}=1\quad \textit{and}\quad \lim_{\xi\to\infty}\phi^-(\xi)=K_*.$$

For a given number $\lambda>0$, let

$$X_\lambda:=\{\phi\in C(\mathbb{R},\mathbb{R}):\sup_{\xi\in\mathbb{R}}|\phi(\xi)|e^{-\lambda\xi}<\infty\}$$

and $\|\phi\|_\lambda=\sup_{\xi\in\mathbb{R}}|\phi(\xi)|e^{-\lambda\xi}$. Then it is easy to see that $(X_\lambda,\|\cdot\|_\lambda)$ is a Banach space.

By Lemmas 2.7–2.9 and Proposition 2.11, we have

$$\phi^-(\xi) = T^-(\phi^-)(\xi) \le T^-(\bar\phi^-)(\xi) \le T^+(\bar\phi^-)(\xi) \le T^+(\bar\phi^+)(\xi) \le \bar\phi^+(\xi).$$

Now fix a number $\lambda \in (0, \lambda_1)$. Clearly, both ϕ^- and $\bar\phi^+$ are elements in X_λ. Thus, the set

$$\Gamma := \{\phi \in X_\lambda : \phi^-(\xi) \le \phi(\xi) \le \bar\phi^+(\xi), \text{ for } \xi \in \mathbb{R}\}$$

is a nonempty, closed and convex subset of X_λ. For the operator T defined by (2.3), we have the following lemmas.

Lemma 2.12. *Assume that* (A1), (G1)–(G4) *hold, and* $g'(0) > 1$. *The following assertions hold.*

(i) $T(\Gamma) \subset \Gamma$;
(ii) $T : \Gamma \to \Gamma$ *is continuous with respect to the norm* $\|\cdot\|_\lambda$ *in* X_λ.

Proof. For any $\phi \in \Gamma$, it is obvious that

$$\phi^- = T^-(\phi^-) \le T^-(\phi) \le T(\phi) \le T^+(\bar\phi^+) \le \bar\phi^+,$$

which implies that $T(\Gamma) \subset \Gamma$. For any $\phi, \psi \in \Gamma$, we have

$$\begin{aligned}
\|T(\phi) - T(\psi)\|_\lambda &= \sup_{\xi\in\mathbb{R}} |T(\phi)(\xi) - T(\psi)(\xi)| e^{-\lambda\xi} \\
&\le \sup_{\xi\in\mathbb{R}} \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g'(0) |\phi(\xi - c\tau - j) - \psi(\xi - c\tau - j)| e^{-\lambda\xi} d\tau \\
&\le \Big(g'(0) \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) e^{-\lambda(c\tau + j)} d\tau\Big) \|\phi - \psi\|_\lambda \\
&\le \Big(g'(0) \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) e^{-\lambda j} d\tau\Big) \|\phi - \psi\|_\lambda.
\end{aligned}$$

Thus, $T : \Gamma \to \Gamma$ is continuous with respect to the norm $\|\cdot\|_\lambda$ in X_λ. This completes the proof. □

Lemma 2.13. *Assume that* (A1), (G1)–(G4) *hold, and* $g'(0) > 1$. *Then* $T : \Gamma \to \Gamma$ *is compact.*

Proof. Since $0 \le \phi^- \le T(\phi) \le \bar\phi^+ \le K^*$ for $\phi \in \Gamma$, it is obvious that the family of functions $\{T(\phi)(\xi) : \phi \in \Gamma\}$ is uniformly bounded in $\xi \in \mathbb{R}$. On the other hand, it follows from (A1) and (G1) that $g(u) \le Lu$ for $u \in \mathbb{R}_+$ and there is $M_1 > 0$ such that $\sum_{j\in\mathbb{Z}} A_j(\xi) \le M_1$ for $\xi \in \mathbb{R}$. Therefore, for $\xi_1 > \xi_2$, we have

$$\begin{aligned}
&|T(\phi)(\xi_1) - T(\phi)(\xi_2)| \\
&= \Big| \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g(\phi(\xi_1 - c\tau - j)) d\tau - \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g(\phi(\xi_2 - c\tau - j)) d\tau \Big| \\
&= \Big| \frac{1}{c} \int_{-\infty}^{\xi_1} \sum_{j\in\mathbb{Z}} A_j(\frac{1}{c}(\xi_1 - \tau)) g(\phi(\tau - j)) d\tau \\
&\quad - \frac{1}{c} \int_{-\infty}^{\xi_2} \sum_{j\in\mathbb{Z}} A_j(\frac{1}{c}(\xi_2 - \tau)) g(\phi(\tau - j)) d\tau \Big| \\
&\le \frac{1}{c} \int_{-\infty}^{\xi_2} \Big| \sum_{j\in\mathbb{Z}} A_j(\frac{1}{c}(\xi_2 - \tau)) - \sum_{j\in\mathbb{Z}} A_j(\frac{1}{c}(\xi_1 - \tau)) \Big| g(\phi(\tau - j)) d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{c}\int_{\xi_2}^{\xi_1}\sum_{j\in\mathbb{Z}}A_j(\frac{1}{c}(\xi_1-\tau))g(\phi(\tau-j))d\tau\\
&\le\frac{g'(0)K^*}{c}\int_{-\infty}^{\xi_2}\Big|\sum_{j\in\mathbb{Z}}A_j(\frac{1}{c}(\xi_2-\tau))-\sum_{j\in\mathbb{Z}}A_j(\frac{1}{c}(\xi_1-\tau))\Big|d\tau+\frac{g'(0)K^*M_1}{c}|\xi_1-\xi_2|\\
&=g'(0)K^*\int_0^{\infty}\Big|\sum_{j\in\mathbb{Z}}A_j(\frac{1}{c}(\xi_1-\xi_2)+\tau)-\sum_{j\in\mathbb{Z}}A_j(\tau)\Big|d\tau+\frac{g'(0)K^*M_1}{c}|\xi_1-\xi_2|\\
&=g'(0)K^*\mathbb{A}(\xi_1-\xi_2)+\frac{g'(0)K^*M_1}{c}|\xi_1-\xi_2|,
\end{aligned}
$$

where

$$
\mathbb{A}(\xi)=\int_0^{\infty}\Big|\sum_{j\in\mathbb{Z}}A_j(\frac{1}{c}\xi+\tau)-\sum_{j\in\mathbb{Z}}A_j(\tau)\Big|d\tau\quad\text{for }\xi\in\mathbb{R}. \tag{2.17}
$$

Since $\lim_{\xi\to 0}\mathbb{A}(\xi)=0$, it follows from the above inequality that the family of functions $\{T(\phi)(\xi):\phi\in\Gamma\}$ is equicontinuous in $\xi\in\mathbb{R}$. Thus, by Arzera-Ascoli theorem, for any given sequence $\{\psi_n\}_{n\ge1}$ in $T(\Gamma)$, there exist a subsequence $\{\psi_{n_k}\}_{k\ge1}$ and $\psi\in C(\mathbb{R},\mathbb{R})$ such that $\lim_{k\to\infty}\psi_{n_k}(\xi)=\psi(\xi)$ uniformly for ξ in any compact subset of $\mathbb{R}$. Since $\phi^-(\xi)\le\psi_{n_k}(\xi)\le\bar\phi^+(\xi)$ for $\xi\in\mathbb{R}$, we have $\phi^-(\xi)\le\psi(\xi)\le\bar\phi^+(\xi)$ for $\xi\in\mathbb{R}$ and $\psi\in\Gamma$. Since $\lambda\in(0,\lambda_1)$, it is easy to see that

$$
\lim_{\xi\to\infty}(\bar\phi^+(\xi)-\phi^-(\xi))e^{-\lambda\xi}=0,
$$
$$
\lim_{\xi\to-\infty}(\bar\phi^+(\xi)-\phi^-(\xi))e^{-\lambda\xi}=0
$$

which imply that for any $\epsilon>0$, there exists $B>0$ such that

$$
0\le(\bar\phi^+(\xi)-\phi^-(\xi))e^{-\lambda\xi}\le\epsilon,\quad\text{for }|\xi|\ge B. \tag{2.18}
$$

Since $\lim_{k\to\infty}(\psi_{n_k}(\xi)-\psi(\xi))e^{-\lambda\xi}=0$ uniformly for $\xi\in[-B,B]$, there exists a $k'\in\mathbb{N}_+$ such that

$$
|\psi_{n_k}(\xi)-\psi(\xi)|e^{-\lambda\xi}<\epsilon, \tag{2.19}
$$

for $\xi\in[-B,B]$, $k\ge k'$. It follows from (2.18) and (2.19) that

$$
\|\psi_{n_k}(\xi)-\psi(\xi)\|_\lambda=\sup_{\xi\in\mathbb{R}}|\psi_{n_k}(\xi)-\psi(\xi)|e^{-\lambda\xi}\le\epsilon,\quad\text{for }k\ge k'.
$$

Thus, $\lim_{k\to\infty}\psi_{n_k}=\psi$ in X_λ. This completes the proof. □

Proof of Theorem 2.3(i). By Lemmas 2.9 and 2.12, the Schauder's fixed point theorem implies that there exists $\phi\in\Gamma$ such that $\phi=T(\phi)$ and hence ϕ is a travelling wave of (1.2). Since $0\le\phi^-(\xi)\le\phi(\xi)\le\bar\phi^+(\xi)\le K^*$ for $\xi\in\mathbb{R}$, we can obtain

$$
\lim_{\xi\to-\infty}\phi(\xi)=0,\quad\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda_1\xi}=1.
$$

Moreover,

$$
K_*=\lim_{\xi\to\infty}\phi^-(\xi)\le\liminf_{\xi\to\infty}\phi(\xi)\le\limsup_{\xi\to\infty}\phi(\xi)\le\lim_{\xi\to\infty}\bar\phi^+(\xi)=K^*, \tag{2.20}
$$

that is, $K_*\le\alpha\le\beta\le K^*$. Next, we shall show that $\alpha\le K\le\beta$. Indeed, if $\alpha=\beta$, then $\lim_{\xi\to\infty}\phi(\xi)=\alpha$ exists. Taking $\xi\to\infty$ and applying L'Hospital's rule to (2.1), we can obtain $\alpha=g(\alpha)$ which yields $\alpha=\beta=K$.

Now we consider that $\alpha<\beta$. If there is a large number $M_2>0$ such that $\phi'>0$ on $[M_2,\infty)$ or $\phi'<0$ on $[M_2,\infty)$, then $\lim_{\xi\to\infty}\phi(\xi)$ exists and hence $\alpha=\beta$, which is impossible. Thus, $\phi(\xi)$ is oscillating on positive infinity and then there exist two sequences $\{\xi_j\}_{j\in\mathbb{N}}$ with $\xi_j \to \infty$ and $\{y_i\}_{i\in\mathbb{N}}$ with $y_i \to \infty$ as $i \to \infty$ such that $\phi(\xi_j) \to \alpha$ as $j \to \infty$, and $\phi(y_i) \to \beta$ as $i \to \infty$.

By (A1), for any $\epsilon > 0$, there are sufficiently large numbers $\tau_0 > 0$ and $J_0 > 0$ such that

$$K^* \int_{\tau_0}^{\infty} \sum_{j\in\mathbb{Z}} A_j(\tau)d\tau < \frac{\epsilon}{3} \quad \text{and} \quad K^* \int_0^{\tau_0} \sum_{|j|\geq J_0} A_j(\tau)d\tau < \frac{\epsilon}{3}.$$

Since g is continuous, there exists $\delta > 0$ such that

$$\max\{g(u) : u \in [\alpha-\delta, \beta+\delta]\} < \max\{g(u) : u \in [\alpha,\beta]\} + \frac{\epsilon}{3}.$$

For such a $\delta > 0$, there exists a large enough number $M_3 > 0$ such that

$$\phi(\xi) \in [\alpha-\delta, \beta+\delta], \quad \text{for all } \xi \geq M_3.$$

We take a large enough integer $M_4 > 0$ such that $\xi_m \geq M_3 + J_0 + c\tau_0$ for all $m \geq M_4$. Thus, for $m \geq M_4$, we have

$$\begin{aligned}
&\phi(\xi_m)\\
&= \int_0^{\infty} \sum_{j\in\mathbb{Z}} A_j(\tau)g(\phi(\xi_m - j - c\tau))d\tau\\
&= \int_{\tau_0}^{\infty} \sum_{j\in\mathbb{Z}} A_j(\tau)g(\phi(\xi_m - j - c\tau))d\tau + \int_0^{\tau_0} \sum_{|j|\geq J_0} A_j(\tau)g(\phi(\xi_m - j - c\tau))d\tau\\
&\quad + \int_0^{\tau_0} \sum_{|j|<J_0} A_j(\tau)g(\phi(\xi_m - j - c\tau))d\tau\\
&\leq K^* \int_{\tau_0}^{\infty} \sum_{j\in\mathbb{Z}} A_j(\tau)d\tau + K^* \int_0^{\tau_0} \sum_{|j|\geq J_0} A_j(\tau)d\tau + \max\{g(u) : u \in [\alpha-\delta, \beta+\delta]\}\\
&< \max\{g(u) : u \in [\alpha,\beta]\} + \epsilon.
\end{aligned}$$

Taking the limit as $m \to \infty$, we have

$$\beta \leq \max\{g(u) : u \in [\alpha,\beta]\} + \epsilon.$$

Thus, letting $\epsilon \to 0^+$, it follows that

$$\beta \leq \max\{g(u) : u \in [\alpha,\beta]\}. \tag{2.21}$$

Similarly, it follows that

$$\alpha \geq \min\{g(u) : u \in [\alpha,\beta]\}. \tag{2.22}$$

If $\alpha < \beta \leq K$, then (2.22) and (G2) imply that $\alpha \geq \min\{g(u) : \alpha \leq u \leq \beta\} > \alpha$, which is a contradiction. If $K \leq \alpha < \beta$, then (2.21) and (G2) also imply that $\beta \leq \max\{g(u) : \alpha \leq u \leq \beta\} < \beta$, a contradiction. Hence we conclude that $\alpha < K < \beta$.

(ii). Let $u_1, u_2 \in [\alpha,\beta]$ such that $g(u_1) = \max\{g(u) : \alpha \leq u \leq \beta\}$ and $g(u_2) = \min\{g(u) : \alpha \leq u \leq \beta\}$. Then it is split into three cases.

Case 1. $K \leq u_1 \leq \beta$. If $u_1 = \beta$, according to (2.21), $\beta \leq g(\beta)$, which is invalid since $\beta > K$. Therefore $u_1 < \beta$. By (2.21) and $u_1 \geq K$, we have $\beta \leq g(u_1) \leq u_1 < \beta$, which is contradiction.

Case 2. $\alpha \leq u_2 \leq K$. If $u_2 = \alpha$, according to (2.22), $\alpha \geq g(\alpha)$, which is impossible since $\alpha < K$. Therefore $\alpha < u_2$. By (2.22) and $u_2 \leq K$, we have $\alpha \geq g(u_2) \geq u_2 > \alpha$, which is also contradiction.

Case 3. $u_1 < K < u_2$. By (2.21)-(2.22) and (G5), we have

$$\beta - \alpha \le g(u_1) - g(u_2) < (2K - u_1) - (2K - u_2) = u_2 - u_1,$$

which is impossible.

Thus, $\alpha = \beta$ and hence the limit $\lim_{\xi\to\infty} \phi(y) = \alpha \in [K_*, K^*]$ exists. Taking $y \to \infty$ and applying L'Hospital's rule to (2.1), we can obtain $\alpha = g(\alpha)$ which yields $\alpha = K$.

(iii). In the case where $c = c_*$, we can obtain the existence of travelling waves by using a limiting argument similar to [9, 19, 24]. Let $\{c_m\} \subset (c_*, c_* + 1]$ with $\lim_{m\to\infty} c_m = c_*$. Since $c_m > c_*$, (2.1) with $c = c_m$ admits a solution $\phi_m \in C(\mathbb{R}, [0, K^*])$ such that

$$\lim_{\xi\to-\infty} \phi_m(\xi) = 0, \quad \liminf_{\xi\to\infty} \phi_m(\xi) \ge K_*.$$

Without loss of generality, we may assume that $\phi_m(0) = \frac{1}{2}K_*$ for $m \in \mathbb{N}_+$. Note that

$$\phi_m(\xi) = \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g(\phi_m(\xi - j - c_m\tau))d\tau, \text{ for } \xi \in \mathbb{R},\ m \in \mathbb{N}_+. \tag{2.23}$$

It is easy to see that

$$|\phi_m(\xi_1) - \phi_m(\xi_2)| \le g'(0)K^*\mathbb{A}(\xi_1 - \xi_2) + \frac{g'(0)K^*M_1}{c_m}|\xi_1 - \xi_2|,$$

where $\mathbb{A}(\xi)$ and M_1 are defined in the proof of Lemma 2.12. Therefore, $\{\phi_m(\xi)\}$ is uniformly bounded and equicontinuous in $\xi \in \mathbb{R}$. Using Arzera-Ascoli theorem, we can obtain a subsequence $\{\phi_{m_k}\}$ and ϕ such that $\lim_{k\to\infty} \phi_{m_k}(\xi) = \phi(\xi)$ uniformly for ξ in any compact subset of $\mathbb{R}$. Clearly, $\phi(0) = \frac{1}{2}K_*$ and $\phi \in C(\mathbb{R}, [0, K^*])$. By the dominated convergence theorem and (2.21), it follows that

$$\phi(\xi) = \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g(\phi(\xi - j - c_*\tau))d\tau, \quad \text{for } \xi \in \mathbb{R}.$$

and hence, $\phi(n + c_*t)$ is a travelling wave of (1.2). This completes the proof. □

Remark 2.14. Note that Γ in the present paper and the set Y in [9] are different. We can obtain (2.20) by the construction of a suitable set Γ. The conclusion in [9] similar to (2.20) must be obtained by the property of the spreading speed (see, [9, Theorem 2.2]) since it could not be obtained by the construction of Y.

Proof of Theorem 2.4(i). Assume that (1.2) has a nonnegative bounded solution ϕ with (2.2). It is obvious that $\phi \not\equiv$ constant. Letting $\|\phi\|_\infty = \sup_{\xi\in\mathbb{R}} |\phi(\xi)| > 0$ and according to (2.1), we have

$$\begin{aligned}
\|\phi\|_\infty &= \sup_{\xi\in\mathbb{R}} |\phi(\xi)| \\
&\le \sup_{\xi\in\mathbb{R}} \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau) g'(0)\phi(\xi - j - c\tau)d\tau \\
&< \int_0^\infty \sum_{j\in\mathbb{Z}} A_j(\tau)\|\phi\|_\infty d\tau \\
&= \|\phi\|_\infty
\end{aligned}$$

which is a contradiction. This completes the proof. □

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