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REGULARITY OF GENERALIZED NAVEIR-STOKES EQUATIONS IN TERMS OF DIRECTION OF THE VELOCITY

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Abstract. In this article, the author studies the regularity of 3D generalized Navier-Stokes (GNS) equations with fractional dissipative terms $(-\Delta)^\alpha u$. It is proved that if $\operatorname{div}(u/|u|) \in L^p(0,T;L^q(\mathbb{R}^3))$ with

$$\frac{2\alpha}{p} + \frac{3}{q} \le 2\alpha - \frac{3}{2}, \quad \frac{6}{4\alpha - 3} < q \le \infty.$$

then any smooth on GNS in $[0,T)$ remains smooth on $[0,T]$.

1. Introduction

We consider the incompressible generalized Navier-Stokes equation (GNS)

$$\begin{gathered} u_t + u \cdot \nabla u + (-\Delta)^\alpha u = -\nabla p \\ \nabla \cdot u = 0 \end{gathered} \tag{1.1}$$

Where $u = u(x,t)$ denotes the velocity field, $p = p(x,t)$ the scalar pressure and $u_0(x)$ with $\nabla \cdot u_0 = 0$ in the sense of distribution is the initial velocity field. The fractional power of Laplace operator $(-\Delta)^\alpha$ is defined as in [13]:

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi).$$

For notational convenience, we write $\Lambda = (-\Delta)^{1/2}$.

When $\alpha = 1$, (1.1) become the usual Navier-Stokes equations. Up to now, it is still unknown whether or not there exist global solution for Navier-Stokes equations even if the initial data is sufficiently smooth. This famous problem lead to extensively study the regularity of Navier-Stokes equations.

There are plenty of literatures for usual Navier-Stokes equations, we mentioned some of them. For well-posedness, the readers could refer to Leray[9], Kato[7], Cannone[3], Giga and Miyakawa[6] and Taylor[14]. For regularity results, one could refer to Serrin[12], Kozono and Sohr[8], Beale, Kato and Majda[1], Constantin and Feffernan[5].

For general α, Wu[17] proved that if $u_0 \in L^2$, then the GNS (1.1) posses a weak solution u satisfying $u \in L^\infty([0,T];L^2) \cap L^2([0,T];H^\alpha)$. Moreover, he showed that all solutions are global if $\alpha \ge 1/2 + n/4$, where n is space dimension. For $\alpha < 1/2 + n/4$, Wu[18] studied the local well-posedness of (1.1) in Besov spaces. For

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the regularity of GMHD equations, Wu[19] obtained some Serrin's type criterion. Latter, Zhou[21], Wu[16] and Luo[10] improved some results of Wu. These results can also be used for GNS equations for GMHD equations contains GNS equations.

In this short paper, we studied the regularity to GNS equations in terms of the direction of velocity which is used firstly by Vasseur[15]. He showed that if the initial value $u_0 \in L^2(\mathbb{R}^3)$, and $\operatorname{div}(u/|u|) \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with

$$\frac{2}{p} + \frac{3}{q} \le \frac{1}{2}, q \ge 6, p \ge 4$$

then u is smooth on $(0,\infty) \times \mathbb{R}^3$. Latter, Luo[11] extended this result to MHD equations.

The main result of this paper is as follows.

Theorem 1.1. *Let $\frac{3}{4} < \alpha < \frac{3}{2}, u_0 \in H^1(\mathbb{R}^3)$, u is a smooth solution of (1) in $[0,T)$. If $\operatorname{div}(u/|u|) \in L^p(0,T; L^q(\mathbb{R}^3))$ with*

$$\frac{2\alpha}{p} + \frac{3}{q} \le 2\alpha - \frac{3}{2}, \quad \frac{6}{4\alpha - 3} < q \le \infty.$$

then u remains smooth in $[0,T]$.

To prove this theorem, we need the following result.

Lemma 1.2. *With $0 < \alpha < 2$, $\theta, \Lambda^\alpha\theta \in L^p$ with $p = 2^k$ we obtain*

$$\int |\theta|^{p-2}\theta\Lambda^\alpha\theta dx \ge \frac{1}{p}\int |\Lambda^{\frac{\alpha}{2}}\theta^{\frac{p}{2}}|^2 dx.$$

The proof is similar with Córdoba and Córdoba [4], readers can find the details in Wu[20].

2. Proof of the main result

Multiplying both side of the equations by $|u|^2u$, and integrating by parts, we obtain

$$\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \int_{\mathbb{R}^3} |u|^2u\cdot(-\Delta)^\alpha u dx = 2\int_{\mathbb{R}^3} p|u|u\cdot\nabla|u|dx \tag{2.1}$$

By Lemma 1.2, the left side satisfies

$$\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \int_{\mathbb{R}^3} |u|^2u\cdot(-\Delta)^\alpha u dx \ge \frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \int_{\mathbb{R}^3} |\Lambda^\alpha|u|^2|^2 dx. \tag{2.2}$$

So we obtain

$$\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \|\Lambda^\alpha|u|^2\|_{L^2}^2 dx \le 2\int_{\mathbb{R}^3} p|u|u\cdot\nabla|u|dx. \tag{2.3}$$

Taking the divergence of (1.1), one has

$$-\Delta p = \sum_{i,j}\partial_i\partial_j(u_i, u_j),$$

by Calderon-Zygmund inequality, we have

$$\|p\|_{L^p} \le C\|u\|_{L^{2p}}^2.$$

Then, using Hölder inequality, one obtains

$$\begin{aligned}\int_{\mathbb{R}^3} p|u|u\cdot\nabla|u|dx &= \int_{\mathbb{R}^3}|p||u|^2|\frac{u}{|u|}\cdot\nabla|u||dx\\ &\le \|p\|_{L^r}\|u\|_{L^{2r}}^2\||u|\mathrm{div}(u/|u|)\|_{L^{q_1}}\\ &\le C\|u\|_{L^{2r}}^4\||u|\mathrm{div}(u/|u|)\|_{L^{q_1}},\end{aligned}$$

where $2/r+1/q_1=1$. Here we used the fact

$$|u|\,\mathrm{div}(u/|u|) = -\frac{u}{|u|}\cdot\nabla|u|.$$

By the interpolation inequality and the Sobolev embedding theorem [2], we have

$$\begin{aligned}\|u\|_{L^{2r}} &\le C\|u\|_{L^4}^{1-\theta}\|u\|_{L^{2s}}^{\theta}\\ &= C\|u\|_{L^4}^{1-\theta}\||u|^2\|_{L^s}^{\theta/2}\\ &\le C\|u\|_{L^4}^{1-\theta}\|\Lambda^\alpha|u|^2\|_{L^2}^{\theta/2},\end{aligned}$$

where

$$\frac{1-\theta}{4}+\frac{\theta}{2s}=\frac{1}{2r},\quad s=\frac{6}{3-2\alpha}, \tag{2.4}$$

$$2<r<s,\quad 0<\theta<1. \tag{2.5}$$

So we obtain

$$\begin{aligned}2\int_{\mathbb{R}^3}|p||u|^2|\frac{u}{|u|}\cdot\nabla|u||dx &\le C\|u\|_{L^4}^{4(1-\theta)}\|\Lambda^\alpha|u|^2\|_{L^2}^{2\theta}\||u|\mathrm{div}(u/|u|)\|_{L^{q_1}}\\ &\le C\||u|\mathrm{div}(u/|u|)\|_{L^{q_1}}^{\frac{1}{1-\theta}}\|u\|_{L^4}^4+\frac{1}{2}\|\Lambda^\alpha|u|^2\|_{L^2}^2,\end{aligned} \tag{2.6}$$

where the last inequality is deduced from Young's inequality.

Combining (2.1)-(2.6), we have

$$\frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4+\|\Lambda^\alpha|u|^2\|_{L^2}^2dx \le C\||u|\mathrm{div}(u/|u|)\|_{L^{q_1}}^{\frac{1}{1-\theta}}\|u\|_{L^4}^4+\frac{1}{2}\|\Lambda^\alpha|u|^2\|_{L^2}^2.$$

If $|u|\,\mathrm{div}(u/|u|)\in L^{p_1,q_1}$ and $1/(1-\theta)\le p_1$, then by Gronwell's inequality, we can claim that the smooth solution in $[0,T)$ remains smooth in $[0,T]$. Now we search for the conditions which ensure $|u|\,\mathrm{div}(u/|u|)\in L^{p_1,q_1}$ and $1/(1-\theta)\le p_1$.

Since $\theta\in(0,1)$ and r,q_1,s,θ satisfy

$$\frac{2}{r}+\frac{1}{q_1}=1,\quad \frac{1-\theta}{4}+\frac{\theta}{2s}=\frac{1}{2r},$$

$$2<r<s,\quad s=\frac{6}{3-2\alpha},$$

we obtain

$$\frac{1}{1-\theta}=\frac{2\alpha q_1}{2\alpha q_1-3}.$$

That is, if

$$\frac{2\alpha}{p_1}+\frac{3}{q_1}\le 2\alpha,$$

then $1/(1-\theta)\le p_1$.

Let $\operatorname{div}(u/|u|) \in L^{p,q}$. We know $u \in L^\infty([0,T];L^2) \cap L^2([0,T];H^\alpha)$ and thus $u \in L^{a,b}$ with $2\alpha/a + 3/b = 3/2$. So we obtain $|u| \operatorname{div}(u/|u|) \in L^{p_1,q_1}$ with

$$\frac{1}{p_1} = \frac{1}{a} + \frac{1}{p}, \quad \frac{1}{q_1} = \frac{1}{b} + \frac{1}{q}.$$

From this relation, we obtain, if $2\alpha/p + 3/q \le 2\alpha - 3/2$, then

$$\frac{2\alpha}{p_1} + \frac{3}{q_1} \le 2\alpha.$$

That is, if

$$\frac{2\alpha}{p} + \frac{3}{q} \le 2\alpha - \frac{3}{2},$$

then $|u| \operatorname{div}(u/|u|) \in L^{p_1,q_1}$. And the condition $\frac{6}{4\alpha-3} < q \le \infty$ ensures the inequality $q_1 > \frac{3}{2\alpha}$, which implies

$$2 < r < s, \quad 0 < \theta < 1, \quad s = \frac{6}{3-2\alpha}.$$

This completes the proof.

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