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BOUNDS FOR SOLUTIONS OF NONLINEAR SINGULAR INTERFACE PROBLEMS ON TIME SCALES USING A MONOTONE ITERATIVE METHOD

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Abstract. In this article, we give bounds for solutions to initial-value problems associated with nonlinear singular interface problems. The singular interface problem is described using a pair of dynamic equations on a time scale. The method of upper and lower solutions intertwined with monotone iterative technique is used.

1. Introduction

Solving boundary-value problems with different types of singularities has remained a challenge for mathematicians over the ages. While "regular" problems, those over finite intervals with well-behaved coefficients pose no difficulties, there are applications wherein either the domain of the problem is not well defined, or the continuity and/or smoothness of the functions, coefficients involved are not guaranteed in some parts of the domain, sometimes in the boundary or parts of the boundary. In all such cases the problem is considered to be a "singular" problem. The definition of the problem and therefore the description of the solution becomes a highly difficult task.

In the literature we find a class of interface problems, termed as mixed pair of equations, discussed in the papers [5], [9]–[13], [19]–[25] where two different differential equations are defined on two adjacent intervals and the solutions satisfy a matching condition at the point of interface. These problems are called as matching interface problems. If the boundary is well defined then we call the problem to be a regular interface problem. These interface problems with singularities in the domain are always of great interest.

We see that these interface problems for regular case has been discussed in [19]–[25] and the problem of having singularity at the boundary is discussed in [5]. In [5], authors discuss an application of the classical Weyl limit criterion to define the coefficients with well-known Wronskian boundary conditions to tackle the singularity at the boundary for this class of problems. Though this work is specifically for

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Sturm-Liouville problems, it paves a way to study the problem of singularity at the end boundary points.

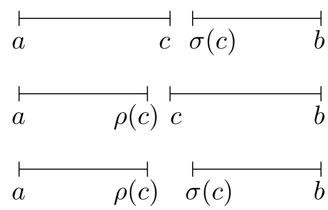
From the above we see that the regular interface problems and interface problems with singularity at the boundary are dealt in detail. But the problem of having a singularity at the point of interface seems to be less explored. Study of these problems using classical analytical tools is tedious. We term these problems as singular interface problems [6]–[8],[16]–[17].

The singularity at the point of interface in the domain of definition of the mixed pair of equations could be of the following three types satisfying certain matching conditions at the singular interface.

Interface 1: $[a, c] \cup [\sigma(c), b]$

Interface 2: $[a, \rho(c)] \cup [c, b]$

Interface 3: $[a, \rho(c)] \cup [\sigma(c), b]$



To describe the singularities in the domain of definition we take help of the terminology used on Time Scale [3]. The new framework of the dynamic equations on time scale with facilities of the two jump operators with various definitions of continuity and derivatives make one's job simple to study the interface problems with mixed operators along with a singular interface. Recently we have worked on the linear singular interface problems as seen in [6]–[8],[16]–[17]. Here we discuss the corresponding nonlinear problem.

The method of lower and upper solutions is one of the commonly used methods for dealing with the second order initial and boundary value problems. It has its origin as early as 1893 [15]. Also this method of lower and upper solutions clubbed with the monotone iterative technique is used in the existence theory for nonlinear problems. A good introduction covering different aspects for the monotone iterative methods is given by Lakshmikantham and others in [4].

Lower and upper solutions give bounds for solutions which are improved iteratively using monotone iterative process. This method of lower and upper solutions for separated BVPs on time scales was developed recently by Akin in [1].

In this article we give bounds for an IVP associated with nonlinear singular interface problems. The singular interface problem is described using a pair of dynamic equations on a time scale. The method of upper and lower solutions intertwined with monotone iterative technique is used. The solution is proved to be bounded between the minimal and maximal solutions.

2. Mathematical Preliminaries

Definitions 2.1-2.4 can be found in [3]. Let $\mathbb{T}$ be a time scale(an arbitrary closed subset of real numbers).

Definition 2.1. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered, while $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t$$

Definition 2.2. $\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{m\} & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty, \end{cases}$ where m is the left scattered maximum.

Definition 2.3. Let f be a function defined on $\mathbb{T}$. We say that f is delta differentiable at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon\, |\sigma(t) - s| \quad \text{for all } s \in \mathcal{N}$$

Definition 2.4.

$$f^\Delta(t) = \begin{cases} \lim_{s \to t, s \in \mathbb{T}} \frac{f(t)-f(s)}{t-s} & \text{if } \mu(t) = 0 \\ \frac{f(\sigma(t))-f(t)}{\mu(t)} & \text{if } \mu(t) > 0 \end{cases}$$

Remark 2.5. For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa 2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa 2} \to \mathbb{R}$. Similarly we define the higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa n} \to \mathbb{R}$.

Definition 2.6. For any $m, n \in \mathcal{C}(\mathbb{T}, \mathbb{R})$ we define the sector $[m, n]$ as

$$[m, n] = \{w \in \mathcal{C}(\mathbb{T}, \mathbb{R}) : m \leq w \leq n\}$$

where $\mathcal{C}(\mathbb{T}, \mathbb{R})$ denotes the space of continuous functions from $\mathbb{T}$ to $\mathbb{R}$.

Definition 2.7. Let $\mathbb{T}_1, \mathbb{T}_2$ be two time scales. Let $(u_1, u_2), (v_1, v_2) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$. By $(u_1, u_2) \leq (v_1, v_2)$ we mean

$$\begin{aligned} u_1(t) &\leq v_1(t) \quad \text{for } t \in \mathbb{T}_1 \\ u_2(t) &\leq v_2(t) \quad \text{for } t \in \mathbb{T}_2 \end{aligned}$$

3. Definition of the initial-value problem

Let $\mathbb{T}_1 = [0, a]_{\mathbb{T}}$(a time scale with end points 0 and a), $K_1 = [\sigma(a), l]_{\mathbb{T}}$(a time scale with end points $\sigma(a)$ and l), $\mathbb{T}_2 = {K_1}^{\kappa^2}$ where $a, \sigma(a)$, $l < +\infty$. Let $\mathcal{C}(\mathbb{T}_i \times \mathcal{C}(\mathbb{T}_i))$ denote the space of continuous functions whose first argument is on the time scale $\mathbb{T}_i$ and the second argument is the from the space of continuous functions $\mathcal{C}(\mathbb{T}_i)$, $i = 1, 2$. Also let (f_1, f_2) be nonlinear function tuple in $\mathcal{C}(\mathbb{T}_1 \times \mathcal{C}(\mathbb{T}_1)) \times \mathcal{C}(\mathbb{T}_2 \times \mathcal{C}(\mathbb{T}_2))$. In this paper we consider the following IVP associated with singular interface problem (IVP-SIP).

$$y_1^{\Delta\Delta}(t) = f_1(t, y_1), \quad t \in \mathbb{T}_1 \tag{3.1}$$

$$y_2^{\Delta\Delta}(t) = f_2(t, y_2), \quad t \in \mathbb{T}_2 \tag{3.2}$$

with the initial conditions

$$y_1(0) = 0 \tag{3.3}$$

$$y_1^{\Delta}(0) = 0 \tag{3.4}$$

followed by the matching interface conditions

$$\rho_1 y_1(a) = \rho_2 y_2(\sigma(a)) \tag{3.5}$$

$$\rho_3 y_1^{\Delta}(a) = \rho_4 y_2^{\Delta}(\sigma(a)), \quad \rho_i > 0,\ i = 1, 2, 3, 4. \tag{3.6}$$

4. Monotone Iterative Methods

We now define the Lower and Upper Solutions for the IVP-SIP in accordance with [2].

Definition 4.1. We call $(\alpha_{01}, \alpha_{02}) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ a lower solution for (3.1)-(3.6) if

$$\begin{gathered}
\alpha_{01}^{\Delta\Delta} \geq f_1(t, \alpha_{01}(t)), \quad t \in \mathbb{T}_1 \\
\alpha_{02}^{\Delta\Delta} \geq f_2(t, \alpha_{02}(t)), \quad t \in \mathbb{T}_2 \\
\alpha_{01}(0) = 0 \\
{\alpha_{01}}^{\Delta}(0) = 0
\end{gathered}$$

and $(\alpha_{01}, \alpha_{02})$ satisfies the interface conditions (3.5)-(3.6).

Definition 4.2. We call $(\beta_{01}, \beta_{02}) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ an upper solution for (3.1)-(3.6) if

$$\begin{gathered}
\beta_{01}^{\Delta\Delta} \leq f_1(t, \beta_{01}(t)), \quad t \in \mathbb{T}_1 \\
\beta_{02}^{\Delta\Delta} \leq f_2(t, \beta_{02}(t)), \quad t \in \mathbb{T}_2 \\
\beta_{01}(0) = 0 \\
{\beta_{01}}^{\Delta}(0) = 0
\end{gathered}$$

and (β_{01}, β_{02}) satisfies the interface conditions (3.5)-(3.6).

Definition 4.3. A pair of functions $(\gamma_1, \gamma_2) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ is called a minimal solution of IVP-SIP (3.1)-(3.6) if the following hold:

(i) (γ_1, γ_2) is a solution of (3.1)-(3.6)
(ii) for any other solution (l_1, l_2) of (3.1)-(3.6) we have

$$\begin{gathered}
\gamma_1(t) \leq l_1(t) \quad \text{for } t \in \mathbb{T}_1, \\
\gamma_2(t) \leq l_2(t) \quad \text{for } t \in \mathbb{T}_2.
\end{gathered}$$

Definition 4.4. A pair of functions $(k_1, k_2) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ is called a maximal solution of IVP-SIP (3.1)-(3.6) if

(i) (k_1, k_2) is a solution of (3.1)-(3.6)
(ii) for any other solution (r_1, r_2) of (3.1)-(3.6) we have

$$\begin{gathered}
r_1(t) \leq k_1(t) \quad \text{for } t \in \mathbb{T}_1, \\
r_2(t) \leq k_2(t) \quad \text{for } t \in \mathbb{T}_2.
\end{gathered}$$

We extend the *maximum* principle in [14] to the present IVP-SIP under consideration. We denote

$$\tilde{\mu}_1 = \sup_{t\in\mathbb{T}_1} \mu(t), \quad \tilde{\mu}_2 = \sup_{t\in\mathbb{T}_2} \mu(t).$$

Lemma 4.5. *Let $M > 0$ be such that if $\tilde{\mu}_1, \tilde{\mu}_2 > 0$,*

$$M < \frac{1}{\tilde{\mu}_1^2}, \quad M < \frac{1}{\tilde{\mu}_2^2}$$

and $(x_1, x_2) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ be such that

$$\begin{gathered} {x_1}^{\Delta\Delta}(t) \le Mx_1(t) \quad \text{for } t \in \mathbb{T}_1 \\ {x_2}^{\Delta\Delta}(t) \le Mx_2(t) \quad \text{for } t \in \mathbb{T}_2, \end{gathered}$$

(x_1, x_2) satisfy the initial and interface conditions (3.3)-(3.6). *Then*

$$\begin{gathered} x_1(t) \ge 0 \quad \text{for } t \in \mathbb{T}_1 \\ x_2(t) \ge 0 \quad \text{for } t \in \mathbb{T}_2. \end{gathered}$$

Proof. *Case I:* Let $t \in \mathbb{T}_1$. Let us assume that there exists a point $m_1 \in \mathbb{T}_1$ such that $x_1(m_1) < 0$. Clearly $m_1 \neq 0$ as $x_1(0) = 0$.

(i) If m_1 is left dense as shown in [14] we obtain the contradiction

$$0 < {x_1}^{\Delta\Delta}(m_1) \le Mx_1(m_1) < 0.$$

(ii) If m_1 is left scattered as shown in [14] we obtain the contradiction

$$M \ge \frac{1}{\tilde{\mu}_1^2}$$

Hence $x_1(t) \ge 0$ for $t \in \mathbb{T}_1$.

Case II: $t \in \mathbb{T}_2$. As in the previous case, it can be shown that $x_2(t) \ge 0$ for $t \in \mathbb{T}_2$. □

Remark 4.6. From [18] we see that the IVP-SIP is equivalent to the operator equation

$$\begin{aligned} \gamma(y_1, y_2) = \Big(&\int_0^{t_1} \int_0^m f_1(s, y_1)\Delta s\Delta m, \int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{m'} f_2(s, y_2)\Delta s\Delta m' \\ &+ \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4} \Big(\int_0^a f_1(s, y_1)\Delta s \Big) \Delta m' \\ &+ \frac{\rho_1}{\rho_2} \Big(\int_0^a \int_0^{m'} f_1(s, y_1)\Delta s\Delta m' \Big) \Big) \end{aligned}$$

where $t_1, m \in \mathbb{T}_1$ and $t_2, m' \in \mathbb{T}_2$.

Definition 4.7. We define

$$\begin{aligned} Ty_1(t) &= \int_0^{t_1} \int_0^m f_1(s, y_1)\Delta s\Delta m, \quad \text{for } t \in \mathbb{T}_1 \\ Ty_2(t) &= \int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{m'} f_2(s, y_2)\Delta s\Delta m' + \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4} \Big(\int_0^a f_1(s, y_1)\Delta s \Big) \Delta m' \\ &\quad + \frac{\rho_1}{\rho_2} \Big(\int_0^a \int_0^{m'} f_1(s, y_1)\Delta s\Delta m' \Big), \quad \text{for } t \in \mathbb{T}_2 \end{aligned}$$

Definition 4.8. Let $(u_1, u_2), (v_1, v_2) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$. We call the operator T to be monotone if $(u_1, u_2) \leq (v_1, v_2)$ implies that

$$\begin{aligned} Tu_1(t) &\leq Tv_1(t) \quad \text{for } t \in \mathbb{T}_1 \\ Tu_2(t) &\leq Tv_2(t) \quad \text{for } t \in \mathbb{T}_2 \end{aligned}$$

Theorem 4.9. *Let $(\alpha_{01}, \alpha_{02}), (\beta_{01}, \beta_{02})$ be lower and upper solutions of IVP-SIP* (3.1)-(3.6). *Let us assume that for $(u_{11}, u_{12}), (v_{11}, v_{12}) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ and*

$$\begin{aligned} \alpha_{01}(t) < u_{11}(t) < v_{11}(t) < \beta_{01}(t) \\ \alpha_{02}(t) < u_{12}(t) < v_{12}(t) < \beta_{02}(t), \end{aligned}$$

we have

$$\begin{aligned} f_1(t, v_{11}) - f_1(t, u_{11}) \leq -M(v_{11} - u_{11}) \\ f_2(t, v_{12}) - f_2(t, u_{12}) \leq -M(v_{12} - u_{12}). \end{aligned}$$

Then the sequences $\{\alpha_{m1}, \alpha_{m2}\}$, $\{\beta_{m1}, \beta_{m2}\} \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ such that

$$\begin{aligned} \alpha_{01} = \alpha_{11}, &\quad \alpha_{n1} = T\alpha_{n1-1}, \\ \alpha_{02} = \alpha_{12}, &\quad \alpha_{n2} = T\alpha_{n2-1}, \\ \beta_{01} = \beta_{11}, &\quad \beta_{n1} = T\beta_{n1-1}, \\ \beta_{02} = \beta_{12}, &\quad \beta_{n2} = T\beta_{n2-1} \end{aligned}$$

converge uniformly to the minimal and maximal solutions of IVP-SIP (3.1)-(3.6) *whenever*

$$\begin{aligned} \alpha_{11} \leq \alpha_{21}, &\quad \beta_{21} \leq \beta_{11}, \\ \alpha_{12} \leq \alpha_{22}, &\quad \beta_{22} \leq \beta_{12}. \end{aligned}$$

Proof. Let $(u_{11}, u_{12}), (v_{11}, v_{12}) \in \mathcal{C}(\mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2, \mathbb{R})$ be such that

$$\begin{aligned} \alpha_{01}(t) < u_{11}(t) < v_{11}(t) < \beta_{01}(t), \\ \alpha_{02}(t) < u_{12}(t) < v_{12}(t) < \beta_{02}(t). \end{aligned}$$

Let us define

$$\begin{aligned} u_{21} = Tu_{11}, &\quad u_{22} = Tu_{12}, \\ v_{21} = Tv_{11}, &\quad v_{22} = Tv_{12}. \end{aligned}$$

We now see that

$$\begin{aligned} ({v_{21}}^{\Delta\Delta} - {u_{21}}^{\Delta\Delta}) - M(v_{21} - u_{21}) &= f_1(t, v_{11}) - f_1(t, u_{11}) - M(v_{21} - u_{21}) \\ &\leq -M(v_{11} - u_{11}) - M(v_{21} - u_{21}) \\ &= -M([v_{11} + v_{21}] - [u_{11} + u_{21}]) \leq 0. \end{aligned}$$

$$\begin{aligned} ({v_{22}}^{\Delta\Delta} - {u_{22}}^{\Delta\Delta}) - M(v_{22} - u_{22}) &= f_1(t, v_{12}) - f_1(t, u_{12}) - M(v_{22} - u_{22}) \\ &\leq -M(v_{12} - u_{12}) - M(v_{22} - u_{22}) \\ &= -M([v_{12} + v_{22}] - [u_{12} + u_{22}]) \leq 0. \end{aligned}$$

$$\begin{aligned} v_{21}(0) - u_{21}(0) = Tv_{11}(0) - Tu_{11}(0) = 0, \\ v_{21}^{\Delta}(0) - u_{21}^{\Delta}(0) = (Tv_{11})^{\Delta}(0) - (Tu_{11})^{\Delta}(0) = 0 \end{aligned}$$

Let us consider

$$\begin{aligned}
\rho_1[v_{21}-u_{21}](a) &= \rho_1[v_{21}(a)-u_{21}(a)] \\
&= \rho_1[Tv_{11}(a)-Tu_{11}(a)] \\
&= \rho_1\Big[T\Big(\frac{\rho_2}{\rho_1}v_{12}(\sigma(a))\Big)-T\Big(\frac{\rho_2}{\rho_1}u_{12}(\sigma(a))\Big)\Big] \\
&= \rho_2[Tv_{12}(\sigma(a))-Tu_{12}(\sigma(a))] \\
&= \rho_2[v_{22}(\sigma(a))-u_{22}(\sigma(a))] \\
&= \rho_2[v_{22}-u_{22}](\sigma(a))
\end{aligned}$$

Also we see that

$$\begin{aligned}
\rho_3[v_{21}^{\Delta}(a)-u_{21}^{\Delta}(a)] &= \rho_3[Tv_{11}^{\Delta}(a)-Tu_{11}^{\Delta}(a)] \\
&= \rho_3\Big[T\Big(\frac{\rho_4}{\rho_3}v_{12}^{\Delta}(\sigma(a))\Big)-T\Big(\frac{\rho_4}{\rho_3}u_{12}^{\Delta}(\sigma(a))\Big)\Big] \\
&= \rho_4[v_{22}^{\Delta}(\sigma(a))-u_{22}^{\Delta}(\sigma(a))]
\end{aligned}$$

Hence from the Lemma 4.5 we have

$$\begin{aligned}
v_{21}-u_{21}\geq 0 \quad &\text{implies} \quad Tv_{11}\geq Tu_{11}, \\
v_{22}-u_{22}\geq 0 \quad &\text{implies} \quad Tv_{12}\geq Tu_{12}.
\end{aligned}$$

Since from our assumption $(u_{11},u_{12})\leq(v_{11},v_{12})$ from Definition 4.8 we see that T is monotone. From the hypothesis we have

$$\begin{aligned}
\alpha_{11}\leq\alpha_{21}, \quad \beta_{21}\leq\beta_{11}, \\
\alpha_{12}\leq\alpha_{22}, \quad \beta_{22}\leq\beta_{12}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
(\alpha_{01},\alpha_{02}) &= (\alpha_{11},\alpha_{12})\leq(\alpha_{21},\alpha_{22})\leq\dots\leq(\alpha_{n1},\alpha_{n2}) \\
&\leq(\beta_{n1},\beta_{n2})\leq\dots(\beta_{21},\beta_{22})\leq(\beta_{11},\beta_{12})=(\beta_{01},\beta_{02}).
\end{aligned}$$

So sequences $(\alpha_{n1},\alpha_{n2})$ and $(\alpha_{n1},\alpha_{n2})$ are bounded and monotone. From [18] we see that T is completely continuous. This and boundedness of the sequences implies that there exists some subsequences such that

$$(\alpha_{n1k},\alpha_{n2k})\to(\gamma_1,\gamma_2), \quad (\beta_{n1k},\beta_{n2k})\to(k_1,k_2)$$

which implies

$$(\alpha_{n1},\alpha_{n2})\to(\gamma_1,\gamma_2), \quad (\beta_{n1},\beta_{n2})\to(k_1,k_2).$$

Taking limits in the definition of $\{\alpha_{n1},\alpha_{n2}\}$, $\{\beta_{n1},\beta_{n2}\}$ we see that (γ_1,γ_2) and (k_1,k_2) are solutions of the IVP-SIP (3.1)-(3.6). We are done through the proof if we can show that (γ_1,γ_2) and (k_1,k_2) are the minimal and maximal solutions of the IVP-SIP respectively. That is we need to show that for any solution of IVP-SIP $(x_1,x_2)\in[\alpha_{01},\beta_{01}]\times[\alpha_{02},\beta_{02}]$ satisfies

$$\begin{aligned}
\alpha_{11}(t)\leq\gamma_1(t)\leq x_1(t)\leq k_1(t)\leq\beta_{11}(t) \quad \text{for } t\in\mathbb{T}_1, \\
\alpha_{12}(t)\leq\gamma_2(t)\leq x_2(t)\leq k_2(t)\leq\beta_{12}(t) \quad \text{for } t\in\mathbb{T}_2.
\end{aligned}$$

We prove by induction that $(\alpha_{n1},\alpha_{n2})\leq(x_1,x_2)\leq(\beta_{n1},\beta_{n2})$.

For $n=0$, we have

$$(\alpha_{01},\alpha_{02})\leq(x_1,x_2)\leq(\beta_{01},\beta_{02}).$$

We assume the result to hold true for n. Hence

$$(\alpha_{n1}, \alpha_{n2}) \le (x_1, x_2) \le (\beta_{n1}, \beta_{n2}).$$

We see that $\beta_{n1+1}(t) - x_1(t)$ and $\beta_{n2+1}(t) - x_2(t)$ satisfy all the conditions of Lemma 4.5. Hence we have

$$\begin{aligned} \beta_{n1+1}(t) &\ge x_1(t) \quad \text{for } t \in \mathbb{T}_1 \\ \beta_{n2+1}(t) &\ge x_2(t) \quad \text{for } t \in \mathbb{T}_2 \end{aligned}$$

Similarly it can be shown that

$$\begin{aligned} \alpha_{n1+1}(t) &\le x_1(t) \quad \text{for } t \in \mathbb{T}_1 \\ \alpha_{n2+1}(t) &\le x_2(t) \quad \text{for } t \in \mathbb{T}_2 \end{aligned}$$

By induction we have

$$(\alpha_{n1}, \alpha_{n2}) \le (x_1, x_2) \le (\beta_{n1}, \beta_{n2}).$$

So

$$\begin{aligned} \alpha_{01}(= \alpha_{11}) \le \alpha_{n1} \le x_1 \le \beta_{n1} \le \beta_{01}(= \beta_{11}), \\ \alpha_{02}(= \alpha_{12}) \le \alpha_{n2} \le x_2 \le \beta_{n2} \le \beta_{02}(= \beta_{12}) \end{aligned}$$

implies

$$\begin{aligned} \alpha_{11} \le \lim_{n\to\infty} \alpha_{n1} \le x_1 \le \lim_{n\to\infty} \beta_{n1} \le \beta_{11}, \\ \alpha_{12} \le \lim_{n\to\infty} \beta_{n2} \le x_2 \le \lim_{n\to\infty} \beta_{n2} \le \beta_{12} \end{aligned}$$

which implies

$$\begin{aligned} \alpha_{11} \le \gamma_1 \le x_1 \le k_1 \le \beta_{11}, \\ \alpha_{12} \le \gamma_2 \le x_2 \le k_2 \le \beta_{12}. \end{aligned}$$

□

Remark 4.10. The results presented here are generalization for the nonlinear problems of corresponding linear problems studied in [9]–[13], [19]–[25]. A pair of nonlinear ordinary differential equations with matching interface conditions is a special case of the problem considered here, and our results hold true by considering $\rho(c) = \sigma(c) = c$ and the delta derivative becomes the ordinary derivative.

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