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ASYMPTOTIC BEHAVIOR OF GROUND STATE SOLUTION FOR HÉNON TYPE SYSTEMS

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Abstract. In this article, we investigate the asymptotic behavior of positive ground state solutions, as $\alpha \to \infty$, for the following Hénon type system

$$-\Delta u = \frac{2p}{p+q}|x|^\alpha u^{p-1}v^q, \quad -\Delta v = \frac{2q}{p+q}|x|^\alpha u^p v^{q-1}, \quad \text{in } B_1(0)$$

with zero boundary condition, where $B_1(0) \subset \mathbb{R}^N$ $(N \geq 3)$ is the unit ball centered at the origin, $p, q > 1$, $p + q < 2^* = 2N/(N-2)$. We show that both components of the ground solution pair (u, v) concentrate on the same point on the boundary $\partial B_1(0)$ as $\alpha \to \infty$.

1. Introduction

In this article, we investigate the asymptotic behavior of positive ground state solution pairs of the following Hénon type system

$$-\Delta u = \frac{2p}{p+q}|x|^\alpha u^{p-1}v^q, \quad -\Delta v = \frac{2q}{p+q}|x|^\alpha u^p v^{q-1}, \quad \text{in } B_1(0) \tag{1.1}$$

with zero boundary condition, where $B_1(0) \subset \mathbb{R}^N$ $(N \geq 3)$ is the unit ball centered at the origin, $\alpha > 0$, $p, q > 1$, $p + q < 2^* = 2N/(N-2)$.

Hénon [6] considered the so called Hénon equation

$$\begin{gathered} -\Delta u = |x|^\alpha u^{p-1}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

which stems from a research of rotating stellar structures. Such a problem enjoys special features. As usual, for arbitrary bounded Ω, critical exponent for problem (1.2) is 2^*, while if Ω is a ball, it was shown in [8] that problem (1.2) has a radially symmetric solution for $p \in (2, \frac{2(N+\alpha)}{N-2})$, the critical exponent $\frac{2(N+\alpha)}{N-2}$ is larger than the critical Sobolev exponent 2^*. Moreover, even in a ball, problem (1.2) possesses non-radial solutions under some conditions, see [9] and references therein. This can also be seen as in [3], where it was shown that the ground state solution of problem (1.2) has a unique maximum point approaching to a point on $\partial B_1(0)$ provided that $\alpha > 0$ fixed, $p \in (2, 2^*)$ and $p \to 2^*$. Similar results for $p \in (2, 2^*)$ fixed and $\alpha \to \infty$ can be found in [1, 2, 4].

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For system (1.1), we proved in [10] that there exists $\alpha^*>0$ such that the ground state solution of problem (1.1) is non-radial if $\alpha>\alpha^*$, $p,q>1$ and $p+q<2^*$; the maximum points of both components u and v of the ground state solution pair (u,v) concentrate at the same point on the boundary $\partial\Omega$ as $p+q\to 2^*$.

In this paper, we investigate the asymptotic behavior of the ground state solution pair of problem (1.1) as $\alpha\to\infty$. Our main result is as follows.

Theorem 1.1. *Let (u_α,v_α) be a positive ground state solution of (1.1) and denote $x_0=(0,\dots,0,1)$. Suppose $x_\alpha,y_\alpha\in B_1(0)$ is a maximum point of u_α, v_α respectively. Then*

$$\begin{gathered}
x_\alpha,y_\alpha\to x\in\partial B_1(0),\\
\lim_{\alpha\to+\infty}\alpha(1-|x_\alpha|),\ \lim_{\alpha\to+\infty}\alpha(1-|y_\alpha|)\ \in(0,+\infty),\\
\alpha^{-\frac{(2-N)(p+q)+2N}{p+q-2}}\int_{B_1(0)}(|\nabla(u_\alpha-\alpha^{\frac{2}{p+q-2}}u(\alpha(x-x_0))|^2\\
+|\nabla(v_\alpha-\alpha^{\frac{2}{p+q-2}}v(\alpha(x-x_0))|^2)dx\to 0
\end{gathered}$$

as $\alpha\to+\infty$, where (u,v) is a ground state solution of the system

$$-\Delta u=\frac{2p}{p+q}e^{x_N}u^{p-1}v^q,\quad -\Delta v=\frac{2q}{p+q}e^{x_N}u^pv^{q-1},\quad in\ \mathbb{R}^N_-\tag{1.3}$$

with $u=v=0$ on $\partial\mathbb{R}^N_-$.

The proof of Theorem 1.1 is inspired by that in [4]. In section 2, we prove that (1.3) has a ground state solution pair. We establish in section 3 an asymptotic estimate for $S_{\alpha,p,q}$ which is defined in the section 3. Then using the blow up argument, we show in section 4 that the maximum points of both components of the ground state solution of (1.1) concentrate on the same point of the boundary of the domain. The proof of Theorem 1.1 is also given in section 4.

2. A variational problem

We consider for $\gamma>0$ the variational problem

$$m_{\gamma,p,q}=\inf_{0\neq u,v\in D_0^{1,2}(\mathbb{R}^N_-)}\frac{\int_{\mathbb{R}^N_-}(|\nabla u|^2+|\nabla v|^2)dx}{(\int_{\mathbb{R}^N_-}e^{\gamma x_N}|u|^p|v|^qdx)^{2/(p+q)}},\tag{2.1}$$

where $p+q\in(2,2^*)$. We will show that $m_{\gamma,p,q}$ is achieved. First, we prove that the problem is well defined. For any $u\in C_0^\infty(\mathbb{R}^N_-)$, by Hölder's inequality,

$$|u(x',x_N)|\le|x_N|^{1/2}\Big(\int_{-\infty}^0|\frac{\partial u(x',t)}{\partial t}|^2\,dt\Big)^{1/2}.$$

If $p_1+q_1=2$ and $u,v\in C_0^\infty(\mathbb{R}^N_-)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N_-}e^{\gamma x_N}|u|^{p_1}|v|^{q_1}\,dx\\
&\le\int_{x_N\le 0}|x_N|e^{\gamma x_N}\,dx_N\int_{\mathbb{R}^{N-1}}\Big(\int_{-\infty}^0|\frac{\partial u(x',t)}{\partial t}|^2dt\Big)^{\frac{p_1}{2}}\Big(\int_{-\infty}^0|\frac{\partial v(x',t)}{\partial t}|^2dt\Big)^{\frac{q_1}{2}}dx'\\
&\le C\Big(\int_{\mathbb{R}^N_-}|\frac{\partial u(x',x_N)}{\partial x_N}|^2\,dx\Big)^{\frac{p_1}{2}}\Big(\int_{\mathbb{R}^N_-}|\frac{\partial v(x',x_N)}{\partial x_N}|^2\,dx\Big)^{\frac{q_1}{2}}
\end{aligned}$$

$$\leq C \int_{\mathbb{R}^N_-} (|\nabla u|^2 + |\nabla v|^2)\, dx.$$

If $p_2 + q_2 = 2^*$, again by Hölder's inequality,

$$\int_{\mathbb{R}^N_-} e^{\gamma x_N} |u|^{p_2} |v|^{q_2}\, dx \leq C\Big(\int_{\mathbb{R}^N_-} (|\nabla u|^2 + |\nabla v|^2)\, dx\Big)^{2^*/2}.$$

Using interpolation inequality for $p+q \in (2, 2^*)$, we have

$$\int_{\mathbb{R}^N_-} e^{\gamma x_N} |u|^{p} |v|^{q}\, dx \leq C\Big(\int_{\mathbb{R}^N_-} (|\nabla u|^2 + |\nabla v|^2)\, dx\Big)^{(p+q)/2}.$$

This implies $m_{\gamma,p,q} > 0$. Next, for every $R > 0$,

$$\int_{x_N \leq -R} \int_{\mathbb{R}^{N-1}} e^{\gamma x_N} |u|^{p} |v|^{q}\, dx \leq C e^{-\gamma R/2}\Big(\int_{\mathbb{R}^N_-} (|\nabla u|^2 + |\nabla v|^2)\, dx\Big)^{(p+q)/2}.$$

Hence, $\int_{x_N \leq -R} e^{\gamma x_N} |u|^p |v|^q\, dx$ is uniformly decay in the x_N-direction. The variational problem $m_{\gamma,p,q}$ is compact in the x_N-direction and it is translation invariant in $x_1, \dots, x_{N-1}$. So we may prove as the proof of [11, Theorem 1.4] the following result.

Proposition 2.1. *Suppose $p+q \in (2, 2^*)$, $\gamma > 0$, $N \geq 3$. Then, $m_{\gamma,p,q}$ is achieved by (u, v) with positive functions $u, v \in D^{1,2}_0(\mathbb{R}^N_-)$.*

3. Estimate for $S_{\alpha,p,q}$

It is known that the problem

$$S_{\alpha,p,q} = \inf_{u,v \in H^1_0(B_1(0))\setminus\{0\}} J_\alpha(u,v) = \inf_{u,v \in H^1_0(B_1(0))\setminus\{0\}} \frac{\int_{B_1(0)} (|\nabla u|^2 + |\nabla v|^2)\, dx}{(\int_{B_1(0)} |x|^\alpha |u|^p |v|^q\, dx)^{2/(p+q)}}$$

is achieved and the minimizer is a solution of problem (1.1) up to a constant. Furthermore, we have the following result.

Proposition 3.1. *Let $p+q \geq 2$. There is $C > 0$ such that*

$$C \leq \frac{S_{\alpha,p,q}}{\alpha^{2-N+\frac{2N}{p+q}}} \leq m_{1,p,q} + o(1),$$

where $o(1) \to 0$ as $\alpha \to \infty$.

Proof. We use the idea in [4]. We establish the upper bound first. For any $\varepsilon > 0$, there exist $w_\varepsilon, h_\varepsilon \in C^\infty_0(\mathbb{R}^N_-)$, $w_\varepsilon, h_\varepsilon \neq 0$, such that

$$J_{\varepsilon,\mathbb{R}^N_-}(w_\varepsilon, h_\varepsilon) = \frac{\int_{\mathbb{R}^N_-} (|\nabla w_\varepsilon|^2 + |\nabla h_\varepsilon|^2) dx}{(\int_{\mathbb{R}^N_-} e^{x_N} |w_\varepsilon|^p |h_\varepsilon|^q dx)^{2/(p+q)}} < m_{1,p,q} + \varepsilon.$$

Let

$$u_\alpha(x) = w_\varepsilon(\alpha x', \alpha(x_N + (1-|x'|^2)^{1/2})), \quad v_\alpha(x) = h_\varepsilon(\alpha x', \alpha(x_N + (1-|x'|^2)^{1/2})),$$

where $x' = (x_1, x_2, \dots, x_{N-1})$. Then $u_\alpha, v_\alpha \in H_0^1(B_1(0))$ if $\alpha > 0$ is large enough. Denote $\tilde{B}_\alpha = \{y : \alpha^{-1}y + x_0 \in B_1(0)\}$, where $x_0 = (0, 0, \dots, 1)$. Then

$$\begin{aligned}\int_{B_1(0)} |\nabla u_\alpha|^2 dx = \alpha^{2-N} \int_{\tilde{B}_\alpha} (\sum_{i=1}^{N-1} \Big| D_i w_\varepsilon(x', x_N + \alpha(1 + (1 - \frac{1}{\alpha^2}|x'|^2)^{1/2})) \\ + \frac{\alpha^{-1}x_i}{(1-\alpha^{-2}|x'|^2)^{1/2}} D_N w_\varepsilon(x', x_N + \alpha(1 + (1 - \frac{1}{\alpha^2}|x'|^2)^{1/2}))\Big|^2 \\ + |D_N w_\varepsilon(x', x_N + \alpha(1 + (1 - \frac{1}{\alpha^2}|x'|^2)^{1/2}))|^2)\, dx.\end{aligned} \tag{3.1}$$

Let

$$y_i = x_i,\ i = 1, 2, \dots, N-1;\ y_N = x_N + \alpha(1 + (1 - \frac{1}{\alpha^2}|x'|^2)^{1/2}),$$

then $|det(\frac{\partial y}{\partial x})| = 1$. By (3.1),

$$\begin{aligned}\int_{B_1(0)} |\nabla u_\alpha|^2 dx &= \alpha^{2-N} \int_{\mathbb{R}^N_-} (\sum_{i=1}^{N-1} |D_i w_\varepsilon + O(\alpha^{-1}) D_N w_\varepsilon|^2 + |D_N w_\varepsilon|^2)\ dy \\ &= \alpha^{2-N}(\int_{\mathbb{R}^N_-} |\nabla w_\varepsilon|^2\ dy + O(\alpha^{-1})).\end{aligned} \tag{3.2}$$

Similarly,

$$\int_{B_1(0)} |\nabla v_\alpha|^2 dx = \alpha^{2-N}(\int_{\mathbb{R}^N_-} |\nabla h_\varepsilon|^2\ dy + O(\alpha^{-1})). \tag{3.3}$$

For any $x \in sptw_\varepsilon \cap spth_\varepsilon$, we have

$$|\frac{x}{\alpha} + x_0|^\alpha = (1 + \frac{2x_N}{\alpha} + O(\alpha^{-2}))^{\alpha/2} = e^{x_N + O(\alpha^{-1})}.$$

Therefore,

$$\begin{aligned}&\int_{B_1(0)} |x|^\alpha |u_\alpha|^p |v_\alpha|^q\, dx \\ &= \alpha^{-N} \int_{\widetilde{B}_\alpha} |\frac{x}{\alpha} + x_0|^\alpha |w_\varepsilon(x', x_N + \alpha(1 + (1 - \frac{1}{\alpha^2|x'|^2})^{1/2}))|^p \\ &\quad \times |h_\varepsilon(x', x_N + \alpha(1 + (1 - \frac{1}{\alpha^2|x'|^2})^{1/2}))|^q\, dx \\ &= \alpha^{-N} \int_{\mathbb{R}^N_-} e^{x_N - \alpha(1-(1-\frac{1}{\alpha^2|x'|^2})^{1/2}) + O(\alpha^{-1})} |w_\varepsilon|^p |h_\varepsilon|^q\, dx \\ &= \alpha^{-N}(\int_{\mathbb{R}^N_-} e^{x_N} |w_\varepsilon|^p |h_\varepsilon|^q\ dy + O(\alpha^{-1})).\end{aligned} \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}J_\alpha(u_\alpha, v_\alpha) &= \alpha^{2-N+\frac{2N}{p+q}} (J_{\varepsilon,\mathbb{R}^N_-}(w_\varepsilon, h_\varepsilon) + O(\alpha^{-1})) \\ &< \alpha^{2-N+\frac{2N}{p+q}} (m_{1,p,q} + \varepsilon + O(\alpha^{-1}))\end{aligned}$$

and then,

$$\frac{S_{\alpha,p,q}}{\alpha^{2-N+\frac{2N}{p+q}}} \le m_{1,p,q} + o(1).$$

Next, we show the lower bound. Let $r \in (0,1]$, $\omega \in S^{N-1}$. For any $u, v \in H_0^1(B_1(0)\setminus\{0\})$, we define $\varphi(r,\omega) = u(r^\beta,\omega), \psi(r,\omega) = v(r^\beta,\omega)$, where $\beta = \frac{N}{N+\alpha}$. Then

$$\int_{B_1(0)} |x|^\alpha |u|^p |v|^q dx = \beta \int_0^1 \int_{\omega\in S^{N-1}} |\varphi(r,\omega)|^p |\psi(r,\omega)|^q r^{N-1}\, dr\, d\omega, \tag{3.5}$$

and

$$\begin{aligned}
&\int_{B_1(0)} |\nabla u|^2 dx\\
&= \beta \int_0^1 \int_{\omega\in S^{N-1}} \Big(\frac{1}{\beta^2 r^{2(\beta-1)}} |\varphi_r(r,\omega)|^2 + \frac{1}{r^{2\beta}} |\nabla_\omega \varphi(r,\omega)|^2\Big) r^{\beta(N-1)+\beta-1}\, dr\, d\omega\\
&= \frac{1}{\beta} \int_0^1 \int_{\omega\in S^{N-1}} \Big(|\varphi_r(r,\omega)|^2 + \frac{\beta^2}{r^2} |\nabla_\omega \varphi(r,\omega)|^2\Big) r^{(2-N)(1-\beta)+N-1}\, dr\, d\omega.
\end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned}
&\int_{B_1(0)} |\nabla v|^2 dx\\
&= \frac{1}{\beta} \int_0^1 \int_{\omega\in S^{N-1}} \Big(|\psi_r(r,\omega)|^2 + \frac{\beta^2}{r^2} |\nabla_\omega \psi(r,\omega)|^2\Big) r^{(2-N)(1-\beta)+N-1}\, dr\, d\omega.
\end{aligned} \tag{3.7}$$

Note that

$$|\nabla_\omega \varphi|^2 = \sum_{i=1}^{N-1} \Big(\frac{\partial \varphi}{\partial x_i} - \frac{x_i}{(1-|x'|^2)^{1/2}} \frac{\partial \varphi}{\partial x_N}\Big)^2 \tag{3.8}$$

and $d\omega = (1+|x'|^2)^{-1/2} dx'$. Let $\bar\varphi(r,x') = \varphi(r,\beta x')$, $\bar\psi(r,x') = \psi(r,\beta x')$, $S_\beta = \{x \in S^{N-1} : |x'| \le \eta\beta\}$, where $\eta > 0$ is small. Then, we may deduce as in [4] that for $\beta > 0$ small,

$$\begin{aligned}
&\int_0^1 \int_{S_\beta} \Big(|\varphi_r(r,\omega)|^2 + \frac{\beta^2}{r^2} |\nabla_\omega \varphi(r,\omega)|^2\Big) r^{(2-N)(1-\beta)+N-1}\, dr\, d\omega\\
&\ge C\beta^{N-1} \int_{B_\eta} |\nabla \bar\varphi(r,x')|^2 r^{(2-N)(1-\beta)}\, dr\, dx'
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\int_0^1 \int_{S_\beta} \Big(|\psi_r(r,\omega)|^2 + \frac{\beta^2}{r^2} |\nabla_\omega \psi(r,\omega)|^2\Big) r^{(2-N)(1-\beta)+N-1}\, dr\, d\omega\\
&\ge C\beta^{N-1} \int_{B_\eta} |\nabla \bar\psi(r,x')|^2 r^{(2-N)(1-\beta)}\, dr\, dx',
\end{aligned} \tag{3.10}$$

where $B_\eta = \{x \in B_1(0) : |x'| \le \eta\}$. Similarly,

$$\begin{aligned}
&\int_0^1 \int_{S_\beta} |\varphi(r,\omega)|^p |\psi(r,\omega)|^q r^{N-1}\, dr\, d\omega\\
&\le C\beta^{N-1} \int_{B_\eta} |\bar\varphi(r,x')|^p |\bar\psi(r,x')|^q r^{N-1}\, dr\, dx'.
\end{aligned} \tag{3.11}$$

Since $\bar\varphi, \bar\psi = 0$ on S^{N-1}, there exists a constant $C > 0$ such that

$$(\int_{B_\eta} |\bar\varphi(r,x')|^{p+q} r^{N-1}\,dr\,dx')^{2/(p+q)} \le C\int_{B_\eta} |\nabla\bar\varphi(r,x')|^2 r^{(2-N)(1-\beta)}\,dr\,dx' \tag{3.12}$$

and

$$\Big(\int_{B_\eta} |\bar\psi(r,x')|^{p+q} r^{N-1}\,dr\,dx'\Big)^{2/(p+q)} \le C\int_{B_\eta} |\nabla\bar\psi(r,x')|^2 r^{(2-N)(1-\beta)}\,dr\,dx'. \tag{3.13}$$

Therefore, by (3.12) and (3.13),

$$\begin{aligned}
&\Big(\int_{B_\eta} |\bar\varphi(r,x')|^p|\bar\psi(r,x')|^q r^{N-1}\,dr\,dx'\Big)^{2/(p+q)}\\
&\le C\int_{B_\eta} (|\bar\varphi(r,x')|^{p+q} r^{N-1}\,dr\,dx')^{2/(p+q)} + C\int_{B_\eta} (|\bar\psi(r,x')|^{p+q} r^{N-1}\,dr\,dx')^{2/(p+q)}\\
&\le C\int_{B_\eta} (|\nabla\bar\varphi(r,x')|^2 + |\nabla\bar\psi(r,x')|^2) r^{(2-N)(1-\beta)}\,dr\,dx'.
\end{aligned} \tag{3.14}$$

We derive from (3.9)-(3.14) that

$$\begin{aligned}
&\Big(\int_0^1\int_{S_\beta} |\varphi(r,\omega)|^p|\psi(r,\omega)|^q r^{N-1}\,dr\,d\omega\Big)^{2/(p+q)}\\
&\le C\beta^{\frac{2(N-1)}{p+q}}(\int_{B_\eta} |\bar\varphi(r,x')|^p|\bar\psi(r,x')|^q r^{N-1}\,dr\,dx')^{\frac{2}{p+q}}\\
&\le C\beta^{\frac{2(N-1)}{p+q}}\int_{B_\eta} (|\nabla\bar\varphi(r,x')|^2 + |\nabla\bar\psi(r,x')|^2) r^{(2-N)(1-\beta)}\,dr\,dx'\\
&\le C\beta^{1-N+\frac{2(N-1)}{p+q}}\int_0^1\int_{S_\beta} (|\varphi_r(r,\omega)|^2 + \frac{\beta^2}{r^2}|\nabla_\omega\varphi(r,\omega)|^2\\
&\quad + |\psi_r(r,\omega)|^2 + \frac{\beta^2}{r^2}|\nabla_\omega\psi(r,\omega)|^2) r^{(2-N)(1-\beta)+N-1}\,dr\,d\omega
\end{aligned} \tag{3.15}$$

Since (1.1) is rotation invariant, we may choose $\beta > 0$ so that S^{N-1} can be covered by finite number S_β up to a rotation, that is $S^{N-1} \subset \cup S_\beta$. Then

$$\begin{aligned}
&\Big(\int_0^1\int_{\omega\in S^{N-1}} |\varphi(r,\omega)|^p|\psi(r,\omega)|^q r^{N-1}\,dr\,d\omega\Big)^{2/(p+q)}\\
&\le \sum\Big(\int_0^1\int_{S_\beta} |\varphi(r,\omega)|^p|\psi(r,\omega)|^q r^{N-1}\,dr\,d\omega\Big)^{2/(p+q)}\\
&\le C\beta^{1-N+\frac{2(N-1)}{p+q}}\int_0^1\int_{S_\beta} (|\varphi_r(r,\omega)|^2 + \frac{\beta^2}{r^2}|\nabla_\omega\varphi(r,\omega)|^2\\
&\quad + |\psi_r(r,\omega)|^2 + \frac{\beta^2}{r^2}|\nabla_\omega\psi(r,\omega)|^2) r^{(2-N)(1-\beta)+N-1}\,dr\,d\omega\\
&\le C\beta^{1-N+\frac{2(N-1)}{p+q}}\int_0^1\int_{S^{N-1}} (|\varphi_r(r,\omega)|^2 + \frac{\beta^2}{r^2}|\nabla_\omega\varphi(r,\omega)|^2\\
&\quad + |\psi_r(r,\omega)|^2 + \frac{\beta^2}{r^2}|\nabla_\omega\psi(r,\omega)|^2) r^{(2-N)(1-\beta)+N-1}\,dr\,d\omega.
\end{aligned} \tag{3.16}$$

Hence, we deduce from (3.6)-(3.7) and (3.16) that

$$\frac{\int_{B_1(0)}(|\nabla u|^2+|\nabla v|^2)\,dx}{(\int_{B_1(0)}|x|^\alpha|u|^p|v|^q\,dx)^{2/(p+q)}}\geq C\beta^{N-2-\frac{2N}{p+q}}=C\alpha^{2-N+\frac{2N}{p+q}}. \tag{3.17}$$

It yields

$$\alpha^{N-2-\frac{2N}{p+q}}\frac{\int_{B_1(0)}(|\nabla u|^2+|\nabla v|^2)\,dx}{(\int_{B_1(0)}|x|^\alpha|u|^p|v|^q\,dx)^{2/(p+q)}}\geq C. \tag{3.18}$$

The proof is complete since u and v are arbitrary. □

4. Asymptotic behavior of ground state solution

Let (U_α,V_α) be a minimizer of $S_{\alpha,p,q}$. Choosing $\lambda_\alpha=(\frac{2}{S_{\alpha,p,q}})^{\frac{1}{2-(p+q)}}$ and defining $u_\alpha=\lambda_\alpha U_\alpha, v_\alpha=\lambda_\alpha V_\alpha$, we see that (u_α,v_α) is a solution pair of (1.1), which is also a minimizer of $S_{\alpha,p,q}$. That is,

$$\int_{B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)\,dx=2\int_{B_1(0)}|x|^\alpha|u_\alpha|^p|v_\alpha|^q\,dx \tag{4.1}$$

and

$$S_{\alpha,p,q}=\frac{\int_{B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)\,dx}{(\int_{B_1(0)}|x|^\alpha|u_\alpha|^p|v_\alpha|^q\,dx)^{2/(p+q)}}. \tag{4.2}$$

It yields

$$\int_{B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)\,dx=2\int_{B_1(0)}|x|^\alpha|u_\alpha|^p|v_\alpha|^q\,dx=2^{-\frac{2}{p+q-2}}S_{\alpha,p,q}^{\frac{p+q}{p+q-2}}. \tag{4.3}$$

By Proposition 3.1

$$C\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}\leq\int_{B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)\,dx\leq C'\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}. \tag{4.4}$$

Let

$$\bar u_\alpha(x)=\alpha^{-\frac{2}{p+q-2}}u_\alpha(\frac{x}{\alpha}),\quad \bar v_\alpha(x)=\alpha^{-\frac{2}{p+q-2}}v_\alpha(\frac{x}{\alpha}),\quad x\in B_\alpha(0).$$

Then

$$C\leq\int_{B_\alpha(0)}(|\nabla\bar u_\alpha|^2+|\nabla\bar v_\alpha|^2)\,dx\leq C'. \tag{4.5}$$

Choose $p_1,q_1>0$ such that $p>p_1$, $q>q_1$ and $p_1+q_1=2$.

Lemma 4.1. *As $\alpha\to+\infty$, we have*

$$0<C\leq C\max_{x\in B_\alpha(0)}|\bar u_\alpha|^{p-p_1}\max_{x\in B_\alpha(0)}|\bar v_\alpha|^{q-q_1}.$$

Proof. By Proposition 3.1,

$$C\alpha^2\leq S_{\alpha,p_1,q_1}=\frac{\int_{B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)\,dx}{\int_{B_1(0)}|x|^\alpha|u_\alpha|^{p_1}|v_\alpha|^{q_1}\,dx}.$$

Equation (4.5) implies

$$\int_{B_\alpha(0)}|\frac{x}{\alpha}|^\alpha|\bar u_\alpha|^{p_1}|\bar v_\alpha|^{q_1}\,dx\leq C\int_{B_\alpha(0)}(|\nabla\bar u_\alpha|^2+|\nabla\bar v_\alpha|^2)\,dx\leq C. \tag{4.6}$$

Hence, by (4.3) and (4.4),

$$0<C\leq\int_{B_\alpha(0)}|\frac{x}{\alpha}|^\alpha|\bar u_\alpha|^p|\bar v_\alpha|^q\,dx$$

$$
\begin{aligned}
&\le \max_{x\in B_\alpha(0)}(|\bar u_\alpha|^{p-p_1}|\bar v_\alpha|^{q-q_1})\int_{B_\alpha(0)}|\frac{x}{\alpha}|^\alpha|\bar u_\alpha|^{p_1}|\bar v_\alpha|^{q_1}\,dx\\
&\le C\max_{x\in B_\alpha(0)}|\bar u_\alpha|^{p-p_1}\max_{x\in B_\alpha(0)}|\bar v_\alpha|^{q-q_1}.
\end{aligned}
$$

The assertion follows. □

Lemma 4.2. *There is $C>0$ such that $|\bar u_\alpha(x)|\le C$, $|\bar v_\alpha(x)|\le C$ for $x\in B_\alpha(0)$.*

Proof. Since (u_α, v_α) is a solution pair of (1.1), then for $x\in B_\alpha(0)$ we have

$$
-\Delta\bar u_\alpha(x)=\frac{2p}{p+q}|\frac{x}{\alpha}|^\alpha\bar u_\alpha^{p-1}(x)\bar v_\alpha^q(x)\le\frac{2p}{p+q}\bar u_\alpha^{p-1}(x)\bar v_\alpha^q(x) \tag{4.7}
$$

and

$$
-\Delta\bar v_\alpha(x)=\frac{2q}{p+q}|\frac{x}{\alpha}|^\alpha\bar u_\alpha^{p}(x)\bar v_\alpha^{q-1}(x)\le\frac{2q}{p+q}\bar u_\alpha^{p}(x)\bar v_\alpha^{q-1}(x).
$$

Now we use the Moser iteration to prove the result. Without confusion, we use (u,v) to denote $(\bar u_\alpha,\bar v_\alpha)$. Let $s\ge1$. Multiplying (4.7) by u^{2s} and integrating by parts, we obtain

$$
s^{-2}(2s-1)\int_{B_\alpha(0)}|\nabla u^s|^2\,dx\le\frac{2p}{p+q}\int_{B_\alpha(0)}u^{p-1+2s}v^q\,dx.
$$

Since $s^{-2}(2s-1)\ge s^{-1}$ if $s\ge1$,

$$
\int_{B_\alpha(0)}|\nabla u^s|^2\,dx\le\frac{2sp}{p+q}\int_{B_\alpha(0)}u^{p-1+2s}v^q\,dx.
$$

By Sobolev inequality and Hölder's inequality, we deduce

$$
\begin{aligned}
&\Big(\int_{B_\alpha(0)}u^{2^*s}\,dx\Big)^{2/2^*}\\
&\le\frac{2sp}{p+q}\int_{B_\alpha(0)}u^{p-1+2s}v^q\,dx\\
&\le\frac{2sp}{p+q}\Big(\int_{B_\alpha(0)}u^{p+q-1+2s}\,dx\Big)^{\frac{p-1+2s}{p+q-1+2s}}\Big(\int_{B_\alpha(0)}v^{p+q-1+2s}\,dx\Big)^{\frac{q}{p+q-1+2s}}\\
&\le\frac{sp}{p+q}\int_{B_\alpha(0)}(u^{p+q-1+2s}+v^{p+q-1+2s})\,dx.
\end{aligned} \tag{4.8}
$$

Similarly, we have

$$
\Big(\int_{B_\alpha(0)}v^{2^*s}\,dx\Big)^{2/2^*}\le\frac{sq}{p+q}\int_{B_\alpha(0)}(u^{p+q-1+2s}+v^{p+q-1+2s})\,dx. \tag{4.9}
$$

Therefore,

$$
\begin{aligned}
\Big(\int_{B_\alpha(0)}(u^{2^*s}+v^{2^*s})\,dx\Big)^{2/2^*}&\le\Big(\int_{B_\alpha(0)}u^{2^*s}\,dx\Big)^{2/2^*}+\Big(\int_{B_\alpha(0)}v^{2^*s}\,dx\Big)^{2/2^*}\\
&\le s\int_{B_\alpha(0)}(u^{p+q-1+2s}+v^{p+q-1+2s})\,dx.
\end{aligned} \tag{4.10}
$$

Now we define $\{s_j\}$ by induction. Let $p+q-1+2s_0 = 2^*$ and $p+q-1+2s_{j+1} = 2^* s_j$, $j = 0, 1, 2, \dots$. We also define $M_0 = 1$, $M_{j+1} = (s_j M_j)^{\frac{2^*}{2}}$, $j = 0, 1, 2, \dots$. We claim that for all $j \geq 0$,

$$\int_{B_\alpha(0)} (u^{p+q-1+2s_j} + v^{p+q-1+2s_j})\, dx \leq C M_j \tag{4.11}$$

and

$$M_j \leq e^{m s_{j-1}}, \tag{4.12}$$

where $C, m > 0$. (4.11) and (4.12) imply

$$\Big(\int_{B_\alpha(0)} (u^{2^* s_j} + v^{2^* s_j})\, dx\Big)^{\frac{1}{2^* s_j}} \leq C M_j^{\frac{1}{2^* s_j}} \leq e^{\frac{m s_{j-1}}{2^* s_j}} \leq C$$

for all j. The assertion then follows. Now, we show (4.11). Obviously, if $j = 0$, (4.11) holds. Suppose it holds for j, we deduce it holds for $j+1$. Indeed,

$$\begin{aligned}
&\int_{B_\alpha(0)} (u^{p+q-1+2s_{j+1}} + v^{p+q-1+2s_{j+1}})\, dx \\
&= \int_{B_\alpha(0)} (u^{2^* s_j} + v^{2^* s_j})\, dx \\
&\leq s_j^{2^*/2} \Big(\int_{B_\alpha(0)} (u^{p+q-1+2s_j} + v^{p+q-1+2s_j})\, dx \Big)^{2^*/2} \\
&\leq \Big(s_j M_j \Big)^{2^*/2} = M_{j+1}.
\end{aligned}$$

Inequality (4.12) can be proved, as in [7]. □

Let $M_\alpha = \max_{x \in B_1(0)} u_\alpha$, $N_\alpha = \max_{x \in B_1(0)} v_\alpha$.

Lemma 4.3. *There holds*

$$C_2 \alpha^{\frac{2}{p+q-2}} \leq M_\alpha, N_\alpha \leq C_3 \alpha^{\frac{2}{p+q-2}}.$$

Proof. By Lemmas 4.1 and 4.2, we have

$$0 < C_1 \leq C \max_{x \in B_\alpha(0)} |\bar{u}_\alpha|^{p-p_1} \quad \text{and} \quad 0 < C_1 \leq C \max_{x \in B_\alpha(0)} |\bar{v}_\alpha|^{q-q_1}.$$

This yields the result. □

Let $x_\alpha \in B_1(0)$ be a maximum point of u_α and $y_\alpha \in B_1(0)$ be a maximum point of v_α.

Lemma 4.4. *The following hold* $\lim_{\alpha \to +\infty} \alpha(1 - |x_\alpha|)$ *and* $\lim_{\alpha \to +\infty} \alpha(1 - |y_\alpha|)$ *are in* $\in (0, +\infty)$.

Proof. We only prove $\lim_{\alpha \to +\infty} \alpha(1 - |x_\alpha|) = L$ and $L \in (0, +\infty)$. The other case can be done in the same way. Let $B_\alpha(-x_\alpha) = \{x : \frac{x}{\alpha} + x_\alpha \in B_1(0)\}$ and define

$$\tilde{u}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} u_\alpha(\frac{x}{\alpha} + x_\alpha), \quad \tilde{v}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} v_\alpha(\frac{x}{\alpha} + x_\alpha).$$

Then, for $x \in B_\alpha(-x_\alpha)$,

$$-\Delta \tilde{u}_\alpha(x) = \frac{2p}{p+q} \Big| \frac{x}{\alpha} + x_\alpha \Big|^\alpha \tilde{u}_\alpha^{p-1}(x) \tilde{v}_\alpha^q(x).$$

By Lemma 4.3,

$$\tilde{u}_\alpha, \tilde{v}_\alpha \le C, \quad \tilde{u}_\alpha(0) = \max_{x \in B_\alpha(-x_\alpha)} \tilde{u}_\alpha(x) \ge C_1 > 0$$

and

$$\int_{B_\alpha(-x_\alpha)} (|\nabla \tilde{u}_\alpha|^2 + |\nabla \tilde{v}_\alpha|^2)\, dx \le C.$$

Suppose that $\alpha(1 - |x_\alpha|) \to +\infty$, we assume that there are $\tilde{u}, \tilde{v} \in D^{1,2}(\mathbb{R}^N)$ such that

$$\begin{gathered} \tilde{u}_\alpha \rightharpoonup \tilde{u}, \quad \tilde{v}_\alpha \rightharpoonup \tilde{v}, \quad \text{in } D^{1,2}(\mathbb{R}^N) \\ \tilde{u}_\alpha \to \tilde{u}, \quad \tilde{v}_\alpha \to \tilde{v}, \quad \text{in } C^1_{\rm loc}(\mathbb{R}^N). \end{gathered}$$

Now, we distinguish two cases: (i) $|x_\alpha| \le l < 1$; (ii) $|x_\alpha| \to 1$ as $\alpha \to +\infty$. For any x with $|x| \le C$, in case (i), we have

$$|\frac{x}{\alpha} + x_\alpha|^\alpha \le (\frac{C}{\alpha} + |x_\alpha|)^\alpha \le (\frac{C}{\alpha} + l)^\alpha \le (l + \varepsilon)^\alpha \to 0, \quad \text{as } \alpha \to +\infty.$$

In case (ii), since $\alpha(1 - |x_\alpha|) \to +\infty$,

$$|\frac{x}{\alpha} + x_\alpha|^\alpha \le e^{\alpha \ln(\frac{|x|}{\alpha} + |x_\alpha| - 1 + 1)} = O(e^{\alpha(\frac{|x|}{\alpha} + |x_\alpha| - 1)}) = O(e^{|x| + \alpha(|x_\alpha| - 1)}) \to 0.$$

So $\tilde{u}$ satisfies

$$-\Delta \tilde{u} = 0, \quad \tilde{u} \in D^{1,2}(\mathbb{R}^N).$$

This implies $\tilde{u} = 0$, a contradiction to $\tilde{u}(0) = \lim_{\alpha \to +\infty} \tilde{u}_\alpha(0) \ge C > 0$. Therefore, $\alpha(1 - |x_\alpha|) \to L < +\infty$.

Now, we claim $L > 0$. Indeed, we have $\tilde{u}(0) = \lim_{\alpha \to +\infty} \tilde{u}_\alpha(0) > 0$. Since (1.1) is invariant under the rotations. After suitably rotating the coordinate system, we may assume that $x_\alpha = (0, \dots, 0, x_N^\alpha)$, where $x_N^\alpha \to 1$, as $\alpha \to +\infty$. Then $(\tilde{u}, \tilde{v})$ is a positive solution pair of (1.3) in $\Omega = \mathbb{R}^N_- + (0, \dots, 0, L)$ with $\tilde{u} = \tilde{v} = 0$ on $\partial\Omega$. If $L = 0$, we would have $\Omega = \mathbb{R}^N_-$, and then we obtain $\tilde{u}(0) = 0$, a contradiction. The proof is complete. □

By Lemma 4.4, we know that $x_\alpha \to x_0 \in \partial B_1(0)$, $y_\alpha \to y_0 \in \partial B_1(0)$ if $\alpha \to +\infty$. In the following, we show that $x_0 = y_0$.

Lemma 4.5. *Both x_α and y_α converge to a point $x_0 \in \partial B_1(0)$ as $\alpha \to +\infty$.*

Proof. We argue by contradiction. Suppose $x_0 \ne y_0$, then there is a $\delta > 0$ such that $B_\delta(x_0) \cap B_\delta(y_0) = \emptyset$. After suitably rotating the coordinate system, we may assume that $x_0 = (0, \dots, 0, 1)$. Applying the blow up argument for

$$\tilde{u}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} u_\alpha(\frac{x}{\alpha} + x_0), \quad \tilde{v}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} v_\alpha(\frac{x}{\alpha} + x_0)$$

in $B_1(0) \cap B_\delta(x_0)$, since $\tilde{u}_\alpha, \tilde{v}_\alpha$ are bounded in $D_0^{1,2}(\mathbb{R}^N_-)$, we may assume that there are $\tilde{u}, \tilde{v} \in D_0^{1,2}(\mathbb{R}^N_-)$ such that

$$\begin{gathered} \tilde{u}_\alpha \rightharpoonup \tilde{u}, \quad \tilde{v}_\alpha \rightharpoonup \tilde{v}, \quad \text{in } D_0^{1,2}(\mathbb{R}^N_-), \\ \tilde{u}_\alpha \to \tilde{u}, \quad \tilde{v}_\alpha \to \tilde{v}, \quad \text{in } C^1_{\rm loc}(\mathbb{R}^N_-). \end{gathered}$$

Moreover, $(\tilde{u}, \tilde{v})$ with $\tilde{u}, \tilde{v} \in D_0^{1,2}(\mathbb{R}^N_-)$ is a positive solution of (1.3). In the same way, we may assume $y_0 = (0, \dots, 0, 1)$. Define

$$\bar{u}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} u_\alpha(\frac{x}{\alpha} + y_0), \quad \bar{v}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} v_\alpha(\frac{x}{\alpha} + y_0).$$

Then

$$\bar{u}_\alpha \rightharpoonup \bar{u}, \quad \bar{v}_\alpha \rightharpoonup \bar{v}, \quad \text{in } D_0^{1,2}(\mathbb{R}^N_-),$$
$$\bar{u}_\alpha \to \bar{u}, \quad \bar{v}_\alpha \to \bar{v}, \quad \text{in } C^1_{\rm loc}(\mathbb{R}^N_-),$$

and $(\bar{u}, \bar{v})$ is a positive solution of (1.3). It implies

$$\int_{\mathbb{R}^N_-} (|\nabla \tilde{u}|^2 + |\nabla \tilde{v}|^2)dx, \quad \int_{\mathbb{R}^N_-} (|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2)dx \geq 2^{-\frac{2}{p+q-2}} m_{1,p,q}^{\frac{p+q}{p+q-2}}$$

By Proposition 3.1,

$$\begin{aligned} I(u_\alpha, v_\alpha) &= (\frac{1}{2} - \frac{1}{p+q}) \int_{B_1(0)} (|\nabla u_\alpha|^2 + |\nabla v_\alpha|^2)dx \\ &= (\frac{1}{2} - \frac{1}{p+q}) 2^{-\frac{2}{p+q-2}} S_{\alpha,p,q}^{\frac{p+q}{p+q-2}} \\ &\leq (\frac{1}{2} - \frac{1}{p+q}) 2^{-\frac{2}{p+q-2}} \alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}} (m_{1,p,q}^{\frac{p+q}{p+q-2}} + o(1)). \end{aligned}$$

On the other hand,

$$\begin{aligned} I(u_\alpha, v_\alpha) &\geq (\frac{1}{2} - \frac{1}{p+q}) \int_{B_1(0)\cap B_\delta(x_0)} (|\nabla u_\alpha|^2 + |\nabla v_\alpha|^2)dx \\ &\quad + (\frac{1}{2} - \frac{1}{p+q}) \int_{B_1(0)\cap B_\delta(y_0)} (|\nabla u_\alpha|^2 + |\nabla v_\alpha|^2)dx \\ &\geq \alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}} (\frac{1}{2} - \frac{1}{p+q}) \int_{B_\alpha(-x_0)\cap B_{\alpha\delta}(0)} (|\nabla \tilde{u}_\alpha|^2 + |\nabla \tilde{v}_\alpha|^2)dx \\ &\quad + \alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}} (\frac{1}{2} - \frac{1}{p+q}) \int_{B_\alpha(-y_0)\cap B_{\alpha\delta}(0)} (|\nabla \bar{u}_\alpha|^2 + |\nabla \bar{v}_\alpha|^2)dx. \end{aligned}$$

So we obtain

$$\begin{aligned} &\int_{B_\alpha(-x_0)\cap B_{\alpha\delta}(0)} (|\nabla \tilde{u}_\alpha|^2 + |\nabla \tilde{v}_\alpha|^2)dx + \int_{B_\alpha(-\theta y_0)\cap B_{\alpha\delta}(0)} (|\nabla \bar{u}_\alpha|^2 + |\nabla \bar{v}_\alpha|^2)dx \\ &\leq 2^{-\frac{2}{p+q-2}} (m_{1,p,q}^{\frac{p+q}{p+q-2}} + o(1)). \end{aligned}$$

Therefore,

$$\begin{aligned} 2^{-\frac{2}{p+q-2}} (m_{1,p,q}^{\frac{p+q}{p+q-2}} + o(1)) &\geq \int_{\mathbb{R}^N_-} (|\nabla \tilde{u}|^2 + |\nabla \tilde{v}|^2)dx + \int_{\mathbb{R}^N_+} (|\nabla \bar{u}|^2 + |\nabla \bar{v}|^2)dx \\ &\geq 2^{-\frac{2}{p+q-2}} (m_{1,p,q}^{\frac{p+q}{p+q-2}} + M_{1,p,q}^{\frac{p+q}{p+q-2}}), \end{aligned}$$

which is impossible. The proof is complete. □

Now, we may assume that $x_0 = (0, \dots, 0, 1)$. Let

$$\hat{u}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} u_\alpha(\frac{x}{\alpha} + x_0), \quad \hat{v}_\alpha(x) = \alpha^{-\frac{2}{p+q-2}} v_\alpha(\frac{x}{\alpha} + x_0), \tag{4.13}$$

which, as before, satisfies

$$\hat{u}_\alpha \rightharpoonup \hat{u}, \quad \hat{v}_\alpha \rightharpoonup \hat{v}, \quad \text{in } D_0^{1,2}(\mathbb{R}^N_-),$$
$$\hat{u}_\alpha \to \hat{u}, \quad \hat{v}_\alpha \to \hat{v}, \quad \text{in } C^1_{\rm loc}(\mathbb{R}^N_-)$$

and $(\hat{u}, \hat{v}) \neq (0, 0)$ is a positive solution of (1.3).

Finally, we have following result.

Proposition 4.6. *The pair $(\hat{u},\hat{v})$ is a minimizer of $m_{1,p,q}$, which satisfies*

$$\int_{R^N_-}(|\nabla(\hat{u}_\alpha-\hat{u})|^2+|\nabla(\hat{v}_\alpha-\hat{v})|^2)dx\to 0,\quad \text{as } \alpha\to+\infty.$$

Proof. By (1.3), we have

$$\int_{\mathbb{R}^N_-}(|\nabla\hat{u}|^2+|\nabla\hat{v}|^2)dx=2\int_{\mathbb{R}^N_-}e^{x_N}\hat{u}^p\hat{v}^q dx,$$

and

$$m_{1,p,q}\le\frac{\int_{\mathbb{R}^N_-}(|\nabla u|^2+|\nabla v|^2)dx}{(\int_{\mathbb{R}^N_-}e^{x_N}|u|^p|v|^q dx)^{2/(p+q)}}.$$

So we obtain

$$\int_{\mathbb{R}^N_-}(|\nabla u|^2+|\nabla v|^2)dx\ge 2^{-\frac{2}{p+q-2}}m_{1,p,q}^{\frac{p+q}{p+q-2}}.$$

For $R>0$ define

$$B_{R,\alpha}=\{x:\frac{x}{\alpha}+x_0\in B_{\frac{R}{\alpha}(x_0)}\cap B_1(0)\},\quad \Omega_\alpha=\{x:\frac{x}{\alpha}+x_0\in B_1(0)\}.$$

By Proposition 3.1, (u_α,v_α) is a minimizer of $S_{\alpha,p,q}$ and satisfies (1.1), then

$$\int_{B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)dx\le 2^{-\frac{2}{p+q-2}}\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}(m_{1,p,q}^{\frac{p+q}{p+q-2}}+o(1)). \tag{4.14}$$

Moreover,

$$\begin{aligned}
&\int_{B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)dx\\
&=\int_{B_{\frac{R}{\alpha}(x_0)}\cap B_1(0)}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)dx+\int_{B_1(0)/B_{\frac{R}{\alpha}(x_0)}}(|\nabla u_\alpha|^2+|\nabla v_\alpha|^2)dx\\
&=\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}\Big(\int_{B_{R,\alpha}}(|\nabla\hat{u}_\alpha|^2+|\nabla\hat{v}_\alpha|^2)dx+\int_{\Omega_\alpha/B_{R,\alpha}}(|\nabla\hat{u}_\alpha|^2+|\nabla\hat{v}_\alpha|^2)dx\Big)\\
&\ge\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}\Big(\int_{\mathbb{R}^N_-\cap B_R(0)}(|\nabla\hat{u}_\alpha|^2+|\nabla\hat{v}_\alpha|^2)dx+o(1)\Big)\\
&\ge\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}\Big(\int_{\mathbb{R}^N_-\cap B_R(0)}(|\nabla u|^2+|\nabla v|^2)dx+o(1)\Big)\\
&\ge\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}(2^{-\frac{2}{p+q-2}}m_{1,p,q}^{\frac{p+q}{p+q-2}}+o(1))\\
&=2^{-\frac{2}{p+q-2}}\alpha^{\frac{(2-N)(p+q)+2N}{p+q-2}}((m_{1,p,q}^{\frac{p+q}{p+q-2}}+o(1)).
\end{aligned} \tag{4.15}$$

By (4.9) and (4.15),

$$\int_{\Omega_\alpha/B_{R,\alpha}}(|\nabla\hat{u}_\alpha|^2+|\nabla\hat{v}_\alpha|^2)dx=o(1)+o_R(1),$$

$$\begin{aligned}
\int_{B_{R,\alpha}}(|\nabla\hat{u}_\alpha|^2+|\nabla\hat{v}_\alpha|^2)dx&=\int_{\mathbb{R}^N_-\cap B_R(0)}(|\nabla u|^2+|\nabla v|^2)dx+o(1)\\
&=2^{-\frac{2}{p+q-2}}m_{1,p,q}^{\frac{p+q}{p+q-2}}+o(1)+o_R(1).
\end{aligned}$$

Let $R \to +\infty$, the above equation yields

$$\int_{\mathbb{R}^N_-} (|\nabla u|^2 + |\nabla v|^2)dx = 2^{-\frac{2}{p+q-2}} m_{1,p,q}^{\frac{p+q}{p+q-2}}. \tag{4.16}$$

An application of the Brezis-Lieb's Lemma gives

$$\int_{\mathbb{R}^N_-} (|\nabla(\hat{u}_\alpha - u)|^2 + |\nabla(\hat{v}_\alpha - v)|^2)dx \to 0$$

as $\alpha \to +\infty$. On the other hand, by (1.3) and (4.16),

$$\frac{\int_{\mathbb{R}^N_-} (|\nabla u|^2 + |\nabla v|^2)dx}{(\int_{\mathbb{R}^N_-} e^{x_N} |u|^p |v|^q dx)^{2/(p+q)}} = 2^{2/(p+q)} \Big(\int_{\mathbb{R}^N_-} (|\nabla u|^2 + |\nabla v|^2)dx\Big)^{\frac{p+q-2}{p+q}} = m_{1,p,q}.$$

This implies that (u, v) achieves $m_{1,p,q}$. As a consequence, we have

$$\begin{aligned} &\alpha^{-\frac{(2-N)(p+q)+2N}{p+q-2}} \int_{B_1(0)} (|\nabla(u_\alpha - \alpha^{\frac{2}{p+q-2}} u(\alpha(x - x_0))|^2 \\ &+ |\nabla(v_\alpha - \alpha^{\frac{2}{p+q-2}} v(\alpha(x - x_0))|^2)dx \to 0, \end{aligned}$$

as $\alpha \to +\infty$. □

Now the the proof of Theorem 1.1 is completed by Lemmas 4.4, 4.5 and Proposition 4.6.

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