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NEUMANN BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. In this article, a fixed point theorem is used to investigate the existence of solutions for differential inclusions, with Neumann boundary conditions, in Banach spaces.

1. Introduction

The aim of this article is to establish the existence of solutions to the boundary-value problem

$$\begin{gathered} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{a.e. on } [0,1] \\ \dot{x}(0) = r, \quad \dot{x}(1) = s, \end{gathered} \tag{1.1}$$

where F is a closed multifunction, measurable in the first argument and Lipschitz continuous in the second argument; and r, s are in a Banach space E.

Neumann boundary-value problems have received the attention of many authors. For review of results on Neumann boundary-value problems for differential equations, we refer the reader to the papers by Boucherif and Al-malki [2], Wang, Cui and Zhang [10], Mawhin and Ruiz [7], Guennoun [6], Granas, Guenther and Lee [5] and the references therein. The techniques involved, in most of these works, are based on the upper and lower solution method, the topological degree, and the topological transversality theorem.

In the literature there are a few papers dealing with the existence of solutions for Neumann boundary-value problems for differential inclusions. Our main purpose, in this paper, is to obtain the existence of solutions to (1.1), in infinite dimensional space, and in the case when the multifunction F is nonconvex. We used the fixed point theorem introduced by Covitz and Nadler for contraction multi-valued maps.

The paper is organized as follows. In Section 2 we recall some preliminary facts that we need in the sequel while in Section 3, we prove our main result.

2. Preliminaries and statement of the main results

Let E be a real separable Banach space with the norm $\|\cdot\|$. We denote by $\mathcal{C}([0,1], E)$ the Banach space of continuous functions from $[0,1]$ to E equipped with the norm $\|x(\cdot)\|_\infty := \sup\{\|x(t)\|; t \in [0,1]\}$. For $x \in E$ and for nonempty sets

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A, B of E we denote $d(x,A) = \inf\{d(x,y); y \in A\}$, $e(A,B) := \sup\{d(x,B); x \in A\}$ and $H(A,B) := \max\{e(A,B), e(B,A)\}$. A multifunction is said to be measurable if its graph is measurable. For more detail on measurability theory, we refer the reader to the book of Castaing-Valadier [3].

For the sake of completeness, we recall the following results that will be used in the sequel.

Definition 2.1. Let $T : E \to 2^E$ be a multi function with closed values

- T is k-Lipschitz if $H\big(T(x), T(y)\big) \leq k\|x-y\|$ for each $x, y \in E$.
- T is a contraction if it is k-Lipschitz with $k < 1$.
- T has a fixed point if there exists $x \in E$ such that $x \in T(x)$.

Lemma 2.2 ([4])**.** *If $T : E \to 2^E$ is a contraction with nonempty closed values, then it has a fixed point.*

Lemma 2.3 ([11])**.** *Assume that $F : [a,b] \times E \to 2^E$ is a multi function with nonempty closed values satisfying:*

- *For every $x \in E$, $F(., x)$ is measurable on $[a,b]$;*
- *For every $t \in [a,b]$, $F(t, .)$ is (Hausdorff) continuous on E.*

Then for any measurable function $x(\cdot) : [a,b] \to E$, the multi function $F(., x(\cdot))$ is measurable on $[a,b]$.

Definition 2.4. A measurable multi-valued function $F : [0,1] \to 2^E$ is said to be integrably bounded if there exists a function $h \in L^1([0,1], E)$ such that for all $v \in F(t)$, $\|v\| \leq h(t)$ for almost every $t \in [0,1]$.

Definition 2.5. A function $x(\cdot) : [0,1] \to E$ is said to be a solution of (1.1) if $x(\cdot)$ is absolutely continuous on $[0,1]$ and satisfies (1.1).

Let $G(t,s)$ be the Green function associated with the problem

$$\begin{gathered} -\ddot{y}(t) + y(t) = 0, \quad t \in [0,1], \\ \dot{y}(0) = \dot{y}(1) = 0, \end{gathered}$$

(see [8, 9]) which is explicitly given by

$$G(t,s) = \begin{cases} \frac{\cosh(1-s)\cosh(t)}{\sinh(1)}, & 0 \leq t \leq s \leq 1, \\ \frac{\cosh(1-t)\cosh(s)}{\sinh(1)}, & 0 \leq s \leq t \leq 1, \end{cases}$$

where $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh(x) = (e^x - e^{-x})/2$. Obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$ and $0 \leq G(t,s) \leq \cosh^2(1)/\sinh(1) = \lambda$, for each $t, s \in [0,1]$. We shall prove the following theorem.

Theorem 2.6. *Let $F : [0,1] \times E \times E \to 2^E$ be a set-valued map with nonempty closed values satisfying*

(i) *For each $(x,y) \in E \times E$, $t \mapsto F(t,x,y)$ is measurable and integrably bounded;*
(ii) *There exists a function $m(\cdot) \in L^1([0,1], \mathbb{R}^+)$ such that for all $t \in [0,1]$ and for all $x_1, x_2, y \in E$*

$$H\big(F(t,x_1,y), F(t,x_2,y)\big) \leq m(t)\|x_1 - x_2\|.$$

Then, if $\int_0^1 (1 + m(s))ds < 1/\lambda$, for all $r, s \in E$, Problem (1.1) has at least one solution on $[0,1]$.

3. Proof of the main result

Let r, s be in E. We introduce first the function $\rho : [0,1] \to E$ defined by

$$\rho(t) = \frac{1}{2}(s-r)t^2 + rt, \quad \forall t \in [0,1],$$

and the multifunction $H : [0,1] \times \mathcal{C}([0,1], E) \to 2^E$ defined by

$$H(t, y(\cdot)) = y(t) - F(t, y(t) + \rho(t), \dot{y}(t) + \dot{\rho}(t)) + (s-r) \tag{3.1}$$

for all $(t, y(\cdot)) \in [0,1] \times \mathcal{C}([0,1], E)$. Consider the problem:

$$\begin{gathered} -\ddot{y}(t) + y(t) \in H(t, y(\cdot)), \quad \text{a.e. on } [0,1] \\ \dot{y}(0) = 0, \quad \dot{y}(1) = 0. \end{gathered} \tag{3.2}$$

We should point out that the function $y(\cdot)$ is a solution of (3.2), if and only if the function $x(t) = y(t) + \rho(t)$ is a solution of (1.1), for all $t \in [0,1]$.

Next, by Lemma 2.3, for $y(\cdot) \in \mathcal{C}([0,1], E)$, $F(\cdot, y(\cdot), \dot{y}(\cdot))$ is closed and measurable, then it has a measurable selection which, by hypothesis (i), belongs to $L^1([0,1], E)$. Thus the set

$$S_{F,y(\cdot)} := \Big\{ f \in L^1([0,1], E) : f(t) \in F(t, y(t), \dot{y}(t)) \text{ for a.e. } t \in [0,1] \Big\}$$

is nonempty. Let us transform problem (3.2) into a fixed point problem. Consider the multi-valued map,

$$T : \mathcal{C}([0,1], E) \to 2^{\mathcal{C}([0,1], E)}$$

defined as follows, for $y(\cdot) \in \mathcal{C}([0,1], E)$,

$$T(y(\cdot)) = \Big\{ z(\cdot) \in \mathcal{C}([0,1], E) : z(t) = \int_0^1 G(t,s) h(s) ds, \forall t \in [0,1], h \in S_{H,y(\cdot)} \Big\},$$

where

$$S_{H,y(\cdot)} := \Big\{ h \in L^1([0,1], E) : h(t) \in H(t, y(\cdot)) \text{ for a.e. } t \in [0,1] \Big\}.$$

We shall show that T satisfies the assumptions of Lemma 2.2. The proof will be given in two steps:

Step 1: T has non-empty closed-values. Indeed, let $(y_p(\cdot))_{p \geq 0} \in T(y(\cdot))$ such that $(y_p(\cdot))_{p \geq 0}$ converges to $\bar{y}(\cdot)$ in $\mathcal{C}([0,1], E)$. Then $\bar{y}(\cdot) \in \mathcal{C}([0,1], E)$ and for each $t \in [0,1]$,

$$y_p(t) \in \int_0^1 G(t,s) H(s, y(\cdot)) ds,$$

where $\int_0^1 G(t,s) H(s, y(\cdot)) ds$ is the Aumann's integral of $G(t, .)H(., y)$, which is defined as

$$\int_0^1 G(t,s) H(s, y(\cdot)) ds = \Big\{ \int_0^1 G(t,s) h(s) ds, \ h \in S_{H,y(\cdot)} \Big\}.$$

Using the fact that the set-valued map F is closed and from (3.1), we conclude that the set

$$\int_0^1 G(t,s) H(s, y(\cdot)) ds$$

is closed for all $t \in [0,1]$. Then

$$\bar{y}(t) \in \int_0^1 G(t,s) H(s, y(\cdot)) ds.$$

So, there exists $h \in S_{H,y(\cdot)}$ such that

$$\bar{y}(t) = \int_0^1 G(t,s)h(s)ds.$$

Hence $\bar{y}(\cdot) \in T(y(\cdot))$. So $T(y(\cdot))$ is closed for each $y(\cdot) \in \mathcal{C}([0,1],E)$.
Step 2: T is a contraction. Indeed, let $y_1(\cdot), y_2(\cdot) \in \mathcal{C}([0,1],E)$ and consider $z_1(\cdot) \in T(y_1(\cdot))$. Then there exists $h_1 \in S_{H,y_1(\cdot)}$ such that

$$z_1(t) = \int_0^1 G(t,s)h_1(s)ds, \quad \forall t \in [0,1].$$

Using (3.1), there exists $f_1 \in S_{F,y_1(\cdot)}$ such that

$$h_1(t) = y_1(t) - f_1(t) + (s-r), \quad \forall t \in [0,1].$$

On the other hand, let $\varepsilon > 0$ and consider the valued map $U_\varepsilon : [0,1] \to 2^E$, given by

$$U_\varepsilon(t) = \big\{x \in E : \|f_1(t) - x\| \le m(t)\|y_1(t) - y_2(t)\| + \varepsilon\big\}.$$

We claim that $U_\varepsilon(t)$ is nonempty, for each $t \in [0,1]$. Indeed, let $t \in [0,1]$, we have

$$H\big(F(t,y_1(t),\dot{y}_1(t)), F(t,y_2(t),\dot{y}_2(t))\big) \le m(t)\|y_1(t) - y_2(t)\|.$$

Hence, there exists $x \in F(t,y_2(t),\dot{y}_2(t))$, such that

$$\|f_1(t) - x\| \le m(t)\|y_1(t) - y_2(t)\| + \varepsilon.$$

By [3, Theorem III.40], the multifunction

$$V : t \to U_\varepsilon(t) \cap F(t,y_2(t),\dot{y}_2(t)) \quad \text{is measurable.} \tag{3.3}$$

Then there exists a measurable selection for V denoted f_2 such that, for all $t \in [0,1]$,

$$f_2(t) \in F(t,y_2(t),\dot{y}_2(t))$$

and

$$\|f_1(t) - f_2(t)\| \le m(t)\|y_1(t) - y_2(t)\| + \varepsilon.$$

Now, for all $t \in [0,1]$, set $h_2(t) = y_2(t) - f_2(t) + (s-r)$ and

$$z_2(t) = \int_0^1 G(t,s)h_2(s)ds.$$

We have

$$\begin{aligned}
\|z_1(t) - z_2(t)\| &\le \int_0^1 \|G(t,s)\|\|h_1(s) - h_2(s)\|ds \\
&\le \lambda \int_0^1 \|y_1(s) - y_2(s)\|ds + \lambda \int_0^1 \|f_1(s) - f_2(s)\|ds \\
&\le \lambda \int_0^1 \|y_1(s) - y_2(s)\|ds + \lambda \int_0^1 m(s)\|y_1(s) - y_2(s)\|ds + \lambda\varepsilon \\
&\le \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s))ds + \lambda\varepsilon.
\end{aligned}$$

So, we conclude that

$$\|z_1(\cdot) - z_2(\cdot)\|_\infty \le \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s))ds + \lambda\varepsilon.$$

By an analogous relation, obtained by interchanging the roles of $y_1(\cdot)$ and $y_2(\cdot)$, it follows that

$$H\big(T(y_1(\cdot)), T(y_2(\cdot))\big) \le \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s))ds + \lambda\varepsilon.$$

By letting $\varepsilon \to 0$, we obtain

$$H\big(T(y_1(\cdot)), T(y_2(\cdot))\big) \le \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s))ds.$$

Consequently, if $\int_0^1 (1 + m(s))ds < \frac{1}{\lambda}$, T is a contraction. By Lemma 2.2, T has a fixed point which is a solution of (3.2). The proof is complete.

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