Electronic Journal of Differential Equations, Vol. 2010(2010), No. 111, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

GENERALIZED PICONE AND RICCATI INEQUALITIES FOR HALF-LINEAR DIFFERENTIAL OPERATORS WITH ARBITRARY ELLIPTIC MATRICES

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Abstract. In the article, we extend the well-known Picone identity for half-linear partial differential equations to equations with anisotropic p-Laplacian.

1. Introduction

The Picone identity appears to be a useful tool in qualitative theory of differential equations. In the simplest case it can be written as

$$\begin{aligned}\Big[\frac{u}{v}(vru'-uRv')\Big]' &= (r-R)u'^2+(Q-q)u^2+R\big(u'-\frac{u}{v}v'\big)^2\\ &\quad+\frac{u}{v}\big[v\big((ru')'+qu\big)-u\big((Rv')'+qv\big)\big]\end{aligned}\tag{1.1}$$

and holds for sufficiently smooth real valued functions u, v, r, R, q and Q. Picone [16] used this identity for a proof of Sturmian comparison theorem for linear second order ODE and other related results. This identity has been extended in several aspects to more general operators than second order linear differential operator. Picone identity is used not only to derive important results in comparison and oscillation theory of related differential equations, but can be also used to get uniqueness or nonexistence results, monotonicity of eigenvalue in domain, results for various eigenvalue problems and inequalities and other results. See [3, 6, 12, 11, 13, 17, 18, 20, 22] for more details. Furthermore, the Piccone identity is closely related to Riccati equation which also appears to be a powerful tool in the general theory of second order linear and half-linear equations.

Equations with p-Laplacian and half-linear equations attracted a wide interest in last years because of their application in various physical and biological phenomena such as flow of non-Newtonian fluids, slow diffusion problem and glaceology, see e.g. [4]. In most applications it is sufficient to consider an isotropic p-Laplacian

$$\operatorname{div}\Big(a(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big),$$

2000 *Mathematics Subject Classification.* 35J92, 35B05.
Key words and phrases. Half-linear differential equation; second order equation; Picone identity; conjugacy; oscillation.

Submitted June 29, 2010. Published August 10, 2010.
Supported by grant GAP201/10/1032 from the Czech Grant Agency.

where $a(x)$ is either identity matrix or a scalar function. However, there are also problems in which anisotropy plays an important role and it is necessary to treat $a(x)$ as a general elliptic matrix function. This includes for example nonlinear dielectric composite, see e.g. [2].

In this article, we establish a suitable replacement for Picone identity in the theory of half-linear partial differential operators

$$l(u) := \operatorname{div}\Big(a(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) + c(x)|u|^{p-2}u, \tag{1.2}$$

$$L(u) := \operatorname{div}\Big(A(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) + C(x)|u|^{p-2}u, \tag{1.3}$$

with anisotropic p-Laplacian, where $\Omega \in \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ for which the Gauss-Ostrogradskii divergence theorem holds, $a \in C^1(\overline{\Omega}, \mathbb{R}^{n\times n})$ and $A \in C^1(\overline{\Omega}, \mathbb{R}^{n\times n})$ are smooth elliptic matrix valued functions, $c \in C^{0,\alpha}(\overline{\Omega})$ and $C \in C^{0,\alpha}(\overline{\Omega})$ are Hölder continuous functions, $\operatorname{div}(\cdot)$ and ∇ are the usual divergence and nabla operators, $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$ and $p > 1$ is a real constant. The notation $\langle\cdot,\cdot\rangle$ is used for the usual scalar product. By $\Lambda_{\max}(x)$ and $\Lambda_{\min}(x)$ we denote the maximal and minimal eigenvalues of the matrix $A(x)$ and similarly $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ denote the maximal and minimal eigenvalues of the matrix $a(x)$.

The domain $D_l(\Omega)$ of operator l is the set of all functions $u(x) \in C^1(\overline{\Omega})$ such that $a(x)\,\|\nabla u\|^{p-2}\,\nabla u \in C^1(\Omega) \cap C(\overline{\Omega})$. In a similar way we define domain $D_L(\Omega)$ of the operator L.

The operators l and L can be viewed as generalization of elliptic linear differential operators and it turns out, that many results proved originally in the linear case which can be obtained from (1.2), (1.3) by letting $p = 2$ can be extended for operators l and L.

As mentioned above, the original Picone identity (1.1) has been generalized in many different directions. These extensions include also isotropic half-linear partial differential operators which have the same form as (1.2) and (1.3) but the matrices $a(x)$ and $A(x)$ are replaced by smooth scalar functions. The Picone identity for this case has been derived in [9] in the form

$$\begin{aligned} &\operatorname{div}\left(\frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}va(x)\,\|\nabla u\|^{p-2}\,\nabla u - |u|^{p-2}uA(x)\|\nabla v\|^{p-2}\nabla v\Big]\right)\\ &= \Big[a(x) - A(x)\Big]\|\nabla u\|^p + \Big[C(x) - c(x)\Big]|u|^p + A(x)Y(u,v)\\ &\quad + \frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}vl(u) - |u|^{p-2}uL(v)\Big], \end{aligned} \tag{1.4}$$

where

$$Y(u,v) = \|\nabla u\|^p + (p-1)\|\frac{u}{v}\nabla v\|^p - p\|\frac{u}{v}\nabla v\|^{p-2}\langle\nabla u, \frac{u}{v}\nabla v\rangle.$$

An important property of the function $Y(u,v)$ is that this function is nonnegative and equals zero if and only if the function u is a constant multiple of v. As shown in [5] and [9], (1.4) can be used to make a connection between ordinary and partial differential equations and allows to embed easily new results from theory of ordinary differential equations into theory of partial differential equations (see also [6] for detailed references about half-linear and related equations). This approach turns out to be valuable to get fast extension of some modern approaches to the conjugacy and oscillation theory. See for example [7] and [8] for some new methods in oscillation theory of half-linear ordinary differential equations, which can be extended in this way to partial differential equations.

As far as we know, nothing is known about possible extension of Picone identity to the case of anisotropic p-Laplacian and partial differential operators (1.2), (1.3). The related results are known in few related cases only, like operators

$$\sum_{k=1}^{m} \operatorname{div}\Big(a_k(x)\|\sqrt{a_k(x)}\nabla u\|^{p-2}\nabla u\Big) + c(x)|u|^{p-2}u,$$

where $a_k(x)$ are positive definite matrices (see [20]) or sublinear-superlinear operator with linear differential part

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(A_{ij}(x,t)\frac{\partial v}{\partial x_j}\Big) + C(x,t)|v|^{\beta-1}v + D(x,t)|v|^{\gamma-1}v, \tag{1.5}$$

where A_{ij} is elliptic matrix, C and D are scalar positive functions, $\beta > 1$, $0 < \gamma < 1$ and the corresponding parabolic equation

$$\frac{\partial v}{\partial t} - L[v] = 0,$$

see [10]. The results for operator (1.5) have been extended to half-linear case by Yoshida [21], however the replacement of the matrix A_{ij} by scalar function is necessary in [21].

Moreover, it seems that direct extension of Picone identity to anisotropic operators (1.2), (1.3) does not exist due to incompatibility between matrix product and nonlinearity in the differential operator. The aim of this paper if to derive suitable replacement for Picone identity which can be used in theory of half-linear partial differential operators (1.2) and (1.3).

If we compare known results for (1.2) and (1.3) and its special cases (obtained using transformation into Riccati type equation or inequality), we see an interesting phenomenon: the results obtained directly for the linear case are sharper than the results obtained from the general case $p > 1$ by letting $p = 2$. A closer discussion related to this phenomenon is in [14] and it appears, that there is a difference between sublinear ($p \le 2$) and superlinear ($p > 2$) case. Hence our task is not only to extend Picone identity to operators (1.2) and (1.3), but we naturally aim to respect this behavior and get results which are in the sublinear case sharper than the results obtained for general p.

2. Main result

The following theorem presents our main result: inequality, which is a replacement for Picone identity for operators (1.2) and (1.3). Note that due to necessity to use some estimates based on minimal and (or) maximal eigenvalues of matrices $a(x)$ and $A(x)$, we get only inequality and not equality like for the linear case or like for equation (1.5). From this reason it is also not reasonable to include the replacement for the term $Y(u,v)$ in (1.4). Despite this fact, we are able to give a common characterization of all cases, when inequality (2.2) below becomes equality. Such a situation corresponds to the case when $Y(u,v) = 0$.

Theorem 2.1. *Let $u \in D_l(\Omega)$ and $v \in D_L(\Omega)$, $v \neq 0$ on Ω. Denote*

$$K(x) = \begin{cases} \left(\frac{\Lambda_{\max}(x)}{\Lambda_{\min}(x)}\right)^{p-1} \Lambda_{\max}(x) & \text{for } p > 2, \\ \Lambda_{\max}(x) & \text{for } 1 < p \le 2. \end{cases} \tag{2.1}$$

The inequality

$$\begin{aligned} &\operatorname{div}\Big(\frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}va(x)\,\|\nabla u\|^{p-2}\,\nabla u - |u|^{p-2}uA(x)\|\nabla v\|^{p-2}\nabla v\Big]\Big) \\ &\ge \Big[\lambda_{\min}(x) - K(x)\Big]\|\nabla u\|^p + \Big[C(x) - c(x)\Big]|u|^p \\ &\quad + \frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}vl(u) - |u|^{p-2}uL(v)\Big] \end{aligned} \tag{2.2}$$

holds for every $x \in \Omega$. The inequality becomes equality if and only if the following conditions hold

(i) *$\nabla u(x)$ is an eigenvector of the matrix $a(x)$ associated with the eigenvalue $\lambda_{\min}(x)$,*
(ii) *$\nabla v(x)$ is an eigenvector of the matrix $A(x)$ associated with the eigenvalue $\Lambda_{\max}(x)$,*
(iii) *if $p > 2$ then $\Lambda_{\max}(x) = \Lambda_{\min}(x)$,*
(iv) *$u(x)$ is a constant multiple of $v(x)$.*

Proof. Direct computations show

$$\begin{aligned} \operatorname{div}\Big(ua(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) &= ul(u) - c(x)|u|^p + \langle \nabla u, a(x)\,\|\nabla u\|^{p-2}\,\nabla u\rangle \\ &\ge ul(u) - c(x)|u|^p + \lambda_{\min}(x)\,\|\nabla u\|^p\,. \end{aligned} \tag{2.3}$$

Evaluating the divergence and using definition of the operator L we obtain

$$\begin{aligned} &-\operatorname{div}\Big(|u|^p\frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big) \\ &= -p|u|^{p-2}u\Big\langle\nabla u, \frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\rangle - \frac{|u|^p}{|v|^{p-2}v}\operatorname{div}\big(A(x)\|\nabla v\|^{p-2}\nabla v\big) \\ &\quad - (1-p)\frac{|u|^p}{|v|^p}\big\langle A(x)\|\nabla v\|^{p-2}\nabla v, \nabla v\big\rangle \\ &= -\frac{|u|^p}{|v|^{p-2}v}L(v) + C(x)|u|^p - p|u|^{p-2}u\Big\langle\nabla u, \frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\rangle \\ &\quad + (p-1)\frac{|u|^p}{|v|^p}\big\langle A(x)\|\nabla v\|^{p-2}\nabla v, \nabla v\big\rangle\,. \end{aligned} \tag{2.4}$$

To estimate the last two terms, in terms of the product $K(x)\,\|\nabla u\|^p$, we use the Young inequality

$$\frac{p-1}{p}X^{\frac{p}{p-1}} - XY \ge -\frac{1}{p}Y^p \tag{2.5}$$

and split the proof into two cases.

Case 1: First we consider general case $p > 1$. Schwarz inequality and the fact that norm of the matrix A is $\Lambda_{\max}$ imply

$$|u|^{p-2}u\Big\langle\nabla u, \frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\rangle \le |u|^{p-1}\,\|\nabla u\|\,\Big\|\frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\|$$

$$\leq |u|^{p-1}\|\nabla u\|\Lambda_{\max}(x)\frac{\|\nabla v\|^{p-1}}{|v|^{p-1}}.$$

Using this inequality and Young inequality we can find an apriori bound for last two terms at the right-hand side of (2.4) as follows

$$\begin{aligned}
&\frac{p-1}{p}\frac{|u|^p}{|v|^p}\left\langle A(x)\|\nabla v\|^{p-2}\nabla v,\nabla v\right\rangle - |u|^{p-2}u\Big\langle\nabla u,\frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\rangle\\
&\geq \frac{p-1}{p}\Big|\frac{u}{v}\Big|^p\|\nabla v\|^p\Lambda_{\min}(x) - |u|^{p-1}\|\nabla u\|\,\Lambda_{\max}(x)\frac{\|\nabla v\|^{p-1}}{|v|^{p-1}}\\
&= \frac{p-1}{p}\Big[\Big(\Big|\frac{u}{v}\Big|\|\nabla v\|\Lambda_{\min}^{1/p}(x)\Big)^{p-1}\Big]^{\frac{p}{p-1}} - |u|^{p-1}\|\nabla u\|\,\Lambda_{\max}(x)\frac{\|\nabla v\|^{p-1}}{|v|^{p-1}}\\
&= \frac{p-1}{p}\Big[\Big(\Big|\frac{u}{v}\Big|\|\nabla v\|\Lambda_{\min}^{1/p}(x)\Big)^{p-1}\Big]^{\frac{p}{p-1}}\\
&\quad - \Big(\Big|\frac{u}{v}\Big|\|\nabla v\|\Lambda_{\min}^{1/p}(x)\Big)^{p-1}\|\nabla u\|\,\Lambda_{\max}(x)\Lambda_{\min}^{\frac{1-p}{p}}(x)\\
&\geq -\frac{1}{p}\Lambda_{\max}^p(x)\Lambda_{\min}^{1-p}(x)\,\|\nabla u\|^p.
\end{aligned}$$

Case 2: In this second case we consider $1 < p \leq 2$ and use more careful estimates. From the fact that the minimal eigenvalue of the inverse matrix $A^{-1}(x)$ is $\Lambda_{\max}^{-1}(x)$ we get the estimate

$$\begin{aligned}
&\frac{p-1}{p}\frac{|u|^p}{|v|^p}\left\langle A(x)\|\nabla v\|^{p-2}\nabla v,\nabla v\right\rangle\\
&= \frac{p-1}{p}\frac{|u|^p}{|v|^p}\|\nabla v\|^{p-2}\left\langle A(x)\nabla v, A^{-1}(x)A(x)\nabla v\right\rangle\\
&\geq \frac{p-1}{p}\frac{|u|^p}{|v|^p}\|\nabla v\|^{p-2}\frac{1}{\Lambda_{\max}(x)}\|A(x)\nabla v\|^2\\
&= \frac{p-1}{p}\Big[\Big(\frac{1}{\Lambda_{\max}(x)}\Big)^{\frac{p-1}{p}}\Big(\frac{|u|}{|v|}\Big)^{p-1}\|\nabla v\|^{\frac{(p-2)(p-1)}{p}}\|A(x)\nabla v\|^{\frac{2(p-1)}{p}}\Big]^{\frac{p}{p-1}}.
\end{aligned}$$

Schwarz inequality implies

$$|u|^{p-2}u\Big\langle\nabla u,\frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\rangle \leq |u|^{p-1}\|\nabla u\|\frac{\|\nabla v\|^{p-2}}{|v|^{p-1}}\|A(x)\nabla v\|,$$

and if we multiply by (-1) and rewrite the right hand side into the form suitable for Young inequality, we get

$$\begin{aligned}
&-|u|^{p-2}u\Big\langle\nabla u,\frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\rangle\\
&\qquad\geq -|u|^{p-1}\|\nabla u\|\frac{\|\nabla v\|^{p-2}}{|v|^{p-1}}\|A(x)\nabla v\|\\
&\qquad= -\Big[\Big(\frac{1}{\Lambda_{\max}(x)}\Big)^{\frac{p-1}{p}}\Big(\frac{|u|}{|v|}\Big)^{p-1}\|\nabla v\|^{\frac{(p-2)(p-1)}{p}}\|A(x)\nabla v\|^{\frac{2(p-1)}{p}}\Big]\\
&\qquad\quad\times\Big(\frac{1}{\Lambda_{\max}(x)}\Big)^{\frac{1-p}{p}}\|\nabla u\|\,\|\nabla v\|^{\frac{p-2}{p}}\|A(x)\nabla v\|^{\frac{2-p}{p}}.
\end{aligned}$$

Summing up the last two inequalities, using Young inequality and using obvious inequality

$$\|A(x)\nabla v\| \le \Lambda_{\max}(x)\|\nabla v\|$$

which for $p \le 2$ implies

$$\|A(x)\nabla v\|^{2-p} \le \Lambda_{\max}^{2-p}(x)\|\nabla v\|^{2-p}$$

we have

$$\begin{aligned}
&\frac{p-1}{p}\frac{|u|^p}{|v|^p}\left\langle A(x)\|\nabla v\|^{p-2}\nabla v, \nabla v\right\rangle - |u|^{p-2}u\Big\langle \nabla u, \frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big\rangle \\
&\ge -\frac{1}{p}\Big(\frac{1}{\Lambda_{\max}(x)}\Big)^{1-p}\|\nabla u\|^p\,\|\nabla v\|^{p-2}\,\|A(x)\nabla v\|^{2-p} \\
&\ge -\frac{1}{p}\Big(\frac{1}{\Lambda_{\max}(x)}\Big)^{1-p}\|\nabla u\|^p\,\|\nabla v\|^{p-2}\Lambda_{\max}^{2-p}(x)\|\nabla v\|^{2-p} \\
&= -\frac{1}{p}\|\nabla u\|^p\,\Lambda_{\max}(x).
\end{aligned}$$

Summarizing both cases we have

$$-\operatorname{div}\Big(|u|^p\frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big) \ge -\frac{|u|^p}{|v|^{p-2}v}L(v) + C(x)|u|^p - K(x)\,\|\nabla u\|^p \tag{2.6}$$

and adding to (2.3) we obtain

$$\begin{aligned}
&\operatorname{div}\Big(\frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}va(x)\,\|\nabla u\|^{p-2}\,\nabla u - |u|^{p-2}uA(x)\|\nabla v\|^{p-2}\nabla v\Big]\Big) \\
&\ge ul(u) - c(x)|u|^p + \lambda_{\min}(x)\,\|\nabla u\|^p \\
&\quad - \frac{|u|^p}{|v|^{p-2}v}L(v) + C(x)|u|^p - K(x)\,\|\nabla u\|^p,
\end{aligned}$$

which implies (2.2).

It remains to investigate, when inequality becomes equality. A closer investigation of the proof shows, that to get equality in (2.2), all inequalities used in the proof have to reduce into equalities. Remark that (2.5) becomes equality if and only if $X^{\frac{p}{p-1}} = Y^p$. We distinguish two cases again.

Case 1: $p > 2$. In this case (2.2) becomes equality if and only if all the following equalities hold:

$$\langle \nabla u, a(x)\nabla u\rangle = \lambda_{\min}\,\|\nabla u\|^2, \tag{2.7}$$

$$uv\,\langle \nabla u, \nabla v\rangle = |uv|\,\|\nabla u\|\,\|\nabla v\|, \tag{2.8}$$

$$\|A(x)\nabla v\| = \Lambda_{\max}(x)\|\nabla v\|, \tag{2.9}$$

$$\langle \nabla v, A(x)\nabla v\rangle = \Lambda_{\min}\|\nabla v\|^2, \tag{2.10}$$

$$\Big|\frac{u}{v}\Big|^p\|\nabla v\|^p\Lambda_{\min}(x) = \Lambda_{\max}^p(x)\Lambda_{\min}^{1-p}(x)\,\|\nabla u\|^p. \tag{2.11}$$

Equations (2.7) and (2.10) imply that ∇u and ∇v are eigenvectors of matrices $a(x)$ and $A(x)$ belonging to $\lambda_{\min}(x)$ and $\Lambda_{\min}(x)$, respectively. Equation (2.9) implies that ∇v is also an eigenvector of $A(x)$ belonging to $\Lambda_{\max}(x)$ and hence $\Lambda_{\max}(x) = \Lambda_{\min}(x)$. Using this fact, (2.11) becomes

$$\Big|\frac{u}{v}\Big|\|\nabla v\| = \|\nabla u\|. \tag{2.12}$$

Equation (2.8) implies that ∇u is a scalar multiple of ∇v, i.e., there exists a function $\rho(x)$ such that $\nabla u = \rho \nabla v$. Now (2.12) and (2.8) imply that $u(x) = \rho(x)v(x)$. Evaluating the gradient we get

$$\nabla u = v\nabla \rho + \rho \nabla v$$

and

$$\|\nabla u\| \le |v|\,\|\nabla \rho\| + |\rho|\|\nabla v\|.$$

From here and from the fact that v does not have zeros on Ω we conclude that $\|\nabla \rho\| = 0$ and ρ is a constant function. Hence all conditions (i)–(iv) hold. On the other hand, it is easy to see, that if (i)–(iv) hold, then (2.7)–(2.11) are satisfied and hence equality holds in (2.2).

Case 2: $p \le 2$. Similarly to the previous case we find that (2.2) becomes equality if and only if (2.7), (2.8), (2.9) and the following equations hold:

$$\left\langle A(x)\nabla v, A^{-1}(x)A(x)\nabla v\right\rangle = \Lambda_{\max}^{-1}(x)\,\|A(x)\nabla v\|^2\,, \tag{2.13}$$

$$\left|\frac{u}{v}\right|^p \|\nabla v\|^{p-2}\frac{\|A(x)\nabla v\|^2}{\Lambda_{\max}(x)} = \Lambda_{\max}^{p-1}(x)\,\|\nabla u\|^p\,\|\nabla v\|^{p-2}\,\|A(x)\nabla v\|^{2-p}\,. \tag{2.14}$$

As in the previous case (2.7) and (2.9) imply that ∇u and ∇v are eigenvectors of matrices $a(x)$ and $A(x)$ belonging to $\lambda_{\min}(x)$ and $\Lambda_{\max}(x)$, respectively. This also implies that (2.13) holds and (2.14) reduces into (2.12). The remaining part is identical to the previous case and the proof that (i), (ii), and (iv) imply (2.7)–(2.9), (2.13) and (2.14) is easy. Theorem is proved. □

Remark 2.2 (Riccati inequality)**.** If $L(v) = 0$ and $u \equiv 1$, then (2.4) becomes the generealized Riccati equation for the vector variable $\vec{w}(x) := A(x)\frac{\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}$:

$$\operatorname{div}\vec{w} + C(x) + (p-1)\left\|A^{-1}(x)\vec{w}\right\|^{q-2}\left\langle \vec{w}, A^{-1}(x)\vec{w}\right\rangle = 0.$$

Recall that the eigenvalues of the matrix $A^{-1}(x)$ are reciprocal values of the eigenvalues of the matrix $A(x)$ and thus

$$\frac{1}{\Lambda_{\max}(x)}\,\|\vec{w}\| \le \left\|A^{-1}(x)\vec{w}\right\| \le \frac{1}{\Lambda_{\min}(x)}\,\|\vec{w}\|\,,$$
$$\frac{1}{\Lambda_{\max}(x)}\,\|\vec{w}\|^2 \le \left\langle \vec{w}, A^{-1}(x)\vec{w}\right\rangle \le \frac{1}{\Lambda_{\min}(x)}\,\|\vec{w}\|^2\,.$$

Combinations of these estimates allow to derive various types of Riccati inequalities.

Remark 2.3. Remark that if the matrices $a(x)$, $A(x)$ are scalar multiples of identity matrix (say $a(x) = \tilde{a}(x)I$ and $A(x) = \tilde{A}(x)I$ where $\tilde{a}$ and $\tilde{A}$ are scalar functions) as in [9], then $\lambda_{\max}(x) = \lambda_{\min}(x) = \tilde{a}(x)$ and $K(x) = \Lambda_{\max}(x) = \Lambda_{\min}(x) = \tilde{A}(x)$. In this case we have the following identity for the first term from the right hand side of (2.2): $[\lambda_{\min}(x) - K(x)]\,\|\nabla u\|^p = [\tilde{a}(x) - \tilde{A}(x)]\,\|\nabla u\|^p$.

Immediately from the proof of Theorem 2.1 we obtain the following statement, where only the "second part" (2.6) of the Picone inequality (2.2) is considered. A closer examination of the proof reveals that condition (i) in Theorem 2.1 is needed only for the equality in the "first part" (2.3) of (2.2), while the other three conditions (ii)–(iv) mean the equality in (2.6).

Corollary 2.4. *Let $u \in C^1(\overline{\Omega})$, $v \in D_L(\Omega)$, $v \neq 0$ on Ω and let K be the function defined in* (2.1). *Then the following inequality*

$$\operatorname{div}\Big(|u|^p \frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big) \leq \frac{|u|^p}{|v|^{p-2}v} L(v) - C(x)|u|^p + K(x)\,\|\nabla u\|^p \tag{2.15}$$

holds for every $x \in \Omega$. The inequality in (2.15) *can be replaced by equality if and only if conditions (ii)–(iv) of Theorem 2.1 hold.*

Remark 2.5. Note that

$$\Big(\frac{\Lambda_{\max}(x)}{\Lambda_{\min}(x)}\Big)^{p-1}\Lambda_{\max}(x) \geq \Lambda_{\max}(x) \tag{2.16}$$

The quotient $\frac{\Lambda_{\max}(x)}{\Lambda_{\min}(x)}$ is conditioned number of the matrix $A(x)$ and this number shows, that the inequality for the case $p \leq 2$ is sharper than inequality for $p > 2$ (which holds in fact for every $p > 1$). In addition, if $\Lambda_{\max}(x) = \Lambda_{\min}(x)$, then there is no difference between cases $p > 2$ and $p \leq 2$ in Theorem 2.1 and Corollary 2.4.

Remark 2.6. If $p > 2$ and A is not a scalar multiple of identity matrix, then condition (iii) of Theorem 2.1 fails and (2.2) never becomes equality.

3. Applications of Picone inequality

As a consequence of the Picone inequality derived in the previous section we have the following version of Leighton-type comparison theorem.

Theorem 3.1. *Let u be a nontrivial solution of $l(u) = 0$ such that $u = 0$ on $\partial\Omega$ and let*

$$\int_\Omega [(\lambda_{\min}(x) - K(x))\,\|\nabla u\|^p + (C(x) - c(x))|u|^p]\,\mathrm{d}x \geq 0.$$

Then every solution of $L(v) = 0$ has a zero in $\overline{\Omega}$.

Proof. Suppose, by contradiction, that v is a solution of $L(v) = 0$ such that $v \neq 0$ in $\overline{\Omega}$. The functions u, v satisfy assumptions of Theorem 2.1 and since v is not a constant multiple of u, the Picone inequality (2.2) holds strict. Integrating this inequality with using the Gauss-Ostrogradskii theorem we obtain

$$\begin{aligned}&\int_{\partial\Omega} \Big\langle \frac{u}{|v|^{p-2}v}\Big[|v|^{p-2}va(x)\,\|\nabla u\|^{p-2}\,\nabla u - |u|^{p-2}uA(x)\|\nabla v\|^{p-2}\nabla v\Big], \nu\Big\rangle\,\mathrm{d}S\\ &> \int_\Omega \Big[\lambda_{\min}(x) - K(x)\Big]\|\nabla u\|^p + \Big[C(x) - c(x)\Big]|u|^p\,\mathrm{d}x,\end{aligned}$$

where ν denotes the outside unit normal. The fact that $u = 0$ on $\partial\Omega$ gives

$$\int_\Omega [(\lambda_{\min}(x) - K(x))\,\|\nabla u\|^p + (C(x) - c(x))|u|^p]\,\mathrm{d}x < 0,$$

a contradiction. □

The next two statements follow directly from Theorem 3.1.

Corollary 3.2. *Let u be a nontrivial solution of $l(u) = 0$ such that $u = 0$ on $\partial\Omega$.*

(i) *If $\lambda_{\min}(x) \geq K(x)$ and $C(x) \geq c(x)$ in Ω, then every solution of $L(v) = 0$ has a zero in $\overline{\Omega}$.*
(ii) *If $\lambda_{\min}(x) = \lambda_{\max}(x)$, then every solution of $l(u) = 0$ has a zero in $\overline{\Omega}$.*

Similarly as Theorem 3.1 follows from Theorem 2.1, the next theorem can be obtained from Corollary 2.4.

Theorem 3.3. *Suppose that there exists a nontrivial function $u \in C^1(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and*

$$\int_\Omega [K(x)\,\|\nabla u\|^p - C(x)|u|^p]\;\mathrm{d}x \le 0.$$

Then every solution of $L(v) = 0$ has a zero in $\overline{\Omega}$.

Proof. Let v be a nontrivial solution of $L(v) = 0$ and suppose, by contradiction, that $v \neq 0$ in $\overline{\Omega}$. The functions u, v satisfy the (strict) inequality (2.15). Integrating this inequality and using the Gauss-Ostrogradskii theorem similarly as in the proof of Theorem 3.1, we have

$$\int_\Omega [K(x)\,\|\nabla u\|^p - C(x)|u|^p]\;\mathrm{d}x > 0,$$

a contradiction. □

As a direct consequence of the above theorem we obtain the following Wirtinger-type inequality.

Corollary 3.4. *If there exists a solution v of $L(v) = 0$ such that $v \neq 0$ in $\overline{\Omega}$, then*

$$\int_\Omega [K(x)\,\|\nabla u\|^p - C(x)|u|^p]\;\mathrm{d}x > 0,$$

for any nontrivial function $u \in C^1(\overline{\Omega}, \mathbb{R})$ such that $u = 0$ on $\partial\Omega$.

We finish this section with two statements which show that if the integral inequality in Theorem 3.1 or Theorem 3.3 is strict, then the zero of the solution of $L(v) = 0$ occurs in Ω. The method of the proof is similar to that used in [9] or [19].

Theorem 3.5. *Let $\partial\Omega \in C^1$ and suppose that there exists a nontrivial function $u \in C^1(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and*

$$\int_\Omega [K(x)\,\|\nabla u\|^p - C(x)|u|^p]\;\mathrm{d}x < 0. \tag{3.1}$$

Then every solution of $L(v) = 0$ has a zero in Ω.

Proof. Conditions imposed on u and Ω imply that $u \in W_0^{1,p}(\Omega)$. Hence (see e.g. [1]) there exists a sequence of functions $u_k \in C_0^\infty(\Omega)$ converging to u in the norm

$$\|w\|_p := \Big(\int_\Omega [\|w\|^p + \|\nabla w\|^p]\,\mathrm{d}x\Big)^{1/p}.$$

Suppose, by contradiction, that there exists a solution v of $L(v) = 0$ such that $v \neq 0$ in Ω. From Corollary 2.4 it follows that for $x \in \Omega$,

$$\operatorname{div}\Big(|u_k|^p \frac{A(x)\|\nabla v\|^{p-2}\nabla v}{|v|^{p-2}v}\Big) \le K(x)\,\|\nabla u_k\|^p - C(x)|u_k|^p.$$

Integrating this inequality over the set $\Omega_k \subset \Omega$ containing the (compact) support of u_k and using the Gauss-Ostrogradskii theorem we have

$$0 \le \int_{\Omega_k} [K(x)\,\|\nabla u_k\|^p - C(x)|u_k|^p]\;\mathrm{d}x = \int_\Omega [K(x)\,\|\nabla u_k\|^p - C(x)|u_k|^p]\;\mathrm{d}x. \tag{3.2}$$

Next, it can be shown in the same way as in [19, Theorem 8.1.3] that

$$\begin{aligned}&\Big|\int_\Omega [K(x)\,\|\nabla u_k\|^p - C(x)|u_k|^p]\,\mathrm{d}x - \int_\Omega [K(x)\,\|\nabla u\|^p - C(x)|u|^p]\,\mathrm{d}x\Big|\\&\le M\Big(\|u_k\|_p + \|u\|_p\Big)^{p-1}\|u_k-u\|_p\,,\end{aligned}$$

where M is a positive constant. Since $\|u_k - u\|_p \to 0$ as $k\to\infty$, we have

$$\lim_{k\to\infty}\int_\Omega [K(x)\,\|\nabla u_k\|^p - C(x)|u_k|^p]\,\mathrm{d}x = \int_\Omega [K(x)\,\|\nabla u\|^p - C(x)|u|^p]\,\mathrm{d}x,$$

which, together with (3.2), contradicts assumption (3.1). □

Corollary 3.6. *Let $\partial\Omega \in C^1$ and suppose that there exists a nontrivial solution u of $l(u) = 0$ such that $u = 0$ on $\partial\Omega$ and*

$$\int_\Omega [(\lambda_{\min}(x) - K(x))\,\|\nabla u\|^p + (C(x) - c(x))|u|^p]\,\mathrm{d}x > 0. \tag{3.3}$$

Then every solution of $L(v) = 0$ has a zero in Ω.

Proof. Inequality (3.3), computation used in (2.3) and Gauss-Ostrogradskii theorem imply

$$\begin{aligned}\int_\Omega [K(x)\,\|\nabla u\|^p - C(x)|u|^p]\,\mathrm{d}x &< \int_\Omega [\lambda_{\min}(x)\,\|\nabla u\|^p - c(x)|u|^p]\,\mathrm{d}x\\&\le \int_\Omega \Big[\operatorname{div}\Big(ua(x)\,\|\nabla u\|^{p-2}\,\nabla u\Big) - ul(u)\Big]\,\mathrm{d}x\\&= 0.\end{aligned}$$

The statement now follows from Theorem 3.5. □

4. An oscillation result

Recall that the half-linear differential equation

$$\big(s(t)|y'|^{p-2}y'\big)' + q(t)|y|^{p-2}y = 0,$$

where $s > 0$, q are real-valued continuous functions on $t \in [t_0,\infty)$, is said to be oscillatory if any nontrivial solution of this equation has a sequence of zeros tending to ∞.

Concerning linear and half-linear partial equations, there are two types of oscillation: *oscillation* (sometimes also weak oscillation) and *nodal oscillation* (strong oscillation).

Denote

$$\Omega(r_0) = \{x \in \mathbb{R}^n : \|x\| \ge r_0\}$$

and assume that the coefficients of the operator L satisfy $A \in C^1(\Omega(r_0), \mathbb{R}^{n\times n})$, $C \in C^{0,\alpha}(\Omega(r_0))$. We say that a solution v of $L(v) = 0$ is *oscillatory* if it has a zero in $\Omega(r)$ for every $r \ge r_0$. Equation $L(v) = 0$ is said to be *oscillatory* if every solution of this equation is oscillatory. The equation $L(v) = 0$ is said to be *nonoscillatory* if it is not oscillatory.

Similarly, the equation $L(v) = 0$ is said to be *nodally oscillatory*, if every its solution has a nodal domain outside of every ball in $\mathbb{R}^n$ and *nodally nonoscillatory* in the opposite case.

It is known that nodal oscillation implies oscillation. The opposite implication is known to be valid only in the linear case $p = 2$ (see [15]) and remains an open question in the half-linear multidimensional case (the case $n = 1$ is trivial). While Riccati technique is suitable to study weak oscillation, Picone identity and variational technique are suitable to study both types of oscillation (and hence both techniques overlap for weak oscillation). In the remaining part of this paper we deal (for simplicity) with the weak oscillation (referred to as oscillation) and show one simple but important application of Picone inequality.

The following oscillation theorem compares oscillation of the PDE $L(u) = 0$ with oscillation of a certain ordinary differential equation. This enables to extend many oscillation criteria from theory of ordinary equations to partial differential equations. Note that the statement we present has been proved using the Riccati technique in [14]. Using Picone identity the proof is simple and straightforward.

Theorem 4.1. *Suppose that the half-linear ordinary differential equation*

$$\tilde{l}(y) := \Big(\tilde{K}(r)|y'|^{p-2}y'\Big)' + \tilde{C}(r)|y|^{p-2}y = 0, \tag{4.1}$$

where

$$\tilde{K}(r) := \int_{\|x\|=r} K(x)\,\mathrm{d}S, \quad \tilde{C}(r) := \int_{\|x\|=r} C(x)\,\mathrm{d}S,$$

is oscillatory. Then the equation $L(v) = 0$ is also oscillatory.

Proof. Let $y = y(r)$ be an (oscillatory) solution of (4.1) and let $r_1 < r_2 < \cdots$, $r_k \to \infty$, be the sequence of its zeros. Denote

$$D_k = \{x \in \mathbb{R}^n;\ r_k \le \|x\| \le r_{k+1}\}, \quad k \in \mathbb{N}.$$

Then the function defined by $u(x) = y(\|x\|)$ satisfies $u(x) = 0$ on ∂D_k for $k \in \mathbb{N}$. By a direct computation we have

$$\begin{aligned}
&\int_{D_k} \left(K(x)\,\|\nabla u(x)\|^p - C(x)|u(x)|^p\right)\,\mathrm{d}x \\
&= \int_{r_k}^{r_{k+1}} \Big[\Big(\int_{\|x\|=r} K(x)\,\mathrm{d}S\Big)|y'(r)|^p - \Big(\int_{\|x\|=r} C(x)\,\mathrm{d}S\Big)|y(r)|^p\Big]\,\mathrm{d}r \\
&= \int_{r_k}^{r_{k+1}} \Big(\tilde{K}(r)|y'(r)|^p - \tilde{C}(r)|y(r)|^p\Big)\,\mathrm{d}r.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
&\int_{r_k}^{r_{k+1}} \Big(\tilde{K}(r)|y'(r)|^p\Big)\,\mathrm{d}r \\
&= \tilde{K}(r)|y'(r)|^{p-2}y'(r)y(r)\Big|_{r_k}^{r_{k+1}} - \int_{r_k}^{r_{k+1}} \Big(\tilde{K}(r)|y'(r)|^{p-2}y'(r)\Big)' y(r)\,\mathrm{d}r \\
&= -\int_{r_k}^{r_{k+1}} \Big(\tilde{K}(r)|y'(r)|^{p-2}y'(r)\Big)' y(r)\,\mathrm{d}r.
\end{aligned}$$

Hence

$$\int_{D_k} \left(K(x)\,\|\nabla u(x)\|^p - C(x)|u(x)|^p\right)\,\mathrm{d}x = -\int_{r_k}^{r_{k+1}} \tilde{l}(y(r))y(r)\,\mathrm{d}r = 0, \quad k \in \mathbb{N}.$$

Now, it follows from Theorem 3.3 that every solution of $L(v) = 0$ has a zero in D_k, $k \in \mathbb{N}$, that is, $L(v) = 0$ is oscillatory. □

Remark 4.2. The statement of the previous theorem remains to hold if we replace the coefficients $\tilde K(r)$, $\tilde C(r)$ in (4.1) by $r^{n-1}\bar K(r)$, $r^{n-1}\bar C(r)$, respectively, where

$$\bar K(r) := \frac{1}{\omega_n r^{n-1}} \int_{\|x\|=r} K(x)\,\mathrm{d}S, \quad \bar C(r) := \frac{1}{\omega_n r^{n-1}} \int_{\|x\|=r} C(x)\,\mathrm{d}S$$

are the spherical means of $K(x)$, $C(x)$, respectively, over the sphere $S_r = \{x \in \mathbb{R}^n;\ \|x\| = r\}$ and where ω_n is the area of the unit sphere in $\mathbb{R}^n$.

Summary. The Picone identity is a very powerful tool in comparison and oscillation theory of equations with p-Laplacian. In this paper we extended this identity to the case of anisotropic p-Laplacian. Despite the fact that we get inequality instead of equality, our extension is shown to be sharp (for isotropic p-Laplacian reduces to the known Picone identity) and we believe, that it can be used to extend most of the results based on the Picone identity to equations with anisotropic p-Laplacian. As an application, we proved related comparison theorems and new variational inequalities. We also showed how to use our result to deduce oscillation of partial differential equations with anisotropic p-Laplacian from oscillation of ordinary differential equations.

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