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OBSERVABILITY AND CONTROLLABILITY FOR A VIBRATING STRING WITH DYNAMICAL BOUNDARY CONTROL

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Abstract. We consider the exact controllability of a wave equation by means of dynamical boundary control. Unlike the classical control, a difficulty is due to the presence of the dynamical type. First, we establish a new weak observability results. Next, by the HUM method, we prove that the system is exactly controllable by means of regular dynamical boundary control.

1. Introduction and statement of the main result

The aim of this paper is to investigate the observability and the exact controllability of the one-dimensional system

$$\begin{gathered} y_{tt} - y_{xx} = 0 \quad 0 < x < 1,\ t > 0, \\ y(0,t) = 0 \quad t > 0, \\ y_x(1,t) + \eta(t) = 0 \quad t > 0, \\ \eta_t(t) - y_t(1,t) = v(t) \quad t > 0 \end{gathered} \tag{1.1}$$

with the initial conditions

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad 0 < x < 1, \quad \eta(0) = \eta_0 \in \mathbb{R} \tag{1.2}$$

where $v(t)$ denotes the dynamical boundary control.

In a previous paper [14], we have considered the energy decay rate of the following one-dimensional wave equation with dynamical boundary control

$$\begin{gathered} y_{tt} - y_{xx} = 0 \quad 0 < x < 1,\ t > 0, \\ y(0,t) = 0 \quad t > 0, \\ y_x(1,t) + \eta(t) = 0 \quad t > 0, \\ \eta_t(t) - y_t(1,t) = -\eta(t) \quad t > 0 \end{gathered} \tag{1.3}$$

with the initial conditions

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad 0 < x < 1, \quad \eta(0) = \eta_0 \in \mathbb{R} \tag{1.4}$$

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where $\eta(t)$ denotes the dynamical boundary control. We proved that the uniform decay rate of the system (1.3)-(1.4) is not true in the energy space. In addition, using a spectral approach, we established the optimal energy decay rate $1/t$ for smooth initial data.

A physical implementation of the dynamic control may be used in pressurized gas tanks with servo controlled actors, as well as in standard mass-spring dampers (see [3] and the references herein). We mention that the dynamical controls form a part of indirect mechanisms proposed by Russell (see [13] and the references herein), see also [15] and [12].

Now let y be a smooth solution of the system (1.1). We define the associated energy

$$E(t) = \frac{1}{2}\Big\{ \int_0^1 (y_x^2 + y_t^2)dx + \eta^2 \Big\}. \tag{1.5}$$

Denoting by $Y(x,t) = (y(x,t), y_t(x,t), \eta(t))$ the state of the system (1.1) and by $V = (0,0,v)$ the control. we can formulate the system (1.1)-(1.2) as an abstract problem

$$Y_t = AY + V, \quad Y(0) = Y_0 \in \mathcal{H} \tag{1.6}$$

where A is an m-dissipative operator on an appropriate Hilbert space $\mathcal{H}$. We obtain thus a weak formulation of the original problem (1.1).

In this paper, our aim is to study the exact controllability of the system (1.1). For this aim, we will adapt the Hilbert Uniqueness Method [5, 6, 7] to the abstract problem (1.6).

First, by a multiplier method, we establish an inverse observability inequality with the usual norm for initial data in $\mathcal{H}$ and consequently, by the HUM method, we prove that the problem (1.6) is exactly controllable by means of singular control $v \in H^1(0,T)'$.

Next, to prove the exact controllability of (1.6) by means of regular control $v \in L^2(0,T)$, we have to establish observability results with a weaker norm (see [6]). Here lies the main difficulty in this paper. In fact, the operator A of the problem (1.6) is not invertible in the energy space, so the method used by Rao in [11] can not be adapted in this case. Indeed, the observability inequalities obtained with the usual norm can not be extended, directly using A^{-1}, to initial data in $D(A)'$. To overcome this difficulty, we establish new intermediate observability results with the usual norm and then, by a suitable change of variable, we extend these results to initial data in $D(A)'$.

In the case of static feedback, the two conditions $y_x(1,t) + \eta(t) = 0$ and $\eta_t(t) - y_t(1,t) = v(t)$ are replaced by the condition $y_x(1,t) + g(y(1,t)) = v(t)$, the exact controllability of the system (1.1)-(1.2) was well studied by different approaches (see [5, 6, 7, 4] and the references herein).

The paper is organized as follows: In section 2, we consider the homogeneous problem associated to (1.1). Using a multiplier method, we first establish direct and inverse observability results with the usual norm; i.e, for initial data in $\mathcal{H}$. Next, we establish new intermediate observability results which leads, by a suitable change, to extend these observability inequalities to initial data in $D(A)'$. In section 3, using the HUM method, we prove that the abstract problem (1.6) is exactly controllable by means of either a singular control $v \in H(0,T)'$ for usual initial data $Y_0 \in \mathcal{H}$ and $T > 2$ or by means of regular control $v \in L^2(0,T)$ for smooth initial data $Y_0 \in D(A)$ and $T > 2$.

2. Observability results

In this section, our aim to establish all observability results necessary to the controllability of the system (1.1) by singular and regular control v. For this purpose we consider the following homogeneous system ($v = 0$):

$$\begin{gathered} \phi_{tt} - \phi_{xx} = 0 \quad 0 < x < 1,\ t > 0, \\ \phi(0,t) = 0 \quad t > 0 \\ \phi_x(1,t) + \xi(t) = 0 \quad t > 0, \\ \xi_t(t) - \phi_t(1,t) = 0 \quad t > 0 \end{gathered} \tag{2.1}$$

with the initial conditions

$$\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad 0 < x < 1, \quad \xi(0) = \xi_0 \in \mathbb{R}. \tag{2.2}$$

First, we will study the well-posedness of the problem (2.1).

2.1. **Well-posedness of the problem.**

To write formally the system (2.1)-(2.2), we, first, introduce

$$V = \{\phi \in H^1(0,1) : \phi(0) = 0\}$$

and define the energy space $\mathcal{H} = V \times L^2(0,1) \times \mathbb{R}$, endowed with the inner product

$$(\Phi, \widetilde{\Phi})_{\mathcal{H}} = \int_0^1 \phi_x \widetilde{\phi}_x dx + \int_0^1 \psi \widetilde{\psi} dx + \xi \widetilde{\xi}, \quad \Phi = (\phi, \psi, \xi), \quad \widetilde{\Phi} = (\widetilde{\phi}, \widetilde{\psi}, \widetilde{\xi}) \in \mathcal{H}.$$

Next we define the linear unbounded operator A on

$$\begin{gathered} D(A) = \{\Phi = (\phi, \psi, \xi) \in \mathcal{H} : \phi \in H^2(0,1), \quad \psi \in V \text{ and } \ \phi_x(1) + \xi = 0\}, \\ A\Phi = (\psi, \phi_{xx}, \psi(1)), \quad \forall \Phi = (\phi, \psi, \xi) \in D(A). \end{gathered}$$

Then setting $\Phi(x,t) = (\phi(x,t), \phi_t(x,t), \xi(t)) \in D(A)$, the state of the system (2.1), we formally transform the problem (2.1)-(2.2) into an evolutionary equation:

$$\Phi_t = A\Phi, \quad \Phi(0) = \Phi_0 \in \mathcal{H}. \tag{2.3}$$

It is easy to check that A is skew adjoint and m-dissipative on $\mathcal{H}$ and therefore generates a strongly continuous group of isometries $S_A(t)$ on the energy space $\mathcal{H}$ (see [9, 2]). So, we have the following existence and uniqueness result.

Proposition 2.1. *(a-) Assume that $\Phi_0 \in \mathcal{H}$. The system* (2.3) *admits a unique weak solution $\Phi(t)$ satisfying*

$$\Phi(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

(b-) Assume that $\Phi_0 \in D(A)$. The system (2.3) *admits a unique strong solution $\Phi(t)$ satisfying*

$$\Phi(t) \in C^0(\mathbb{R}^+; D(A)) \cap C^1(\mathbb{R}^+; \mathcal{H})$$

and we have

$$\|\Phi(t)\|_{\mathcal{H}} = \|\Phi_0\|_{\mathcal{H}}, \quad \forall t \in \mathbb{R}^+.$$

Then we will establish two observability results for usual initial data.

2.2. **Observability results for initial data in $\mathcal{H}$.** In this part, by a multiplier method, we establish the following observability results.

Theorem 2.2. *Let $T > 2$ be arbitrarily. Then for every $\Phi_0 \in \mathcal{H}$ the solution Φ of the system* (2.3) *satisfies the following inequalities*

$$\frac{1}{T+2}\int_0^T \big[|\phi_x(1,t)|^2 + |\phi_{xt}(1,t)|^2\big]dt \leq \|\Phi_0\|^2_{\mathcal{H}}, \tag{2.4}$$

$$\|\Phi_0\|^2_{\mathcal{H}} \leq \frac{2}{T-2}\int_0^T \big[|\phi_x(1,t)|^2 + |\phi_{xt}(1,t)|^2\big]dt. \tag{2.5}$$

Proof. Assume that $\Phi_0 \in D(A^2)$. Multiplying the equation (2.1) by $2x\phi_x$ and integrating by parts, we obtain

$$\int_0^1\int_0^T \Big(\phi_t^2 + \phi_x^2\Big)\,dx\,dt = -2\Big[\int_0^1 \phi_t x\phi_x dx\Big]_0^T + \int_0^T \Big(|\phi_{xt}(1,t)|^2 + |\phi_x(1,t)|^2\Big)dt.$$

This implies

$$T\|\Phi_0\|^2_{\mathcal{H}} + 2\Big[\int_0^1 \phi_t x\phi_x dx\Big]_0^T = \int_0^T \Big(|\phi_{xt}(1,t)|^2 + 2|\phi_x(1,t)|^2\Big)dt. \tag{2.6}$$

On the other hand, using Cauchy-Schwartz inequality, we deduce that

$$|2\int_0^1 \phi_t x\phi_x dx| \leq \|\Phi_0\|^2_{\mathcal{H}}, \quad \forall t \in \mathbb{R}^+.$$

Finally we have

$$-2\|\Phi_0\|^2_{\mathcal{H}} \leq 2\Big[\int_0^1 \phi_t x\phi_x dx\Big]_0^T \leq 2\|\Phi_0\|^2_{\mathcal{H}}. \tag{2.7}$$

Inserting (2.7) in (2.6) we obtain (2.4) and (2.5) for every $\Phi_0 \in D(A^2)$. By a density argument we prove (2.4) and (2.5) for every $\Phi_0 \in \mathcal{H}$. The proof is thus complete. □

Remark 2.3. (i) There exists no constant $c > 0$ such that

$$\|\Phi_0\|^2_{\mathcal{H}} \leq c\int_0^T |\phi_{xt}(1,t)|^2 dt.$$

In fact, it easy to see that the operator A has 0 as an eigenvalue, with an associated eigenfunction $\Psi_0 = (x, 0, -1)$. Let $\Phi_0 = \Psi_0$ then $\Phi = \Phi_0$ is the solution of the problem (2.3) and we have

$$\|\Phi_0\|^2_{\mathcal{H}} = 2, \quad \text{and} \quad \int_0^T |\phi_{xt}(1,t)|^2 dt = 0.$$

(ii) There exists no constant $c > 0$ such that

$$\|\Phi_0\|^2_{\mathcal{H}} \leq c\int_0^T |\phi_x(1,t)|^2 dt.$$

In fact, the skew operator A has $i\mu_n \in i\mathbb{R}$, $n \in \mathbb{Z}$, isolated eigenvalues with algebraic multiplicity one and $|\mu_n|$ goes to infinity as n goes to infinity. Moreover, μ_n has the following asymptotic expansion (see [14])

$$\mu_n = n\pi + \frac{\pi}{2} + \frac{1}{n\pi} - \frac{1}{2n^2\pi} + O(\frac{1}{n^3}), \quad \text{as } n \to \infty.$$

The associated eigenvectors:

$$\Psi_0 = (x, 0, -1), \quad \Psi_n = (\frac{1}{\mu_n}\sin(\mu_n x), i\sin(\mu_n x), -\cos(\mu_n)), \quad \forall n \in \mathbb{Z}^*.$$

Let $\Phi_0^n = \Psi_n$, $n \in \mathbb{Z}^*$, then $\Phi^n = e^{i\mu_n t}\Psi_n$ is the solution of the problem (2.3) and we have

$$\|\Phi_0^n\|_{\mathcal{H}}^2 = 1 + |\cos(\mu_n)|^2 \to 1, \quad \text{and} \quad \int_0^T |\phi_x(1,t)|^2 dt = T|\cos(\mu_n)|^2 \to 0.$$

We conclude that the usual inverse observability inequalities obtained for the classical wave equation (see [6, 5]) does not hold in this case.

(iii) The observability inequality (2.5) leads, by the HUM method, to the exact controllability of (1.1) by means of singular control $v \in H^1(0,1)'$.

2.3. **Observability results for initial data in** $D(A)'$**.** To prove that the system (1.1) is exactly controllable by means of regular control $v \in L^2(0,T)$, we have to establish an inverse observability inequality with a weaker norm in [6, pp. 122-127]. For this aim, we will extend the inverse observability inequality, obtained for usual initial data, to initial data in $D(A)'$. Since the operator A is not invertible, the classical methods based on using A^{-1} to obtain the extension (see [11]) can not be adapted for this system. To overcome this difficulty, we first establish two intermediate observability inequalities based on the following theorem.

Theorem 2.4. *Let $T > 2$ and $\alpha > 1$ be a real number. Then there exist constants $c_1(T) > 0$ and $c_2(T) > 0$ such that for every $\Phi_0 \in \mathcal{H}$ the solution Φ of the system (2.3) satisfies the following inequalities*

$$c_1 \int_0^T \Big[\Big(\phi_x(1,t) - e^{-\alpha t}\phi_x(1,0)\Big)^2 + \Big(\alpha\phi_x(1,t) + \phi_{xt}(1,t)\Big)^2\Big] dt \le \|\Phi_0\|_{\mathcal{H}}^2, \tag{2.8}$$

$$\|\Phi_0\|_{\mathcal{H}}^2 \le c_2 \int_0^T \Big[\Big(\phi_x(1,t) - e^{-\alpha t}\phi_x(1,0)\Big)^2 + \Big(\alpha\phi_x(1,t) + \phi_{xt}(1,t)\Big)^2\Big] dt. \tag{2.9}$$

Proof. It is sufficient to prove the estimates (2.8) and (2.9) for $\Phi_0 \in D(A)$ the case of $\Phi_0 \in \mathcal{H}$, then follows by a density argument. First, a direct computation gives

$$\begin{aligned} &\int_0^T \Big[\Big(\phi_x(1,t) - e^{-\alpha t}\phi_x(1,0)\Big)^2 + \Big(\alpha\phi_x(1,t) + \phi_{xt}(1,t)\Big)^2\Big] dt \\ &\le 2(1+\alpha^2)\int_0^T \Big[|\phi_x(1,t)|^2 + |\phi_{xt}(1,t)|^2\Big] dt + \Big(\frac{1}{\alpha} - \frac{1}{\alpha}e^{-2\alpha T}\Big)\phi_x^2(1,0). \end{aligned} \tag{2.10}$$

On the other hand, using the definition of the norm we have $\phi_x^2(1,0) = \xi_0^2 \le \|\Phi_0\|_{\mathcal{H}}^2$. Then inserting (2.4) into (2.10), we obtain the direct inequality (2.8), and we have

$$c_1^{-1} = 2(T+2)(1+\alpha^2) + (\frac{1}{\alpha} - \frac{1}{\alpha}e^{-2\alpha T}).$$

Next, we verify the inverse inequality (2.9) by contradiction. Assume that (2.9) fails, then there exists a sequence $(\Phi^n)_{n\in\mathbb{N}}$ such that

$$\|\Phi^n(t)\|_{\mathcal{H}} = \|\Phi_0^n\|_{\mathcal{H}} = 1, \quad \forall t \in \mathbb{R} \tag{2.11}$$

and

$$\int_0^T \Big[\Big(\phi_x^n(1,t) - e^{-\alpha t}\phi_x^n(1,0)\Big)^2 + \Big(\alpha\phi_x^n(1,t) + \phi_{xt}^n(1,t)\Big)^2\Big] dt \to 0 \tag{2.12}$$

where $\Phi^n(t) = (\phi^n(x,t), \phi^n_t(x,t), \xi^n(t))$ is the solution of the problem

$$\Phi^n_t = A\Phi^n, \quad \Phi^n(0) = \Phi^n_0. \tag{2.13}$$

Since $|\phi^n_x(1,0)|^2 \le \|\Phi^n_0\|^2_{\mathcal{H}} = 1$ then there exists a subsequence $\phi^n_x(1,0)$, still indexed by n for convenience, that converges to a constant $-1 \le c \le 1$ as $n \to +\infty$. From (2.12) we deduce that

$$\phi^n_x(1,t) \to ce^{-\alpha t}, \quad \text{in } L^2(0,T) \tag{2.14}$$

$$\varepsilon_n = \int_0^T \Big(\alpha\phi^n_x(1,t) + \phi^n_{xt}(1,t)\Big)^2 dt \to 0. \tag{2.15}$$

Since $\alpha > 1$, then using the inequality (2.5) we have

$$\frac{T-2}{2}\|\Phi^n_0\|^2_{\mathcal{H}} \le \varepsilon_n + \alpha|\phi^n_x(1,0)|^2. \tag{2.16}$$

Using the linearity of the problem, (2.14)-(2.16) and the trace theorem, we conclude that, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \ge n_0$,

$$\frac{T-2}{2}\|\Phi^n(t) - \Phi^m(t)\|^2_{\mathcal{H}} \le 2(\varepsilon_n + \varepsilon_m) + \alpha|\phi^n_x(1,0) - \phi^m_x(1,0)|^2 \le \varepsilon.$$

Then $(\Phi^n(t))$ is a cauchy sequence in $\mathcal{H}$. This implies that

$$\Phi^n(t) \to \Phi(t), \quad \text{strongly in } \mathcal{H}.$$

Using (2.5), (2.11), (2.13) and (2.14) we deduce that $\Phi(t) = (\phi(x,t), \phi_t(x,t), \xi(t))$ solves the problem

$$\Phi_t = A\Phi, \quad \Phi(0) = \Phi_0 \tag{2.17}$$

and the supplementary conditions

$$\phi_x(1,t) = ce^{-\alpha t}, \quad t > 0, \tag{2.18}$$

$$\|\Phi(t)\|_{\mathcal{H}} = 1, \quad t > 0. \tag{2.19}$$

Now, let $\Phi = (\phi, \phi_t, \xi)$ be the solution of (2.17)-(2.18) then, using Remark 2.3 (ii), we have

$$\Phi(x,t) = \sum a_n \Psi_n(x) e^{i\mu_n t}, \quad a_n \in \mathbb{C}$$

where the sequence μ_n satisfies $\sin\mu_n = -\mu_n \cos\mu_n$. This implies

$$\xi(t) = -\phi_x(1,t) = -ce^{-\alpha t} = -\sum a_n \cos(\mu_n) e^{i\mu_n t}. \tag{2.20}$$

Noting that $\mu_n \in \mathbb{R}$, then using (2.19) we deduce that

$$\sum |a_n \cos(\mu_n)|^2 < \infty. \tag{2.21}$$

Using (2.20) and (2.21) we conclude, from Riesz-Fisher theorem [1, pp. 110], that the function $ce^{-\alpha t}$ is a $\mathcal{B}^2$ almost periodic function. Then the Parseval equation is true for $ce^{-\alpha t}$ [1, pp. 109]; i.e.,

$$\sum |a_n \cos(\mu_n)|^2 = M\{c^2 e^{-2\alpha t}\}$$

where the mean value $M\{c^2e^{-2\alpha t}\}$ is given by

$$M\{c^2 e^{-2\alpha t}\} = \lim_{X\to+\infty} \frac{1}{X}\int_0^X c^2 e^{-2\alpha t} dt = 0.$$

This, together the fact that $\cos\mu_n \ne 0$, implies that $a_n = 0$ for all $n \in \mathbb{Z}^\star$ and $c = 0$. Applying Holmgren's theorem [8], the system (2.17)-(2.18) admits the unique trivial solution $\Phi = 0$, this contradicts (2.19). The proof is thus complete. □

Next, by a suitable change of variable, we will establish a direct and inverse observability inequality for initial data in $D(A)'$.

Theorem 2.5. *Let $T > 2$ and $\alpha > 1$ be an arbitrarily real number. Then there exist constants $c_3(T) > 0$ and $c_4(T) > 0$ such that the solution of the system (2.3) satisfies the following inequalities*

$$c_3 \int_0^T \Big[\Big(\int_0^t \phi_x(1,s)e^{\alpha s}ds\Big)^2 + e^{2\alpha t}|\phi_x(1,t)|^2\Big]dt \le \|\Phi_0\|^2_{D(A)'}, \tag{2.22}$$

$$\|\Phi_0\|^2_{D(A)'} \le c_4 \int_0^T \Big[\Big(\int_0^t \phi_x(1,s)e^{\alpha s}ds\Big)^2 + e^{2\alpha t}|\phi_x(1,t)|^2\Big]dt. \tag{2.23}$$

Proof. It is sufficient to prove (2.22) and (2.23) for $\Phi_0 \in D(A)$ the general case follows by a density argument. Let $\Phi_0 \in D(A)$ then the problem (2.3) has a unique solution $\Phi \in D(A)$. We define a new function $\Psi(x,t)$ by

$$\Psi(x,t) = e^{\alpha t}\Phi(x,t).$$

It easy to see that Ψ solve the equation

$$\Psi_t = (\alpha I + A)\Psi, \quad \Psi(0) = \Psi_0 = \Phi_0 \in \mathcal{H}. \tag{2.24}$$

Replacing ϕ by $e^{-\alpha t}\psi$ in (2.8)-(2.9) we obtain

$$c_1 e^{-2\alpha T}\int_0^T \Big[\Big(\psi_x(1,t) - \psi_x(1,0)\Big)^2 + |\psi_{xt}(1,t)|^2\Big]dt \le \|\Psi_0\|^2_{\mathcal{H}} \tag{2.25}$$

and

$$\|\Psi_0\|^2_{\mathcal{H}} \le c_2 \int_0^T \Big[\Big(\psi_x(1,t) - \psi_x(1,0)\Big)^2 + |\psi_{xt}(1,t)|^2\Big]dt. \tag{2.26}$$

Defining

$$\widetilde{\Psi}_0 = (\alpha I + A)^{-1}\Psi_0 = (\alpha I + A)^{-1}\Phi_0.$$

Then

$$\|\widetilde{\Psi}_0\|^2_{\mathcal{H}} = \|(\alpha I + A)^{-1}\Phi_0\|^2_{\mathcal{H}} = \|\Phi_0\|^2_{D(\alpha I + A)'}. \tag{2.27}$$

Now, let $\widetilde{\Psi}$ the solution of the equation

$$\widetilde{\Psi}_t = (\alpha I + A)\widetilde{\Psi}, \quad \widetilde{\Psi}(0) = (\alpha I + A)^{-1}\Phi_0. \tag{2.28}$$

Applying the inequalities (2.25)-(2.26) to $\widetilde{\Psi}$ we obtain

$$c_1 e^{-2\alpha T}\int_0^T \Big[\Big(\widetilde{\psi}_x(1,t) - \widetilde{\psi}_x(1,0)\Big)^2 + |\widetilde{\psi}_{xt}(1,t)|^2\Big]dt \le \|\Phi_0\|^2_{D(\alpha I + A)'} \tag{2.29}$$

and

$$\|\Phi_0\|^2_{D(\alpha I + A)'} \le c_2 \int_0^T \Big[\Big(\widetilde{\psi}_x(1,t) - \widetilde{\psi}_x(1,0)\Big)^2 + |\widetilde{\psi}_{xt}(1,t)|^2\Big]dt. \tag{2.30}$$

Using (2.28) we have

$$\widetilde{\Psi}_t(0) = (\alpha I + A)\widetilde{\Psi}(0) = \Phi_0.$$

Then $\widetilde{\Psi}_t$ solve the equation

$$\widetilde{\Psi}_{tt} = (\alpha I + A)\widetilde{\Psi}_t, \quad \widetilde{\Psi}_t(0) = \Phi_0. \tag{2.31}$$

This implies that $\widetilde{\Psi}_t = \Psi$ and

$$\widetilde{\psi}_x(1,t) - \widetilde{\psi}_x(1,0) = \int_0^t \psi_x(1,s)ds, \quad \widetilde{\psi}_{xt}(1,t) = \psi_x(1,t).$$

Using (2.29) and (2.30) we obtain

$$c_1 e^{-2\alpha T}\int_0^T \Big[\Big(\int_0^t \psi_x(1,s)ds\Big)^2 + |\psi_x(1,t)|^2\Big]dt \le \|\Phi_0\|^2_{D(\alpha I+A)'} \tag{2.32}$$

and

$$\|\Phi_0\|^2_{D(\alpha I+A)'} \le c_2\int_0^T \Big[\Big(\int_0^t \psi_x(1,s)ds\Big)^2 + |\psi_x(1,t)|^2\Big]dt. \tag{2.33}$$

On the other hand, we have

$$\|\Phi_0\|^2_{D(A)} \le \|\Phi_0\|^2_{D(\alpha I+A)} \le (1+\alpha)\|\Phi_0\|^2_{D(A)}.$$

This implies that $\|\cdot\|_{D(\alpha I+A)'}$ and $\|\cdot\|_{D(A)'}$ are equivalent. Replacing $\psi_x(1,t)$ by $e^{\alpha t}\phi_x(1,t)$, we obtain (2.22) and (2.23) with

$$c_3 = c_1 e^{-2\alpha T}, \quad c_4 = c_2.$$

The proof is complete. □

3. Exact controllability of the system

In this section we study the exact controllability result in $\mathcal{H}$ the controlled system

$$\begin{gathered} y_{tt} - y_{xx} = 0 \quad 0<x<1, \quad t>0,\\ y(0,t) = 0 \quad t>0,\\ y_x(1,t) + \eta(t) = 0 \quad t>0,\\ \eta_t(t) - y_t(1,t) = v(t) \quad t>0 \end{gathered} \tag{3.1}$$

with the initial conditions

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad 0<x<1, \quad \eta(0) = \eta_0 \in \mathbb{R}. \tag{3.2}$$

Setting $Y(x,t) = (y(x,t), y_t(x,t), \eta(t))$ the state of the system (3.1)-(3.2) we formally transform the problem into an evolutionary problem

$$Y_t = AY + V, \quad Y(0) = Y_0 \in \mathcal{H} \tag{3.3}$$

where $V = (0,0,v)$.

3.1. **Exact controllability for initial data in $\mathcal{H}$.** The observability inequalities for usual initial data obtained in the subsection 2.2 leads, by the HUM method, to the exact controllability of the system (3.3) by means of singular control $v \in H^1(0,T)'$. Now, let $\Phi = (\phi, \phi_t, \xi)$ be a solution of the homogeneous problem (2.3). Multiplying the equation (3.3) by Φ and integrating by parts so that we obtain formally

$$(Y_0, \Phi_0)_{\mathcal{H}} + \int_0^t v(s)\xi(s)ds = (Y(x,t), \Phi(x,t))_{\mathcal{H}}. \tag{3.4}$$

Identify the Hilbert space $\mathcal{H}$ with its dual and define the linear form L by setting

$$L(\Phi_0) = (Y_0, \Phi_0)_{\mathcal{H}} + \int_0^t v(s)\xi(s)ds, \quad \forall \Phi_0 \in \mathcal{H} \tag{3.5}$$

we obtain a weak formulation of the problem (3.3).

$$L(\Phi_0) = (Y(x,t), \Phi(x,t))_{\mathcal{H}} = (Y(x,t), S_A(t)\Phi_0)_{\mathcal{H}}, \quad \forall \Phi_0 \in \mathcal{H} \tag{3.6}$$

where $S_A(t)$ the group of isometries associated to the homogeneous problem (2.3).

Next, we consider the exact controllability of the equation (3.3) for usual initial data $Y_0 \in \mathcal{H}$. We choose the control

$$v(t) = v_0(t) - \frac{d}{dt}v_1(t), \quad v_0 \in L^2(0,T), \quad \frac{d}{dt}v_1(t) \in H^1(0,T)' \tag{3.7}$$

where the derivative $\frac{d}{dt}$ is defined in the sense of $H^1(0,T)'$

$$-\int_0^T \frac{d}{dt}v_1(t)\mu(t)dt = \int_0^T v_1(t)\frac{d}{dt}\mu(t)dt, \quad \forall \mu \in H^1(0,T). \tag{3.8}$$

Theorem 3.1. *Let $T > 0$ and v be chosen in (3.7). For every $Y_0 \in \mathcal{H}$, the controlled system (3.3) admits a unique weak solution $Y(x,t)$ such that*

$$Y(x,t) \in C^0([0,T];\mathcal{H}) \tag{3.9}$$

defined in the sense that the equation (3.6) is satisfied for all $\Phi_0 \in \mathcal{H}$ and all $0 < t < T$. Moreover the linear mapping

$$(Y_0, v_0, v_1) \to Y \tag{3.10}$$

is continuous form $\mathcal{H} \times L^2(0,T) \times L^2(0,T)$ into $\mathcal{H}$.

Proof. Let $\Phi_0 \in \mathcal{H}$ and $\Phi = (\phi, \phi_t, \xi)$ be the solution of the system (2.3). We have

$$\begin{aligned}
&|\int_0^t v(s)\xi(s)ds| \\
&= |\int_0^t (v_0 - \frac{d}{ds}v_1(s))\phi_x(1,s)ds| \\
&= |\int_0^t v_0\phi_x(1,s)ds + \int_0^t v_1(s)\phi_{xs}(1,s)ds| \\
&\le \|v_0\|_{L^2(0,T)}\|\phi_x(1,.)\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)}\|\phi_{xt}(1,.)\|_{L^2(0,T)} \\
&\le (\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)})(\|\phi_x(1,.)\|_{L^2(0,T)} + \|\phi_{xt}(1,.)\|_{L^2(0,T)}).
\end{aligned} \tag{3.11}$$

Using (2.4), (3.5) and (3.11) we obtain

$$|L(\Phi_0)| \le \Big[\sqrt{2(T+2)}(\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)}) + \|Y_0\|_{\mathcal{H}}\Big]\|\Phi_0\|_{\mathcal{H}}$$

for all $\Phi_0 \in \mathcal{H}$. This implies that the linear form L is continuous in the space $\mathcal{H}$. And we have

$$\|L\|_{\mathcal{L}(\mathcal{H},\mathbb{R})} \le \sqrt{2(T+2)}(\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)}) + \|Y_0\|_{\mathcal{H}}.$$

From Riesz's representation theorem, there exist a unique $Z(x,t) \in \mathcal{H}$ solution of the following problem

$$L(\Phi_0) = (Z(x,t), \Phi_0)_{\mathcal{H}}, \quad \forall \Phi_0 \in \mathcal{H}.$$

Finally, we define $Y(x,t)$ by $S_A(t)Y(x,t) = -Z(x,t)$ and we deduce that $Y(x,t)$ is the unique solution of the problem (3.6). And we have

$$\|Y(x,t)\|_{\mathcal{H}} \le \sqrt{2(T+2)}(\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)}) + \|Y_0\|_{\mathcal{H}}, \quad \forall t \in [0,T].$$

This implies that the linear application (3.10) is continuous from $\mathcal{H} \times L^2(0,T) \times L^2(0,T)$ into $\mathcal{H}$. The proof is thus complete. □

Theorem 3.2. *Let $T > 2$. For all $Y_0 \in \mathcal{H}$, there exists a control $v(t) = v_0(t) - \frac{d}{dt}v_1(t)$, $v_0, v_1 \in L^2(0,T)$ such that the weak solution $Y(x,t)$ of the controlled problem* (3.3) *satisfies the final condition*

$$Y(T) = 0. \tag{3.12}$$

Proof. Let Φ be the solution of the homogeneous system (2.3) with initial data $\Phi_0 \in \mathcal{H}$. We define the semi-norm

$$\|\Phi_0\|_1^2 = \int_0^T (|\phi_x(1,t)|^2 + |\phi_{xt}(1,t)|^2)dt, \quad \forall \Phi_0 \in \mathcal{H}. \tag{3.13}$$

Thanks to inequalities (2.4) and (2.5), we know that (3.13) defines an equivalent norm in the energy space $\mathcal{H}$. Now, choosing the controller $v(t)$ as

$$v(t) = v_0(t) - \frac{d}{dt}v_1(t) =: -\phi_x(1,t) + \frac{d}{dt}\phi_{xt}(1,t) \tag{3.14}$$

where the derivative $\frac{d}{dt}$ is defined in the sense of (3.8). Using the direct inequality (2.4), we have

$$\|v_0(t)\|_{L^2(0,T)} + \|v_1(t)\|_{L^2(0,T)} \leq \sqrt{2(T+2)}\|\Phi_0\|_{\mathcal{H}}. \tag{3.15}$$

Now solve the backward problem

$$\Psi_t = A\Psi + V, \quad \Psi(T) = 0. \tag{3.16}$$

Using Theorem 2.4 the problem (3.16) admits a unique weak solution $\Psi(x,t) \in C^0([0,T];\mathcal{H})$, and we have

$$\|\Psi\|_{\mathcal{H}} \leq \sqrt{2(T+2)}(\|v_0(t)\|_{L^2(0,T)} + \|v_1(t)\|_{L^2(0,T)}). \tag{3.17}$$

Next we define the operator Λ as

$$\Lambda\Phi_0 = -\Psi(0), \quad \forall \Phi_0 \in \mathcal{H}. \tag{3.18}$$

By virtue of inequalities (3.15) and (3.17) we obtain

$$\|\Lambda\Phi_0\|_{\mathcal{H}} \leq \sqrt{2(T+2)}(\|v_0(t)\|_{L^2(0,T)} + \|v_1(t)\|_{L^2(0,T)}) \leq 2(T+2)\|\Phi_0\|_{\mathcal{H}}.$$

This implies that Λ is a linear continuous operator from $\mathcal{H}$ into $\mathcal{H}$. Multiplying the backward problem (3.16) by Φ and integrating by parts we obtain

$$-(\Psi_0, \Phi_0)_{\mathcal{H}} = \int_0^T (|\phi_x(1,t)|^2 + |\phi_{xt}(1,t)|^2)dt. \tag{3.19}$$

This implies

$$(\Lambda\Phi_0, \Phi_0)_{\mathcal{H}} = \|\Phi_0\|_1^2. \tag{3.20}$$

Thanks to the Lax-Milgram theorem, we deduce that Λ is an isomorphism from $\mathcal{H}$ onto $\mathcal{H}$. In particular, given any $-Y_0 \in \mathcal{H}$, there exists a unique $\Phi_0 \in \mathcal{H}$ such that

$$\Lambda\Phi_0 = -Y_0. \tag{3.21}$$

This equality implies that the weak solution $Y(x,t)$ of backward problem (3.16), with v given by (3.14) satisfy the initial value condition $Y(x,0) = Y_0$ and that final condition $Y(x,T) = 0$. The proof is complete. □

3.2. **Exact controllability for initial data in $D(A)$.** Now we consider the exact controllability of the equation (3.3) by means of a regular control $v \in L^2(0,T)$

$$v(t) = -e^{\alpha t} v_0(t) + e^{\alpha t} \int_T^t \Big(\int_0^s v_0(\tau) e^{\alpha\tau} d\tau \Big) ds, \quad v_0(t) \in L^2(0,T). \tag{3.22}$$

For the wellposedness of the equation (3.3) with the control (3.22) we first interpret (3.6) into the following form

$$L(\Phi_0) = \langle Y(x,t), S_A(t)\Phi_0 \rangle_{D(A)\times D(A)'}, \quad \forall \Phi_0 \in D(A)' \tag{3.23}$$

where the linear form L is defined by

$$L(\Phi_0) = \langle Y_0, \Phi_0 \rangle_{D(A)\times D(A)'} + \int_0^t v(s)\xi(s) ds, \quad \forall \Phi_0 \in D(A)'. \tag{3.24}$$

Theorem 3.3. *Let $T > 0$ and $v \in L^2(0,T)$ defined by (3.22). For every $Y_0 \in D(A)$ the controlled system (3.3) admits a unique weak solution satisfying*

$$Y(x,t) \in C^0\Big([0,T]; D(A)\Big) \tag{3.25}$$

defined in the sense that the equation (3.23) is satisfied for all $\Phi_0 \in D(A)'$ and all $0 < t < T$. Moreover the linear application

$$(Y_0, v_0) \to Y \tag{3.26}$$

is continuous form $D(A) \times L^2(0,T)$ into $D(A)$.

Proof. Let $\Phi_0 \in D(A)'$ and $\Phi = (\phi, \phi_t, \xi)$ be the solution of the system (2.3). It easy to see that

$$\begin{aligned} &| \int_0^t v(s)\xi(s) ds| \\ &= | \int_0^t e^{\alpha s} v_0(s) \phi_x(1,s) ds + \int_0^t \Big(\int_0^s e^{\alpha\tau} \phi_x(1,\tau) d\tau \Big) \Big(\int_0^s e^{\alpha\tau} v_0(\tau) d\tau \Big) ds| \\ &\le e^{\alpha T} \|v_0\|_{L^2(0,T)} \|\phi_x(1,.)\|_{L^2(0,T)} + \|\tilde{v}_0\|_{L^2(0,T)} \|\tilde{\phi}_x(1,.)\|_{L^2(0,T)} \end{aligned}$$

where

$$\tilde{\phi}_x(1,t) = \int_0^t e^{\alpha s} \phi_x(1,s) ds, \quad \tilde{v}_0(t) = \int_0^t e^{\alpha s} v_0(s) ds.$$

We deduce that

$$| \int_0^t v(s)\xi(s) ds \big| \le c_5 \|v_0\|_{L^2(0,T)} (\|\phi_x(1,.)\|_{L^2(0,T)} + \|\tilde{\phi}_x(1,.)\|_{L^2(0,T)})$$

where c_5 is a constant given by

$$c_5 = 1 + \sqrt{\frac{T(e^{2\alpha T} - 1)}{2\alpha}}.$$

Using (2.22) and (3.24) we obtain

$$|L(\Phi_0)| \le (\sqrt{2} c_5 c_3^{-1/2} \|v_0\|_{L^2(0,T)} + \|Y_0\|_{D(A)}) \|\Phi_0\|_{D(A)'}, \quad \forall \Phi_0 \in D(A)'.$$

This implies that the linear form L is continuous in the space $D(A)'$, and we have

$$\|L\|_{\mathcal{L}(D(A)',\mathbb{R})} \le \sqrt{2} c_5 c_3^{-1/2} \|v_0\|_{L^2(0,T)} + \|Y_0\|_{D(A)}.$$

From the Riesz representation theorem, there exist a unique $Z(x,t) \in D(A)$ solution of the following problem

$$L(\Phi_0) =< Z(x,t), \Phi_0\rangle_{D(A)\times D(A)'}, \quad \forall \Phi_0 \in D(A)'.$$

Finally, we define $Y(x,t)$ by $S_A(t)Y(x,t) = -Z(x,t)$ and we deduce that $Y(x,t)$ is the unique solution of the problem (3.23), and we have

$$\|Y(x,t)\|_{D(A)} \leq \sqrt{2}c_5 c_3^{-1/2}\|v_0\|_{L^2(0,T)} + \|Y_0\|_{D(A)}, \quad \forall t \in [0,T].$$

This implies that the linear application (3.26) is continuous from $D(A) \times L^2(0,T)$ into $D(A)$. The proof is complete. □

Theorem 3.4. *Let $T > 2$. For all $Y_0 \in D(A)$, there exists a control $v(t) \in L^2(0,T)$ such that the weak solution $Y(x,t)$ of the controlled problem* (3.3) *satisfies the final condition*

$$Y(T) = 0. \tag{3.27}$$

Proof. Let $\Phi_0 \in D(A)'$ and Φ be the solution of the homogeneous system (2.3). We define the semi-norm

$$\|\Phi_0\|_2 = \int_0^T [(\int_0^t \phi_x(1,s)e^{\alpha s}ds)^2 + e^{2\alpha t}|\phi_x(1,t)|^2]dt, \quad \forall \Phi_0 \in D(A)'. \tag{3.28}$$

Thanks to inequalities (2.22) and (2.23), we know that (3.28) defines an equivalent norm in the energy space $D(A)'$. Now, choosing the controller $v(t)$ by

$$v(t) = -e^{\alpha t}\phi_x(1,t) + e^{\alpha t}\int_T^t \Big(\int_0^s \phi_x(1,\tau)e^{\alpha\tau}d\tau\Big)ds \in L^2(0,T). \tag{3.29}$$

From the direct inequality (2.22), we have

$$\|v_0\|_{L^2(0,T)} \leq c_1^{-1/2}\|\Phi_0\|_{D(A)'}. \tag{3.30}$$

Next we solve the backward problem

$$\Psi_t = A\Psi + V, \quad \Psi(T) = 0. \tag{3.31}$$

Using Theorem 2.4 the problem (3.31) admits a unique weak solution $\Psi(x,t) \in C^0([0,T]; D(A))$. And we have

$$\|\Psi\|_{D(A)} \leq \sqrt{2}c_5 c_3^{-1/2}\|v_0\|_{L^2(0,T)}. \tag{3.32}$$

Next we define the operator Λ as

$$\Lambda\Phi_0 = -\Psi(0), \quad \forall \Phi_0 \in D(A)'. \tag{3.33}$$

By virtue of inequalities (3.30) and (3.32) we obtain

$$\|\Lambda\Phi_0\|_{D(A)} \leq \sqrt{2}c_5 c_1^{-1}e^{\alpha T}\|\Phi_0\|_{D(A)'}.$$

This implies that Λ is a linear continuous operator from $D(A)'$ into $D(A)$. Now multiplying the backward problem (3.31) by Φ and integrating by parts we obtain

$$-\langle\Psi_0, \Phi_0\rangle_{D(A)\times D(A)'} = \int_0^T \Big[\Big(\int_0^t \phi_x(1,s)e^{\alpha s}ds\Big)^2 + e^{2\alpha t}|\phi_x(1,t)|^2\Big]dt. \tag{3.34}$$

This implies

$$\langle\Lambda\Phi_0, \Phi_0\rangle_{D(A)\times D(A)'} = \|\Phi_0\|_2^2. \tag{3.35}$$

Thanks to the the Lax-Milgram theorem, we deduce that Λ is an isomorphism from $D(A)'$ into $D(A)$. In particular, given any $-Y_0 \in D(A)$, there exists a unique $\Phi_0 \in D(A)'$ such that

$$\Lambda\Phi_0 = -Y_0. \tag{3.36}$$

This equality implies that the weak solution $Y(x,t)$ of backward problem (3.33), with v given by (3.29) satisfy the initial value condition $Y(x,0) = Y_0$ and that final condition $Y(x,T) = 0$. The proof is thus complete. □

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