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PULLBACK ATTRACTORS FOR NON-AUTONOMOUS PARABOLIC EQUATIONS INVOLVING GRUSHIN OPERATORS

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Abstract. Using the asymptotic a priori estimate method, we prove the existence of pullback attractors for a non-autonomous semilinear degenerate parabolic equation involving the Grushin operator in a bounded domain. We assume a polynomial type growth on the nonlinearity, and an exponential growth on the external force. The obtained results extend some existing results for non-autonomous reaction-diffusion equations.

1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for dissipative dynamical systems is to analyze the existence and structure of its global attractor, which in the autonomous case, is an invariant compact set which attracts all the trajectories of the system, uniformly on bounded sets. This set has, in general, a very complicated geometry which reflects the complexity of the long-time behavior of the system (see e.g. [4, 17] and references therein).

However, non-autonomous equations are also of great importance and interest as they appear in many applications in the natural sciences. On some occasions, some phenomena are modelled by nonlinear evolutionary equations which do not take into account all the relevant information of the real systems. Instead some neglected quantities can be modelled as an external force which in general becomes time-dependent (sometimes periodic, quasiperiodic or almost periodic due to seasonal regimes).

The long-time behavior of solutions of such equations have been studied extensively in the last years. The first attempt was to extend the notion of global attractor to the non-autonomous case which led to the concept of the so-called uniform attractor (see [4]). It is remarkable that the conditions ensuring the existence of the uniform attractor parallel those for autonomous case. However, one disadvantage of the uniform attractor is that it need not to be "invariant" unlike the global attractor for autonomous systems. Moreover, it is well-known that the trajectories may be unbounded for many non-autonomous systems when time tends

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to infinity and there does not exist the uniform attractor for these systems. In order to overcome this drawback, a new concept, called pullback attractor, has been introduced for non-autonomous case. The theory of pullback attractors has been developed for both the non-autonomous and random dynamical systems and has shown to be very useful in the understanding of the dynamics of non-autonomous dynamical systems (see [3] and references therein). In the recent years, the existence of pullback attractors has been proved for some partial differential equations (see, for instance, [3, 9, 10, 16, 19, 21]). However, to the best of our knowledge, little seems to be known for the asymptotic behavior of solutions of non-autonomous degenerate equations.

One of the class of degenerate equations that has been studied widely in recent years is the class of equations involving an operator of Grushin type

$$G_s u = \Delta_{x_1} u + |x_1|^{2s} \Delta_{x_2} u, \quad (x_1, x_2) \in \Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},\ s \geq 0.$$

This operator was first introduced in [7]. Noting that $G_0 = \Delta$ and G_s, when $s > 0$, is not elliptic in domains in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ intersecting with the hyperplane $\{x_1 = 0\}$. The existence and nonexistence results for the elliptic equation

$$\begin{gathered} -G_s u + f(u) = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered}$$

were proved in [18]. Furthermore, the semilinear elliptic systems with the Grushin type operator, which are in the Hamilton form or in the potential form, were also studied in [5, 6].

In this paper we study the following non-autonomous semilinear degenerate parabolic equation in a bounded domain $\Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ $(N_1, N_2 \geq 1)$ with smooth boundary $\partial\Omega$,

$$\begin{gathered} \frac{\partial u}{\partial t} - G_s u + f(u) = g(t), \quad x \in \Omega, t > \tau, \\ u|_{t=\tau} = u_\tau(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{1.1}$$

where $u_\tau \in L^2(\Omega)$ is given, $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying

$$C_1|u|^p - k_1 \leq f(u)u \leq C_2|u|^p + k_2, \quad p \geq 2, \tag{1.2}$$

$$f'(u) \geq -\ell, \quad \text{for all } u \in \mathbb{R}, \tag{1.3}$$

and the external force g satisfies

$$g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega)), \quad \int_{-\infty}^0 e^{\lambda_1 s}|g(s)|_2^2 ds < \infty, \quad \int_{-\infty}^0 \int_{-\infty}^s e^{\lambda_1 r}|g(r)|_2^2\, dr\, ds < \infty, \tag{1.4}$$

$$g' \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega)), \quad \int_{-\infty}^0 e^{\lambda_1 s}|g'(s)|_2^2 ds < \infty, \tag{1.5}$$

where C_1, C_2, k_1, k_2, ℓ are positive constants, λ_1 is the first eigenvalue of the operator $-G_s$ in Ω with the homogeneous Dirichlet boundary condition.

The nonlinearity f is assumed to satisfy the polynomial type growth and a standard dissipative condition. A typical example of a function satisfying conditions (1.2)-(1.3) is

$$f(u) \sim u|u|^{p-2}, \quad p \geq 2 \ \ (\text{ for } |u| \text{ large}).$$

The conditions in (1.4) hold if $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ and there exist $\gamma \in (0, \lambda_1), \tau \in \mathbb{R}$ (w.l.o.g. $\tau < 0$) and $M_\tau > 0$ such that $|g(t)|_2^2 \le M_\tau e^{-\gamma t}$ for all $t \le \tau$.

The existence and long-time behavior of solutions to problems of type (1.1) in the autonomous case, that is the case of g independent of time t, has been studied in [1, 2]. In this paper we continue studying the long-time behavior of solutions to problem (1.1) by allowing the external force g to depend on time t. The natural energy space for problem (1.1) involves the space $S^1_0(\Omega)$ (see Section 2 for its definition). The main aim of this paper is to prove the following result.

Theorem 1.1. *Assume that f and g satisfy conditions* (1.2)-(1.5). *Then problem* (1.1) *generates a process $U(t, \tau)$ in $L^2(\Omega)$, which possesses a pullback $\mathcal{D}$-attractor in $S^1_0(\Omega) \cap L^p(\Omega)$.*

Let us describe the methods used in the paper. First, we use the compactness method [11] to prove the global existence of a weak solution and use *a priori* estimates to show the existence of a family of pullback $\mathcal{D}$-absorbing sets $\hat{B} = \{B(t) : t \in \mathbb{R}\}$ in $S^1_0(\Omega) \cap L^p(\Omega))$ for the process. By the compactness of the embedding $S^1_0(\Omega) \hookrightarrow L^2(\Omega)$, the process is pullback $\mathcal{D}$-asymptotically compact in $L^2(\Omega)$. This immediately implies the existence of a pullback $\mathcal{D}$-attractor in $L^2(\Omega)$. When proving the existence of pullback $\mathcal{D}$-attractors in $L^p(\Omega)$ and in $S^1_0(\Omega) \cap L^p(\Omega)$, to overcome the difficulty since lack of embbeding results, we use the asymptotic *a priori* estimate method initiated in [13] for autonomous equations and developed in [12] for non-autonomous equations. It is noticed that, to prove the existence of pullback attractors in $L^2(\Omega)$ and in $L^p(\Omega)$, we need only assume the external force g satisfies condition (1.4), while to prove the existence of a pullback attractor in $S^1_0(\Omega) \cap L^p(\Omega)$ we need the additional assumption (1.5). Thus, in particular, our results improve the recent results in [16, 9, 10] for the non-autonomous Laplacian equations in bounded domains.

The content of the paper is as follows. In Section 2, for the convenience of the reader, we recall some concepts and results on function spaces and pullback attractors which we will use. Section 3 is devoted to the proof of main results. First, the global existence and uniqueness of a weak solution to problem (1.1) are proved by using the compactness method. Then we prove the existence of pullback attractors in various spaces by using the asymptotic *a priori* estimate method.

2. Preliminaries

Function space and operator. The natural energy space for problem (1.1) involves the space $S^1_0(\Omega)$, defined as the closure of $C^1_0(\bar{\Omega})$ with respect to the norm

$$\|u\| := \Big(\int_\Omega (|\nabla_{x_1} u|^2 + |x_1|^{2s} |\nabla_{x_2} u|^2) dx \Big)^{1/2}.$$

The space $S^1_0(\Omega)$ is a Hilbert space with respect to the scalar product

$$((u, v)) := \int_\Omega (\nabla_{x_1} u \nabla_{x_1} v + |x_1|^{2s} \nabla_{x_2} u \nabla_{x_2} v) dx.$$

The following lemma comes from [18].

Lemma 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ ($N_1, N_2 \ge 1$). Then the following embeddings hold:*

(i) $S^1_0(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$ *continuously,*

(ii) $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ *compactly if* $p \in [1, 2_s^*)$, *where* $2_s^* = \frac{2N(s)}{N(s)-2}, N(s) = N_1 + (s+1)N_2$.

For the rest of this article, for the sake of brevity, we denote by $|\cdot|_2$, $(\cdot,\cdot)$, $\|\cdot\|$, $((\cdot,\cdot))$ the norms and scalar products in $L^2(\Omega)$ and $S_0^1(\Omega)$ respectively, and by $|\cdot|_p$ the norm in $L^p(\Omega)$.

It is known (see [2]) that there exists a complete orthonormal system of eigenvectors $\{e_j\}$ corresponding to the eigenvalues $\{\lambda_j\}$ such that

$$\begin{gathered}(e_j, e_k) = \delta_{jk}, \quad -G_s e_j = \lambda_j e_j, \quad j,k = 1,2,\dots,\\ 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots, \quad \lambda_j \to +\infty \text{ as } j \to \infty.\end{gathered}$$

Noting that

$$\lambda_1 = \inf\Big\{\frac{\|u\|^2}{|u|_2^2} : u \in S_0^1(\Omega), u \neq 0\Big\},$$

we have

$$\|u\|^2 \ge \lambda_1 |u|_2^2, \quad \text{for all } u \in S_0^1(\Omega). \tag{2.1}$$

Pullback $\mathcal{D}$-attractors. Let X be a metric space with metric d. Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X. For $A, B \subset X$, the Hausdorff semi-distance between A and B is defined by

$$\operatorname{dist}(A,B) = \sup_{x\in A} \inf_{y \in B} d(x,y).$$

Let $\{U(t,\tau) : t \ge \tau, \tau \in \mathbb{R}\}$ be a process in X; i.e., $U(t,\tau) : X \to X$ such that $U(\tau,\tau) = Id$ and $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \ge s \ge \tau, \tau \in \mathbb{R}$. The process $\{U(t,\tau)\}$ is said to be norm-to-weak continuous if $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$, as $x_n \to x$ in X, for all $t \ge \tau, \tau \in \mathbb{R}$. The following result is useful for verifying that a process is norm-to-weak continuous.

Proposition 2.2. [22] *Let X, Y be two Banach spaces, X^*, Y^* be respectively their dual spaces. Assume that X is dense in Y, the injection $i : X \to Y$ is continuous and its adjoint $i^* : Y^* \to X^*$ is dense, and $\{U(t,\tau)\}$ is a continuous or weak continuous process on Y. Then $\{U(t,\tau)\}$ is norm-to-weak continuous on X iff for $t \ge \tau$, $\tau \in \mathbb{R}$, $U(t,\tau)$ maps a compact set of X to be a bounded set of X.*

Suppose that $\mathcal{D}$ is a nonempty class of parameterised sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.

Definition 2.3. The process $\{U(t,\tau)\}$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\hat{\mathcal{D}} \in \mathcal{D}$, and any sequence $\{\tau_n\}_n$ with $\tau_n \le t$ for all n, and $\tau_n \to -\infty$, any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Definition 2.4. A process $\{U(t,\tau)\}$ is called pullback ω-$\mathcal{D}$-limit compact if for any $\varepsilon > 0$, any $t \in \mathbb{R}$, and $\hat{\mathcal{D}} \in \mathcal{D}$, there exists a $\tau_0 = \tau_0(\hat{\mathcal{D}}, \varepsilon, t) \le t$ such that

$$\alpha\Big(\cup_{\tau \le \tau_0} U(t,\tau)D(\tau)\Big) \le \varepsilon,$$

where α is the Kuratowski measure of noncompactness of $B \in \mathcal{B}(X)$,

$$\alpha(B) = \inf\{\delta > 0 \mid B \text{ has a finite open cover of sets of diameter } \le \delta\}.$$

Lemma 2.5 ([9]). *A process $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact if and only if it is ω-$\mathcal{D}$-limit compact.*

Definition 2.6. A family of bounded sets $\hat{\mathcal{B}} \in \mathcal{D}$ is called pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}$ if for any $t \in \mathbb{R}$ and any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t) \le t$ such that

$$\cup_{\tau \le \tau_0} U(t,\tau)D(\tau) \subset B(t).$$

Definition 2.7. A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a pullback $\mathcal{D}$-attractor for $\{U(t,\tau)\}$ if

(1) $A(t)$ is compact for all $t \in \mathbb{R}$;
(2) $\hat{\mathcal{A}}$ is invariant; i.e., $U(t,\tau)A(\tau) = A(t)$, for all $t \ge \tau$;
(3) $\hat{\mathcal{A}}$ is pullback $\mathcal{D}$-attracting; i.e.,
$$\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0,$$
for all $\hat{\mathcal{D}} \in \mathcal{D}$ and all $t \in \mathbb{R}$;
(4) If $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets then $A(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Theorem 2.8. [9] *Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, then $\{U(t,\tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \cap_{s \le t} \overline{\cup_{\tau \le s} U(t,\tau)B(\tau)}.$$

3. Proof of the main result

Existence of global solutions. Putting

$$V = L^2(\tau, T; S_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega)),$$
$$V^* = L^2(\tau, T; S^{-1}(\Omega)) + L^{p'}(\tau, T; L^{p'}(\Omega)),$$

where p' is the conjugate of p.

Definition 3.1. A function u is called a weak solution of (1.1) on (τ, T) iff

$$u \in V, \quad \frac{\partial u}{\partial t} \in V^*,$$
$$u|_{t=\tau} = u_\tau \quad \text{a.e. in } \Omega,$$

and

$$\int_\tau^T \int_\Omega \Big(\frac{\partial u}{\partial t}\varphi + \nabla_{x_1} u \nabla_{x_1}\varphi + |x_1|^{2s} \nabla_{x_2} u \nabla_{x_2}\varphi + f(u)\varphi \Big)\, dx\, dt = \int_\tau^T \int_\Omega g\varphi\, dx\, dt$$

for all test functions $\varphi \in V$.

The following proposition makes the initial condition in problem (1.1) meaningful. Its proof is exactly the same as the proof of [2, Lemma 2.3].

Proposition 3.2. *If $u \in V$ and $\frac{\partial u}{\partial t} \in V^*$, then $u \in C([\tau, T]; L^2(\Omega))$.*

Theorem 3.3. *Under assumptions (1.2)-(1.4), for any $\tau \in \mathbb{R}, u_\tau \in L^2(\Omega)$ given, problem (1.1) has a unique weak solution u defined on (τ, ∞). Moreover, the following inequality holds:*

$$|u(t)|_2^2 \le e^{-\lambda_1(t-\tau)}|u_\tau|_2^2 + \frac{2k_1}{\lambda_1}|\Omega| + \frac{e^{-\lambda_1 t}}{\lambda_1}\int_{-\infty}^t e^{\lambda_1 s}|g(s)|_2^2 ds, \ \forall t \ge \tau. \tag{3.1}$$

Proof. The existence and uniqueness of a weak solution are proved similarly to the autonomous case analyzed in [2], so it is omitted here. We now prove (3.1). Multiplying (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \|u\|^2 + \int_\Omega f(u)u dx = \int_\Omega g(t)u\, dx.$$

Using (1.2) and the Cauchy inequality, we obtain

$$\frac{d}{dt}|u|_2^2 + 2\|u\|^2 + 2C_1|u|_p^p \le 2k_1|\Omega| + \frac{1}{\lambda_1}|g(t)|_2^2 + \lambda_1|u|_2^2. \tag{3.2}$$

Noting that $\|u\|^2 \ge \lambda_1|u|_2^2$, we have

$$\frac{d}{dt}|u|_2^2 + \lambda_1|u|_2^2 \le 2k_1|\Omega| + \frac{1}{\lambda_1}|g(t)|_2^2. \tag{3.3}$$

Applying the Gronwall lemma, we get (3.1). Hence it follows that the solution u can be extended to $[\tau, +\infty)$. $\square$

Thanks to Theorem 3.3, we can define the family of maps

$$U(t,\tau) : L^2(\Omega) \to S_0^1(\Omega) \cap L^p(\Omega),$$

where $U(t,\tau)u_\tau$ is the unique solution of (1.1) with the initial datum u_τ at time τ. Then U defines a continuous process on $L^2(\Omega)$. Moreover, U also defines processes on $L^p(\Omega)$ and on $S_0^1(\Omega) \cap L^p(\Omega)$, which are norm-to-weak continuous since Proposition 2.2 and Lemma 3.4 below.

Existence of a family of pullback $\mathcal{D}$-absorbing sets. Let $\mathcal{R}$ be the set of all function $r : \mathbb{R} \to (0, +\infty)$ such that $\lim_{t\to-\infty} te^{\lambda_1 t} r^2(t) = 0$ and denote by $\mathcal{D}$ the class of all families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(S_0^1(\Omega) \cap L^p(\Omega))$ such that $D(t) \subset \overline{B}(r(t))$ for some $r(t) \in \mathcal{R}$, where $\overline{B}(r(t))$ denotes the closed ball in $S_0^1(\Omega) \cap L^p(\Omega)$ with radius $r(t)$.

Lemma 3.4. *Assume that f and g satisfy conditions* (1.2)-(1.4)*, and u is the weak solution of* (1.1)*. Then there exists a constant $c > 0$ such that the following inequality holds for $t > \tau$:*

$$\begin{aligned} |u|_2^2 + \|u\|^2 + |u|_p^p \le c\Big(&\Big(1 + (t-\tau) + \frac{1}{t-\tau}\Big)e^{-\lambda_1(t-\tau)}|u_\tau|_2^2 + \big(1 + \frac{1}{t-\tau}\big) \\ &+ \big(1 + \frac{1}{t-\tau}\big)e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}|g(s)|_2^2 ds \\ &+ \big(1 + \frac{1}{t-\tau}\big)e^{-\lambda_1 t}\int_{-\infty}^t\int_{-\infty}^s e^{\lambda_1 r}|g(r)|_2^2\, dr\, ds\Big). \end{aligned} \tag{3.4}$$

This implies that there exists a family of pullback $\mathcal{D}$-absorbing sets in $S_0^1(\Omega) \cap L^p(\Omega)$ for the process $\{U(t,\tau)\}$.

Proof. Multiplying (3.1) by $e^{\lambda_1 t}$ and integrating from τ to t, we get

$$\int_\tau^t e^{\lambda_1 s}|u|_2^2 ds \le (t-\tau)e^{\lambda_1\tau}|u_\tau|_2^2 + \frac{2k_1}{\lambda_1^2}|\Omega|e^{\lambda_1 t} + \frac{1}{\lambda_1}\int_{-\infty}^t\int_{-\infty}^s e^{\lambda_1 r}|g(r)|^2\, dr\, ds. \tag{3.5}$$

Using (3.2) and the fact that $\|u\|^2 \ge \lambda_1|u|_2^2$, we have

$$\frac{d}{dtt}|u|_2^2 + \|u\|^2 + 2C_1|u|_p^p \le 2k_1|\Omega| + \frac{1}{\lambda_1}|g(t)|_2^2, \tag{3.6}$$

thus

$$\frac{d}{dt}(e^{\lambda_1 t}|u|_2^2) + e^{\lambda_1 t}(\|u\|^2 + 2C_1|u|_p^p) \le \lambda_1 e^{\lambda_1 t}|u|_2^2 + 2k_1|\Omega|e^{\lambda_1 t} + \frac{e^{\lambda_1 t}}{\lambda_1}|g(t)|_2^2. \quad (3.7)$$

Integrating from τ to t and using (3.5), we have

$$\begin{aligned}
&\int_\tau^t e^{\lambda_1 s}(\|u\|^2 + 2C_1|u|_p^p)ds \\
&\le (1+\lambda_1(t-\tau))e^{\lambda_1\tau}|u_\tau|_2^2 + \frac{4k_1}{\lambda_1}|\Omega|e^{\lambda_1 t} + \frac{1}{\lambda_1}\int_{-\infty}^t e^{\lambda_1 s}|g(s)|_2^2 ds \\
&\quad + \int_{-\infty}^t \int_{-\infty}^s e^{\lambda_1 r}|g(r)|^2\,dr\,ds.
\end{aligned} \quad (3.8)$$

Combining (3.5) and (3.8), we obtain

$$\begin{aligned}
&\int_\tau^t e^{\lambda_1 s}(\|u\|^2 + 2C_1|u|_p^p + |u|_2^2)ds \\
&\le (1+(\lambda_1+1)(t-\tau))e^{\lambda_1\tau}|u_\tau|_2^2 \\
&\quad + \frac{2k_1(2\lambda_1+1)}{\lambda_1^2}|\Omega|e^{\lambda_1 t} + \frac{1}{\lambda_1}\int_{-\infty}^t e^{\lambda_1 s}|g(s)|_2^2 ds \\
&\quad + \Big(1+\frac{1}{\lambda_1}\Big)\int_{-\infty}^t \int_{-\infty}^s e^{\lambda_1 r}|g(r)|_2^2\,dr\,ds.
\end{aligned} \quad (3.9)$$

From (1.2) we deduce that there exist constants $\tilde{C_1}, \tilde{C_2}, \tilde{k_1}, \tilde{k_2}$ such that

$$\tilde{C_1}|s|^p - \tilde{k_1} \le F(s) \le \tilde{C_1}|s|^p + \tilde{k_2}, \quad (3.10)$$

where $F(s) = \int_0^s f(\tau)d\tau$. Combining (3.6) and (3.10), we get

$$\frac{d}{dt}|u|_2^2 + \|u\|^2 + C_5\int_\Omega F(u)dx \le \frac{1}{\lambda_1}|g(t)|_2^2 + C_6. \quad (3.11)$$

We now give some formal calculations, a rigorous proof is done by using Galerkin approximations and Lemma 11.2 in [15]. Mutiplying (1.1) by u_t and integrating over Ω, we have

$$|u_t|_2^2 + \frac{1}{2}\frac{d}{dt}\Big(\|u\|^2 + 2\int_\Omega F(u)dx\Big) = \int_\Omega g(t)u_t dx \le \frac{1}{2}|g(t)|_2^2 + \frac{1}{2}|u_t|_2^2,$$

thus

$$\frac{d}{dt}\Big(\|u\|^2 + 2\int_\Omega F(u)dx\Big) \le |g(t)|_2^2. \quad (3.12)$$

Using (3.11), (3.12) and (2.1), we have (without loss of generality $C_8 < \lambda_1$)

$$\frac{d}{dt}G(u) + C_8 G(u) \le C_9|g(t)|_2^2 + C_7,$$

where $G(u) = |u|_2^2 + \|u\|^2 + 2\int_\Omega F(u)dx$, that implies

$$\frac{d}{dt}((t-\tau)e^{\lambda_1 t}G(u)) \le (1+(\lambda_1 - C_8)(t-\tau))G(u)e^{\lambda_1 t} + (C_7 + C_9|g(t)|_2^2)(t-\tau)e^{\lambda_1 t}.$$

Integrating from τ to t, we get

$$
\begin{aligned}
&(t-\tau)G(u)\\
&\le (1+C_{11}(t-\tau))\int_\tau^t G(u)e^{\lambda_1 s}ds + C_{10}(t-\tau)e^{\lambda_1 t} + C_9(t-\tau)\int_\tau^t e^{\lambda_1 s}|g(s)|_2^2 ds.
\end{aligned}
$$

Using (3.9) we get the desired inequality (3.4) for a suitable constant $c>0$.

Let

$$
r_0(t) = 2c\Big(1+e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}|g(s)|_2^2 ds + e^{-\lambda_1 t}\int_{-\infty}^t\int_{-\infty}^s e^{\lambda_1 r}|g(r)|_2^2\, dr\, ds\Big),
$$

and $\overline{B}_0(r_0(t))$ be the closed ball in $S_0^1(\Omega)\cap L^p(\Omega)$ centered at 0 with radius $r_0(t)$. Obviously for any $\hat{\mathcal{D}}\in\mathcal{D}$ and any $t\in\mathbb{R}$, by (3.4) there exists $\tau_0=\tau_0(\hat{\mathcal{D}},t)\le t$ such that the solution u with initial datum $u_\tau\in\mathcal{D}(\tau)$ at time τ satisfies

$$
|u|_2^2 + \|u\|^2 + |u|_p^p \le r_0(t) \text{ for all } \tau\le\tau_0;
$$

i.e., $\hat{\mathcal{B}} = \{\overline{B}_0(r_0(t)) : t\in\mathbb{R}\}$ is a family of bounded pullback $\mathcal{D}$-absorbing sets in $S_0^1(\Omega)\cap L^p(\Omega)$. □

From the above lemma we deduce that the process $\{U(t,\tau)\}$ maps a compact set of $S_0^1(\Omega)\cap L^p(\Omega)$ to be a bounded set of $S_0^1(\Omega)\cap L^p(\Omega)$, and thus by Proposition 2.2, the process $\{U(t,\tau)\}$ is norm-to-weak continuous in $S_0^1(\Omega)\cap L^p(\Omega)$. Since $\{U(t,\tau)\}$ has a family of pullback $\mathcal{D}$-absorbing sets in $S_0^1(\Omega)\cap L^p(\Omega)$, in order to prove the existence of pullback $\mathcal{D}$-attractors, we only need to check that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact.

Existence of pullback $\mathcal{D}$-attractors. Because $S_0^1(\Omega)\hookrightarrow L^2(\Omega)$ compactly, we immediately get the following result.

Theorem 3.5. *Assume that f and g satisfy* (1.2)-(1.4). *Then the process corresponding to* (1.1) *has a pullback $\mathcal{D}$-attractor in $L^2(\Omega)$.*

To prove that the process $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in $L^p(\Omega)$, we need the following lemma.

Lemma 3.6. *Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process in $L^2(\Omega)$ and $L^p(\Omega)$, and $\{U(t,\tau)\}$ satisfy the following two conditions:*

(i) *$\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in $L^2(\Omega)$;*

(ii) *for any $\varepsilon>0, \hat{\mathcal{B}}\in\mathcal{D}$, there exist constants $M(\varepsilon,\hat{\mathcal{B}})$ and $\tau_0(\varepsilon,\hat{\mathcal{B}})\le t$ such that:*

$$
\Big(\int_{\Omega(|U(t,\tau)u_\tau|\ge M)} |U(t,\tau)u_\tau|^p\Big)^{\frac1p} < \varepsilon, \quad \text{for any } u_\tau\in B(\tau), \text{ and } \tau\le\tau_0.
$$

Then $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in $L^p(\Omega)$.

Proof. For any fixed $\varepsilon>0$, and $\hat{\mathcal{B}}\in\mathcal{D}$, it follows from condition (i) and Lemma 2.5 that there exists $\tau_1=\tau_1(\hat{\mathcal{B}},\varepsilon)\le\tau_0$ such that

$$
\alpha(\cup_{\tau\le\tau_1}U(t,\tau)B(\tau)) \le (3M)^{\frac{2-p}{2}}(\frac{\varepsilon}{2})^{\frac p2} \text{ in } L^2(\Omega);
$$

i.e., $\cup_{\tau\le\tau_1}U(t,\tau)B(\tau)$ has a finite $(3M)^{\frac{2-p}{2}}(\frac{\varepsilon}{2})^{\frac{p}{2}}$-net in $L^2(\Omega)$. From the above, and by (ii), using [22, Lemma 5.3], $\cup_{\tau\le\tau_1}U(t,\tau)B(\tau)$ has a finite ε-net in $L^p(\Omega)$. By the definition of the measure of noncompactness, we obtain

$$\alpha\big(\cup_{\tau\le\tau_1}U(t,\tau)B(\tau)\big)\le\varepsilon\quad\text{in } L^p(\Omega);$$

i.e., $\{U(t,\tau)\}$ is pullback ω-$\mathcal{D}$-limit compact in $L^p(\Omega)$. Hence $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in $L^p(\Omega)$ thanks to Lemma 2.5 again. □

Theorem 3.7. *Assume that f and g satisfy conditions* (1.2)-(1.4). *Then the process corresponding to problem* (1.1) *has a pullback $\mathcal{D}$-attractor in $L^p(\Omega)$.*

Proof. It is sufficient to show that the process $\{U(t,\tau)\}$ satisfies the condition (ii) in Lemma 3.6. We will give some formal calculations, a rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [15].

Take M large enough such that there exists a constant C_1' in $(0,C_1)$ such that $C_1'|u|^{p-1}\le f(u)$ in

$$\Omega_1=\Omega(u(t)\ge M)=\{x\in\Omega: u(x,t)\ge M\},$$

and denote

$$(u-M)^+=\begin{cases}u-M, & u\ge M,\\ 0, & u<M.\end{cases}$$

In Ω_1 we see that

$$\begin{aligned}g(t)((u-M)^+)^{p-1}&\le\frac{C_1'}{2}((u-M)^+)^{2p-2}+\frac{1}{2C_1'}|g(t)|^2\\ &\le\frac{C_1'}{2}((u-M)^+)^{p-1}|u|^{p-1}+\frac{1}{2C_1'}|g(t)|^2,\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}&f(u)((u-M)^+)^{p-1}\ge C_1'|u|^{p-1}((u-M)^+)^{p-1}\\ &\ge\frac{C_1'}{2}((u-M)^+)^{p-1}|u|^{p-1}+\frac{C_1'M^{p-2}}{2}((u-M)^+)^p.\end{aligned}\tag{3.14}$$

Multiplying (1.1) by $|(u-M)^+|^{p-1}$ and using (3.13), (3.14), we deduce that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}|(u-M)^+|_p^p+(p-1)\int_{\Omega_1}\big(|\nabla_{x_1}(u-M)^+|^2\\ &+|x_1|^{2s}|\nabla_{x_2}(u-M)^+|^2\big)|(u-M)^+|^{p-2}dx+C_1'M^{p-2}\int_{\Omega_1}|(u-M)^+|^p dx\\ &\le\int_{\Omega_1}\frac{1}{C_1'}|g(t)|^2dx.\end{aligned}$$

Therefore,

$$\frac{d}{dt}|(u-M)^+|_p^p+CM^{p-2}|(u-M)^+|_p^p\le C|g(t)|_2^2,$$

which implies

$$\frac{d}{dt}(t-\tau)e^{CM^{p-2}t}|(u-M)^+|_p^p\le e^{CM^{p-2}t}|(u-M)^+|_p^p+C(t-\tau)e^{CM^{p-2}t}|g(t)|_2^2.$$

Integrating the above inequality, from τ to t, we get

$$\begin{aligned}
&(t-\tau)e^{CM^{p-2}t}|(u-M)^+|_p^p \\
&\le \int_\tau^t e^{CM^{p-2}t}|(u-M)^+|_p^p + C(t-\tau)\int_\tau^t e^{CM^{p-2}t}|g(t)|_2^2 ds \\
&\le e^{(CM^{p-2}-\lambda_1)t}\int_\tau^t e^{\lambda_1 s}|u|_p^p ds + \frac{C(t-\tau)e^{(CM^{p-2}-\gamma)t}}{CM^{p-2}-\gamma}.
\end{aligned}$$

Then

$$|(u-M)^+|_p^p \le \frac{1}{t-\tau}e^{-\lambda_1 t}\int_\tau^t e^{\lambda_1 s}|u|_p^p ds + \frac{Ce^{-\gamma t}}{CM^{p-2}-\gamma}. \tag{3.15}$$

By (3.15) and (3.9), we obtain

$$\begin{aligned}
|(u-M)^+|_p^p &\le C\Big(\big(1+\frac{1}{t-\tau}\big)e^{-\lambda_1(t-\tau)}|u_\tau|_2^2 + \frac{1}{t-\tau} + \frac{e^{-\lambda_1 t}}{t-\tau}\int_{-\infty}^t e^{\lambda_1 s}|g(s)|_2^2 ds \\
&\quad + \frac{e^{-\lambda_1 t}}{t-\tau}\int_{-\infty}^t\int_{-\infty}^s e^{\lambda_1 r}|g(r)|_2^2\, dr\, ds\Big) + \frac{Ce^{-\gamma t}}{CM^{p-2}-\gamma}.
\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exist $M_1 > 0$ and $\tau_1 < t$ such that for any $\tau < \tau_1$ and any $M \ge M_1$, we have

$$\int_{\Omega(u(t)\ge M)} |(u-M)^+|^p dx \le \varepsilon. \tag{3.16}$$

Repeating the same step above, just taking $(u+M)^-$ instead of $(u-M)^+$, we deduce that there exist $M_2 > 0$ and $\tau_2 < t$ such that for any $\tau < \tau_2$ and any $M \ge M_2$,

$$\int_{\Omega(u(t)\le -M)} |(u+M)^-|^p dx \le \varepsilon, \tag{3.17}$$

where

$$(u+M)^- = \begin{cases} u+M, & u \le -M, \\ 0, & u \ge -M. \end{cases}$$

Let $M_0 = \max\{M_1, M_2\}$ and $\tau_0 = \min\{\tau_1, \tau_2\}$, we obtain

$$\int_{\Omega(|u|\ge M)} (|u|-M)^p dx \le \varepsilon \quad \text{for } \tau \le \tau_0 \text{ and } M \ge M_0.$$

Using (3.16) and (3.17), we have

$$\begin{aligned}
\int_{\Omega(|u|\ge 2M)} |u|^p dx &= \int_{\Omega(|u|\ge 2M)} ((|u|-M)+M)^p dx \\
&\le 2^{p-1}\Big(\int_{\Omega(|u|\ge 2M)} (|u|-M)^p dx + \int_{\Omega(|u|\ge 2M)} M^p dx\Big) \\
&\le 2^{p-1}\Big(\int_{\Omega(|u|\ge M)} (|u|-M)^p dx + \int_{\Omega(|u|\ge M)} (|u|-M)^p dx\Big) \\
&\le 2^p\varepsilon.
\end{aligned} \tag{3.18}$$

This completes the proof. □

To prove the existence of a pullbach $\mathcal{D}$-attractor in $S_0^1(\Omega) \cap L^p(\Omega)$, we need the following lemma.

Lemma 3.8. *Suppose that f and g satisfy (1.2)-(1.5). Then for any $t \in \mathbb{R}$ and any bounded subset $B \subset L^2(\Omega)$, there exists a positive constant $T = T(B,t) \le t$ such that*

$$|u_t(t)|_2^2 \le C\Big(1 + e^{-\lambda_1 t}\int_{-\infty}^{t} e^{\lambda_1 s}(|g(s)|_2^2 + |g'(s)|_2^2)ds\Big), \tag{3.19}$$

for all $\tau \le T(B,t)$ and all $u_\tau \in B$, where $C > 0$ is independent of t and B.

Proof. We give some formal calculations, a rigorous proof is done by use of Galerkin approximations. By differentiating (1.1) in time and setting $v = u_t$, we get

$$v_t - G_s v + f'(u)v = g'(r).$$

Multiplying the above equality by $e^{\lambda_1 r}v$, we get

$$\frac{1}{2}\frac{d}{dr}(e^{\lambda_1 r}|v|_2^2) + e^{\lambda_1 r}\|v\|^2 + e^{\lambda_1 r}(f'(u)v, v) = \frac{\lambda_1}{2}e^{\lambda_1 r}|v|_2^2 + \frac{1}{2}e^{\lambda_1 r}(g'(r), v).$$

Using (1.3) and the Cauchy inequality, we obtain that

$$\frac{d}{dr}(e^{\lambda_1 r}|v|_2^2) \le C\big(e^{\lambda_1 r}|g'(r)|_2^2 + e^{\lambda_1 r}|v|_2^2\big). \tag{3.20}$$

We set $\tau \le r \le t-1$ and $F(s) = \int_0^s f(\xi)d\xi$, then by (1.2), we deduce that

$$\tilde{C}_1\|u\|_{L^p(\Omega)}^p - \tilde{k}_1|\Omega| \le \int_\Omega F(u) \le \tilde{C}_2\|u\|_{L^p(\Omega)}^p + \tilde{k}_2|\Omega|. \tag{3.21}$$

Multiplying the first equation by u, then using (1.3) and the Cauchy inequality (1.1), we get

$$\begin{aligned}
\frac{d}{dr}(e^{\lambda_1 r}|u|_2^2) &= \lambda_1 e^{\lambda_1 r}|u|_2^2 + e^{\lambda_1 r}\frac{d}{dr}|u|_2^2 \\
&\le \lambda_1 e^{\lambda_1 r}|u|_2^2 - e^{\lambda_1 r}\|u\|^2 - 2C_1 e^{\lambda_1 r}|u|_p^p + \frac{1}{\lambda_1}e^{\lambda_1 s}|g(r)|_2^2 + e^{\lambda_1 r}k_1|\Omega| \\
&\le -2C_1 e^{\lambda_1 r}|u|_p^p + \frac{1}{\lambda_1}e^{\lambda_1 r}|g(r)|_2^2 + e^{\lambda_1 r}k_1|\Omega|,
\end{aligned} \tag{3.22}$$

where we have used the fact that $\|u\|^2 \ge \lambda_1|u|_2^2$. By integrating over the interval $[\tau, t]$, we obtain

$$e^{\lambda_1 t}|u|_2^2 \le e^{\lambda_1 \tau}|u_\tau|^2 + C\big(\int_{-\infty}^{t} e^{\lambda_1 s}|g(s)|^2 ds + e^{\lambda_1 t}\big). \tag{3.23}$$

By (1.3) and (3.21), we infer from (3.22) that

$$\frac{d}{ds}(e^{\lambda_1 s}|u|_2^2) + C\Big(e^{\lambda_1 s}\|u\|^2 + 2e^{\lambda_1 s}\int_\Omega F(u)dx\Big) \le C(e^{\lambda_1 s}|g(s)|_2^2 + e^{\lambda_1 s}). \tag{3.24}$$

Integrating this inequality from r to $r+1$ and using (3.23), we obtain

$$\begin{aligned}
&\int_r^{r+1}\Big(e^{\lambda_1 s}\|u\|^2 + 2e^{\lambda_1 s}\int_\Omega F(u)dx\Big)ds \\
&\le C\Big(e^{\lambda_1 r}|u(r)|_2^2 + \int_r^{r+1}(e^{\lambda_1 s}|g(s)|^2 + e^{\lambda_1 s})ds\Big) \\
&\le C\Big(e^{\lambda_1 \tau}|u_\tau|_2^2 + \int_{-\infty}^{t} e^{\lambda_1 s}|g(s)|^2 ds + e^{\lambda_1 t}\Big) < \infty, \quad \forall r \in [\tau, t-1].
\end{aligned} \tag{3.25}$$

Now multiplying the first equation in (1.1) by $e^{\lambda_1 r}u_t = e^{\lambda_1 r}v$, we have

$$\begin{aligned} &e^{\lambda_1 r}|v|_2^2 + \frac{d}{dr}\Big(e^{\lambda_1 r}\|u\|^2 + 2e^{\lambda_1 r}\int_\Omega F(u)dx\Big) \\ &\le \lambda_1\Big(e^{\lambda_1 r}\|u\|^2 + 2e^{\lambda_1 r}\int_\Omega F(u)dx\Big) + e^{\lambda_1 r}|g(r)|_2^2. \end{aligned} \tag{3.26}$$

By (3.25), (3.26), and the uniform Gronwall inequality [17, p. 91], we obtain

$$e^{\lambda_1 r}\|u(r)\|^2 + 2e^{\lambda_1 r}\int_\Omega F(u)dx \le C\Big(e^{\lambda_1 \tau}|u_\tau|_2^2 + \int_{-\infty}^t e^{\lambda_1 s}|g(s)|_2^2 ds + e^{\lambda_1 t}\Big). \tag{3.27}$$

On the other hand, integrating (3.26) from r to $r+1$, by (3.22), (3.25) and (3.27), we have

$$\int_r^{r+1} e^{\lambda_1 s}|v|_2^2 ds \le C\Big(e^{\lambda_1 \tau}|u_\tau|_2^2 + \int_{-\infty}^t e^{\lambda_1 s}(|g(s)|_2^2 + e^{\lambda_1 t}\Big).$$

Then, by (3.20), using the uniform Gronwall lemma once again, we get

$$e^{\lambda_1 t}|v|_2^2 \le C\Big(e^{\lambda_1 \tau}|u_\tau|_2^2 + \int_{-\infty}^t e^{\lambda_1 s}(|g(s)|_2^2 + |g'(s)|_2^2)ds + e^{\lambda_1 t}\Big);$$

that is,

$$|v(t)|_2^2 \le C\Big(e^{-\lambda_1 (t-\tau)}|u_\tau|_2^2 + e^{-\lambda_1 t}\int_{-\infty}^t e^{\lambda_1 s}(|g(s)|_2^2 + |g'(s)|_2^2)ds + 1\Big).$$

This completes the proof. $\square$

We are now in a position to complete the proof of the main theorem.

Proof of Theorem 1.1. By Lemma 3.4, $\{U(t,\tau)\}$ has a family of bounded pullback $\mathcal{D}$-absorbing sets in $S_0^1(\Omega) \cap L^p(\Omega)$. It remains to show that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in $S_0^1(\Omega) \cap L^p(\Omega)$, i.e., for any $t \in \mathbb{R}$, any $\hat{\mathcal{B}} \in \mathcal{D}$, and any sequence $\tau_n \to -\infty$, any sequence $u_{\tau_n} \in B(\tau_n)$, the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ is precompact in $S_0^1(\Omega) \cap L^p(\Omega)$. Thanks to Theorem 3.7, we need only to show that the sequence $\{U(t,\tau_n)u_{\tau_n}\}$ is precompact in $S_0^1(\Omega)$.

Let $u_n(t) = U(t,\tau_n)u_{\tau_n}$. By Theorem 3.5, we can assume that $\{u_n(t)\}$ is a Cauchy sequence in $L^2(\Omega)$. We have

$$\begin{aligned} &\|u_n(t) - u_m(t)\|^2 \\ &= -\langle G_s u_n(t) - G_s u_m(t), u_n(t) - u_m(t)\rangle \\ &= -\langle \frac{du_n}{dt}(t) - \frac{du_m}{dt}(t), u_n(t) - u_m(t)\rangle - \langle f(u_n(t)) - f(u_m(t)), u_n(t) - u_m(t)\rangle \\ &\le |\frac{d}{dt}u_n(t) - \frac{d}{dt}u_m(t)|_2^2 |u_n(t) - u_m(t)|_2^2 + \ell|u_n(t) - u_m(t)|_2^2, \end{aligned}$$

where we have used condition (1.3). Because $\{u_n(t)\}$ is a Cauchy sequence in $L^2(\Omega)$ and by Lemma 3.6, one gets

$$\|u_n(t) - u_m(t)\| \to 0, \quad \text{as } m,n \to \infty.$$

The proof is complete. $\square$

Remark 3.9. The pullback attractor in Theorems 3.5, 3.7 and 1.1 is the same object. Because of the tempered condition in the definition of $\mathcal{D}$, as a corollary of the results in the paper [14], one may establish the existence of a global pullback attractor for a different universe, that of fixed bounded set of $L^2(\Omega)$; this attractor works in the norms $L^2(\Omega), L^p(\Omega)$ and $S_0^1(\Omega) \cap L^p(\Omega)$ (unique, the same in the three frame works) and is a subset of the attractor obtained in Theorems 1.1, 3.5 and 3.7.

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