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EXISTENCE OF THREE POSITIVE PERIODIC SOLUTIONS FOR DIFFERENTIAL SYSTEMS WITH FEEDBACK CONTROLS ON TIME SCALES

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Abstract. Using the Leggett-Williams multiple fixed point theorem, we establish criteria for the existence of three positive periodic solutions of a class of differential systems with feedback controls on time scales.

1. Introduction

Recently, by using the Krasnosel'skii's fixed point theorem for cones, Li and Zhu [12] studied the existence of positive periodic solutions of the following functional differential systems with feedback controls:

$$\begin{aligned} \dot{x}(t) &= -A(t)x(t) + f(t, x_t, x(t-\tau(t,x(t))), u(t-\alpha(t))), \\ \dot{u}(t) &= -B(t)u(t) + C(t)x(h(t,x(t))). \end{aligned} \tag{1.1}$$

Zeng and Zhou [21] considered a class of more general functional differential systems with feedback controls of the form

$$\begin{aligned} \dot{x}(t) &= -A(t,x(t))x(t) + f(t, x_t, x(t-\tau(t,x(t))), u(t-\alpha(t))), \\ \dot{u}(t) &= -B(t,x(t))u(t) + C(t,x(t))x(h(t,x(t))). \end{aligned} \tag{1.2}$$

By means of the Krasnosel'skii's fixed point theorem, they obtained some criteria for the existence of two positive periodic solutions of (1.2).

Also, by applying the continuation theorem of coincidence degree theory, Li and Zhu [13] studied the existence of positive periodic solutions to the difference equations with feedback control of the form

$$\begin{aligned} N(n+1) &= N(n)\exp\Big[r(n)\Big(1 - \frac{N(n-m)}{k(n)} - c(n)\mu(n)\Big)\Big], \\ \Delta\mu(n) &= -a(n)\mu(n) + b(n)N(n-m), \end{aligned} \tag{1.3}$$

where $a : \mathbb{Z} \to (0,1), c, k, r, b : \mathbb{Z} \to \mathbb{R}^+$ are all ω-periodic functions and m is a positive integer.

In the previous ten years, many authors [7, 11, 14, 18] have argued that the discrete time model governed by difference equations are more appropriate than the

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continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Consequently, the studies of dynamic systems governed by difference equations have received great attention from more scholars.

In fact, continuous and discrete systems are very important in implementing and applications. It is well known that the theory of time scales [4, 5] has received a lot of attention which was introduced by Stefan Hilger [8] in order to unify continuous and discrete analysis. Therefore, it is meaningful to study dynamic systems on time scales which can unify differential and difference systems. For the work concerning with the existence of periodic solutions for dynamic systems on time scales, we refer the reader to [2, 3, 6, 15, 16, 17, 19, 20, 22].

Motivated by above statement, in this paper, we will study the following differential systems with feedback controls on time scales:

$$\begin{aligned} x^{\Delta}(t) &= -A(t,x(t))x(\sigma(t)) + \lambda f(t,x_t,x(t-\tau(t,x(t))),u(t-\alpha(t,x(t)))),\\ u^{\Delta}(t) &= -B(t,x(t))u(\sigma(t)) + g(t,x_t,x(h(t,x(t)))), \quad t\in\mathbb{T}, \end{aligned} \tag{1.4}$$

in which $\mathbb{T}$ is a periodic time scales which has the subspace topology inherited from the standard topology on $\mathbb{R}$, $\lambda > 0$ is parameter,

$$A(t,x(t)) = \operatorname{diag}[a_1(t,x(t)),a_2(t,x(t)),\dots,a_n(t,x(t))],$$
$$B(t,x(t)) = \operatorname{diag}[b_1(t,x(t)),b_2(t,x(t)),\dots,b_n(t,x(t))],$$

$a_i(t,y),b_i(t,y),\in C(\mathbb{T}\times\mathbb{R}^n,\mathbb{R})$ satisfy $a_i(t+\omega,y)=a_i(t,y)$, $b_i(t+\omega,y)=b_i(t,y)$ for all $t\in\mathbb{T},y\in\mathbb{R}^n$, $i=1,2,\dots,n$, $t-\tau(t,y)$, $t-\alpha(t,y)$, $h(t,y)\in C(\mathbb{T}\times\mathbb{R}^n,\mathbb{T})$ satisfy $\tau(t+\omega,y)=\tau(t,y)$, $\alpha(t+\omega,y)=\alpha(t,y), h(t+\omega,y)=h(t,y)$ for all $t\in\mathbb{T}$, $y\in\mathbb{R}^n$, $\omega>0$ is a constant, f is a function defined on $\mathbb{T}\times BC\times\mathbb{R}^n\times\mathbb{R}^n$, satisfying $f(t+\omega,x_{t+\omega},y,z)=f(t,x_t,y,z)$ for all $t\in\mathbb{T}$, $x\in BC, y,z\in\mathbb{R}^n$, where BC denotes the Banach space of all bounded continuous functions $\eta:\mathbb{T}\to\mathbb{R}^n$ with the norm $\|\eta\|=\sum_{i=1}^n\max_{\theta\in\mathbb{T}}|\eta_i(\theta)|$, where $\eta=(\eta_1,\eta_2,\dots,\eta_n)^T$, and g is a function defined on $\mathbb{T}\times BC\times\mathbb{R}^n$, satisfying $g(t+\omega,x_{t+\omega},y)=g(t,x_t,y)$ for all $t\in\mathbb{T},x\in BC,y\in\mathbb{R}^n$. If $x\in BC$, then $x_t\in BC$ for any $t\in\mathbb{T}$ is defined by $x_t(\theta)=x_t(t+\theta)$ for $\theta\in\mathbb{T}$. In the sequel, we denote $f=(f_1,f_2,\dots,f_n)^T, g=(g_1,g_2,\dots,g_n)^T$. Let $\mathbb{R}=(-\infty,+\infty),\mathbb{R}_+=[0,+\infty),\mathbb{R}^n_+=\{(x_1,x_2,\dots,x_n)^T: x_i\geq 0, 1\leq i\leq n\}$, respectively. For each $x=(x_1,x_2,\dots,x_n)^T\in\mathbb{R}^n$, the norm of x is defined as $|x|_0=\sum_{i=1}^n|x_i|$.

The main purpose of this paper is to study the existence of at least three nonnegative periodic solutions of (1.4) by using the Leggett-Williams multiple fixed point theorem.

The organization of this paper is as follows. In Section 2, we make some preparations. In Section 3, by using the Leggett-Williams multiple fixed point theorem, we obtain the existence of at least three nonnegative periodic solutions of (1.4). In Section 4, an example is also provided to illustrate the effectiveness of the main results obtained in Section 3.

2. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Definition 2.1 ([4])**.** A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t.$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m, defined $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2 ([9])**.** We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np, f(t + T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t + T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t + T) = f(t)$ for all $t \in \mathbb{T}$.

Definition 2.3 ([4])**.** For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted by $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - t]| \leq \epsilon|\sigma(t) - s|, \quad \forall s \in U.$$

Definition 2.4 ([4])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.5 ([4])**.** A continuous function $F : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and F is differentiable at each $t \in D$.

Definition 2.6 ([4])**.** Assume that $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Suppose further that there exists a function F which is pre-differentiable with region of differentiation D such that

$$F^\Delta(t) = f(t) \quad \text{holds for all } t \in D.$$

We define the Cauchy integral by

$$\int_a^b f(s)\Delta s = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

Definition 2.7 ([4])**.** A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$ is the graininess function. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0\}$ for all $t \in \mathbb{T}$. Let $p \in \mathcal{R}$. The exponential function is defined by

$$e_p(t, s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big),$$

where $\xi_{h(z)}$ is the so-called cylinder transformation.

Lemma 2.8 ([4])**.** *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$*;*
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$, *where* $p \ominus q = p \oplus (\ominus q)$;
(vii) $e_p^{\Delta}(\cdot,s) = pe_p(\cdot,s)$.

For convenience, we introduce the following notation:

$$r_1^i = \sup_{t\in[0,\omega]_{\mathbb{T}}} \frac{1}{|1-e_{\ominus a_i}(t,t-\omega)|}, \quad r_2^i = \inf_{t\in[0,\omega]_{\mathbb{T}}} \frac{1}{|1-e_{\ominus a_i}(t,t-\omega)|},$$
$$\eta_1^i = \sup_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a_i}(t,u), \quad \eta_2^i = \inf_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a_i}(t,u),$$
$$r^M = \max_{1\le i\le n}\{r_1^i\}, \quad r^l = \min_{1\le i\le n}\{r_2^i\},$$
$$\eta^M = \max_{1\le i\le n}\{\eta_1^i\}, \quad \eta^l = \min_{1\le i\le n}\{\eta_2^i\},$$
$$\gamma^i = \sup_{t\in[0,\omega]_{\mathbb{T}}} |\ominus a_i|, \quad \kappa^i = \sup_{t\in[0,\omega]_{\mathbb{T}}} e_{\ominus a_i}(\sigma(t),t),$$
$$\gamma = \max_{1\le i\le n}\{\gamma^i\}, \quad \kappa = \max_{1\le i\le n}\{\kappa^i\}.$$

Lemma 2.9. $(x(t),u(t))^T$ *is an* ω*-periodic solution of* (1.4) *if and only if it is an* ω*-periodic solution of the system*

$$\begin{gathered} x^{\Delta}(t) = -A(t,x(t))x(\sigma(t)) + \lambda f(t,x_t,x(t-\tau(t,x(t))),u(t-\alpha(t,x(t)))), \\ u(t) = \int_{t-\omega}^{t} \overline{G}(t,s)g(s,x_s,x(h(s,x(s))))\Delta s := (\Phi x)(t), \end{gathered} \tag{2.1}$$

where

$$\overline{G}(t,s) = \operatorname{diag}[\overline{G}_1(t,s),\overline{G}_2(t,s),\dots,\overline{G}_n(t,s)]$$

and

$$\overline{G}_i(t,s) = \frac{e_{\ominus b_i}(t,s)}{1-e_{\ominus b_i}(t,t-\omega)}, \quad s\in[t-\omega,t]_{\mathbb{T}},\ i=1,2,\dots,n.$$

Proof. First, assume that $(x(t),u(t))^T$ is an ω-periodic solution of (1.4). From the second equation of (1.4), it follows that

$$u_i^{\Delta}(t) + b_i(t,x(t))u_i(\sigma(t)) = g_i(t,x_t,x(h(t,x(t)))), \quad i=1,2,\dots,n. \tag{2.2}$$

Multiply both sides of this equation by $e_{b_i}(t,0)$ and then integrate them from $t-\omega$ to t to obtain

$$\int_{t-\omega}^{t} [e_{b_i}(s,0)u_i(s)]^{\Delta}\Delta s = \int_{t-\omega}^{t} e_{b_i}(s,0)g_i(s,x_s,x(h(s,x(s))))\Delta s,$$

for $i=1,2,\dots,n$, and then

$$e_{b_i}(t,0)u_i(t) - e_{b_i}(t-\omega,0)u_i(t-\omega) = \int_{t-\omega}^{t} e_{b_i}(s,0)g_i(s,x_s,x(h(s,x(s))))\Delta s,$$

for $i=1,2,\dots,n$. Dividing both sides of the above equation by $e_{b_i}(t,0)$, we have

$$u_i(t) = \int_{t-\omega}^{t} \frac{e_{\ominus b_i}(t,s)}{1-e_{\ominus b_i}(t,t-\omega)} g_i(s,x_s,x(h(s,x(s))))\Delta s, \quad i=1,2,\dots,n.$$

So $(x(t),u(t))^T$ is an ω-periodic solution of (2.1).

Conversely, assume that $(x(t),u(t))^T$ is an ω-periodic solution of (2.1). Then we have

$$e_{b_i}(t,0)u_i(t)-e_{b_i}(t-\omega,0)u_i(t-\omega)=\int_{t-\omega}^{t}e_{b_i}(s,0)g_i(s,x_s,x(h(s,x(s))))\Delta s,$$

for $i=1,2,\dots,n$; that is,

$$\big(1+e_{b_i}(\omega,0)\big)e_{b_i}(t,0)u_i(t)=\int_{t-\omega}^{t}e_{b_i}(s,0)g_i(s,x_s,x(h(s,x(s))))\Delta s,$$

for $i=1,2,\dots,n$. Then

$$\begin{aligned}
&\big[\big(1+e_{b_i}(\omega,0)\big)e_{b_i}(t,0)u_i(t)\big]^{\Delta}\\
&=e_{b_i}(t,0)\big(1+e_{b_i}(\omega,0)\big)\big[u_i^{\Delta}(t)+b_iu_i(\sigma(t))\big]\\
&=\Big(\int_{t-\omega}^{t}e_{b_i}(s,0)g_i(s,x_s,x(h(s,x(s))))\Delta s\Big)^{\Delta}\\
&=e_{b_i}(t,0)g_i(t,x_t,x(h(t,x(t))))-e_{b_i}(t-\omega,0)g_i(t-\omega,x_{t-\omega},x(h(t-\omega,x(t-\omega))))\\
&=e_{b_i}(t,0)\big(1+e_{b_i}(\omega,0)\big)g_i(t,x_t,x(h(t,x(t)))),
\end{aligned}$$

which implies

$$u_i^{\Delta}(t)+b_i(t,x(t))u_i(\sigma(t))=g_i(t,x_t,x(h(t,x(t)))),\quad i=1,2,\dots,n.$$

So $(x(t),u(t))^T$ is an ω-periodic solution of (1.4). The proof of the lemma is complete. □

At the same time, from the definition of $e_p(t,s)$ and the periodicity of b_i, we have $e_{\ominus b_i}(t+\omega,s+\omega)=e_{\ominus b_i}(t,s)$, $i=1,2,\dots,n$, so it is clear that $\overline{G}(t,s)=\overline{G}(t+\omega,s+\omega)$ for all $(t,s)\in\mathbb{T}^2$ and $u(t+\omega)=u(t)$ when x is ω-periodic solution.

Now, (2.1) can be reformulated as

$$x^{\Delta}(t)=-A(t,x(t))x(\sigma(t))+\lambda f(t,x_t,x(t-\tau(t,x(t))),(\Phi x)(t-\alpha(t,x(t)))). \tag{2.3}$$

We proceed from (2.3) and obtain

$$x(t)=\lambda\int_{t-\omega}^{t}G(t,s)f(s,x_s,x(s-\tau(s,x(s))),(\Phi x)(s-\alpha(s,x(s))))\Delta s, \tag{2.4}$$

where

$$G(t,s)=\mathrm{diag}[G_1(t,s),G_2(t,s),\dots,G_n(t,s)]$$

and

$$G_i(t,s)=\frac{e_{\ominus a_i}(t,s)}{1-e_{\ominus a_i}(t,t-\omega)},\quad s\in[t-\omega,t]_{\mathbb{T}},\quad i=1,2,\dots,n.$$

By the periodicity of a_i, $i=1,2,\dots,n$, it is also obvious that $G(t,s)=G(t+\omega,s+\omega)$ for all $(t,s)\in\mathbb{T}^2$.

To obtain our main results, we make the following assumptions throughout this paper.

- (H1) $a_i(t,x(t))>0$ or $a_i(t,x(t))<0$ for all $t\in\mathbb{T}$, $i=1,2,\dots,n$;
- (H2) $f_i(t,\zeta,\xi,\Phi(\eta))a_i(t,x(t))\geq 0$ for all $(t,\zeta,\xi,\eta)\in\mathbb{T}\times BC(\mathbb{T}\times\mathbb{R}_+^n)\times\mathbb{R}_+^n\times\mathbb{R}_+^n$, $i=1,2,\dots,n$;
- (H3) $f(t,\phi_t,\phi(t-\tau(t,\phi(t))),(\Phi\phi)(t-\alpha(t,\phi(t))))$ is a continuous function of t for each $\phi\in BC(\mathbb{T}\times\mathbb{R}_+^n)$;

(H4) for any $L > 0$ and $\epsilon > 0$, there exists a real number $\delta > 0$ such that $\phi, \psi \in BC, \|\phi\| \le L, \|\psi\| \le L, \|\phi - \psi\| < \delta, 0 \le s \le \omega$ imply

$$\begin{aligned}&\|f(s, \phi_s, \phi(s - \tau(s, \phi(s))), (\Phi\phi)(s - \alpha(s, \phi(s))))\\ &- f(s, \psi_s, \psi(s - \tau(s, \psi(s))), (\Phi\psi)(s - \alpha(s, \psi(s))))\| < \epsilon.\end{aligned}$$

Moreover, for the sake of simplicity, let

$$f(t, \phi, \Phi) = f(t, \phi_t, \phi(t - \tau(t, \phi(t))), (\Phi\phi)(t - \alpha(t, \phi(t))))$$

and

$$f(t, x, \Phi) = f(t, x_t, x(t - \tau(t, x(t))), (\Phi x)(t - \alpha(t, x(t)))).$$

Then by (H2),

$$G_i(t, s)f_i(t, x, \Phi) \ge 0 \quad \text{for } (t, s) \in \mathbb{T}^2,\ i = 1, 2, \dots, n. \tag{2.5}$$

Let X be a Banach space and K be a cone in X. A mapping ψ is said to be a concave nonnegative continuous functional on K if $\psi : K \to \mathbb{R}_+$ is continuous and

$$\psi(\mu x + (1 - \mu)y) \ge \mu\psi(x) + (1 - \mu)\psi(y), \quad x, y \in K,\ \mu \in [0, 1].$$

Let $a, b, c > 0$ be constants with K and X as defined above. Define

$$K_a = \{x \in K : \|x\| < a\}, \quad K(\psi, b, c) = \{x \in K : \psi(y) \ge b, \|x\| \le c\}.$$

Theorem 2.10 (Leggett-Williams multiple fixed point theorem [10]). *Let $X = (X, \|\cdot\|)$ be a Banach space and $K \subset X$ a cone, and $c_4 > 0$ a constant. Suppose there exists a concave nonnegative continuous function ψ on K with $\psi(u) \le u$ for $u \in \overline{K}_{c_4}$ and let $T : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers c_1, c_2 and c_3 with $0 < c_1 < c_2 < c_3 \le c_4$ such that*

(i) *$\{u \in K(\psi, c_2, c_3) : \psi(u) > c_2\} \ne \emptyset$ and $\psi(Tu) > c_2$ for all $u \in K(\psi, c_2, c_3)$;*
(ii) *$\|Tu\| < c_1$ for all $u \in \overline{K}_{c_1}$;*
(iii) *$\psi(Tu) > c_2$ for all $u \in K(\psi, c_2, c_4)$ with $\|Tu\| > c_3$.*

Then T has at least three fixed points u_1, u_2 and u_3 in $\overline{K}_{c_4}$. Furthermore, e have $u_1 \in \overline{K}_{c_1}$, $u_2 \in \{u \in K(\psi, c_2, c_4) : \psi(u) > c_2\}$, $u_3 \in \overline{K}_{c_4} \backslash \{K(\psi, c_2,\ c_4) \cup \overline{K}_{c_1}\}$.

Let $X = \{x(t) = (x_1, x_2, \dots, x_n)^T \in C(\mathbb{T}, \mathbb{R}^n) : x(t) = x(t + \omega)\}$ with the norm $\|x\| = \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} |x_i(t)|$, then X is a Banach space with the norm $\|\cdot\|$. Define a cone K in X by

$$K = \{x(t) = (x_1, x_2, \dots, x_n)^T \in X : x_i(t) \ge 0, \forall t \in [0, \omega]_{\mathbb{T}},\ i = 1, 2, \dots, n\}$$

and an operator T_λ on X by

$$(T_\lambda x)(t) = \lambda \int_{t-\omega}^{t} G(t, s)f(s, x, \Phi)\Delta s.$$

And let

$$T_\lambda x = (T_\lambda^1 x, T_\lambda^2 x, \dots, T_\lambda^n x)^T.$$

Lemma 2.11. *$T_\lambda(K) \subset K$ and $T_\lambda : K \to K$ is well-defined.*

Proof. For each $x \in K$, by (H_3), we have $T_\lambda x \in C(\mathbb{T}, \mathbb{R}^n)$, with the periodicity of $G(t, s)$ and $(\Phi x)(t)$, then

$$(T_\lambda x)(t + \omega) = \lambda \int_{t}^{t+\omega} G(t + \omega, s)f(s, x, \Phi)\Delta s$$

$$\begin{aligned}&= \lambda \int_{t-\omega}^{t} G(t+\omega, s+\omega) f(s+\omega, x, \Phi)\Delta s\\ &= \lambda \int_{t-\omega}^{t} G(t,s) f(s,x,\Phi)\Delta s\\ &= (T_\lambda x)(t),\end{aligned}$$

and by (2.5), $T_\lambda x \in K$. The proof is complete. □

Lemma 2.12. *$T_\lambda : K \to K$ is completely continuous.*

Proof. We first show that T_λ is continuous. By (H4), for any $L > 0$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $\phi, \psi \in BC$, $\|\phi\| \le L$, $\|\psi\| \le L$, $\|\phi - \psi\| < \delta$, $0 \le s \le \omega$ imply

$$\begin{aligned}&\|f(s, \phi_s, \phi(s-\tau(s,\phi(s))), (\Phi\phi)(s-\alpha(s,\phi(s))))\\ &- f(s,\psi_s,\psi(s-\tau(s,\psi(s))), (\Phi\psi)(s-\alpha(s,\psi(s))))\| < \frac{\epsilon}{\lambda r^M \eta^M}.\end{aligned}$$

If $x, y \in K$ with $\|x\| \le L$, $\|y\| \le L$, and $\|x-y\| < \delta$, then

$$\begin{aligned}&\|(T_\lambda x)(t) - (T_\lambda y)(t)\|\\ &\le \lambda \int_{t-\omega}^{t} \max_{1\le i\le n} |G_i(t,s)| \|f(s,x_s,x(s-\tau(s,x(s))), (\Phi x)(s-\alpha(s,x(s))))\\ &\quad - f(s,y_s,y(s-\tau(s,y(s))),(\Phi y)(s-\alpha(s,y(s))))\|\Delta s\\ &\le \lambda r^M \eta^M \int_0^{\omega} \|f(s,x_s,x(s-\tau(s,x(s))),(\Phi x)(s-\alpha(s,x(s))))\\ &\quad - f(s,y_s,y(s-\tau(s,y(s))),(\Phi y)(s-\alpha(s,y(s))))\|\Delta s\\ &< \lambda r^M \eta^M \frac{\epsilon}{\lambda r^M \eta^M} = \epsilon\end{aligned}$$

for all $t \in [0,\omega]_{\mathbb{T}}$. This yields $\|T_\lambda x - T_\lambda y\| < \epsilon$. Thus, T_λ is continuous.

Next, we show that T_λ maps any bounded sets in K into relatively compact sets. Now, we first prove that f maps bounded sets into bounded sets. Indeed, let $\epsilon = 1$. By (H4), for any $\mu > 0$, there exists $\delta > 0$ such that $x, y \in BC$, $\|x\| \le \mu$, $\|y\| \le \mu$, $\|x-y\| < \delta$, $0 \le s \le \omega$ imply

$$\begin{aligned}&\|f(s,x_s,x(s-\tau(s,x(s))),(\Phi x)(s-\alpha(s,x(s))))\\ &- f(s,y_s,y(s-\tau(s,y(s))),(\Phi y)(s-\alpha(s,y(s))))\| < 1.\end{aligned}$$

Choose a positive integer N such that $\frac{\mu}{N} < \delta$. Let $x \in BC$ and define $x^k(t) = \frac{k}{N}x(t)$ for $k = 0, 1, 2, \dots, N$. If $\|x\| \le \mu$, then

$$\|x^k - x^{k-1}\| = \|\frac{k}{N}x(t) - \frac{k-1}{N}x(t)\| \le \frac{1}{N}\|x\| \le \frac{\mu}{N} < \delta.$$

Thus,

$$\begin{aligned}&\|f(s,x_s^k,x^k(s-\tau(s,x^k(s))),(\Phi x^k)(s-\alpha(s,x^k(s))))\\ &- f(s,x_s^{k-1},x^{k-1}(s-\tau(s,x^{k-1}(s))),(\Phi x^{k-1})(s-\alpha(s,x^{k-1}(s))))\| < 1\end{aligned}$$

for all $s \in [0,\omega]_{\mathbb{T}}$. This yields

$$\begin{aligned}&\|f(s,x_s,x(s-\tau(s,x(s))),(\Phi x)(s-\alpha(s,x(s))))\|\\ &= \|f(s,x_s^N,x^N(s-\tau(s,x^N(s))),(\Phi x^N)(s-\alpha(s,x^N(s))))\|\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=1}^{N} \|f(s, x_s^k, x^k(s-\tau(s,x^k(s))), (\Phi x^k)(s-\alpha(s,x^k(s))))\\
&\quad - f(s, x_s^{k-1}, x^{k-1}(s-\tau(s,x^{k-1}(s))), (\Phi x^{k-1})(s-\alpha(s,x^{k-1}(s))))\|\\
&\quad + \|f(s,0,0,0\|\\
&< N + \|f(s,0,0,0\| =: Q.
\end{aligned}
$$

For $t \in [0,\omega]_{\mathbb{T}}$, we have

$$
\begin{aligned}
\|T_\lambda x\| &= \sum_{i=1}^{n} \max_{t\in[0,\omega]_{\mathbb{T}}} |(T_\lambda^i x)(t)|\\
&\le \lambda r^M \eta^M \sum_{i=1}^{n} \int_0^\omega |f_i(s, x_s, x(s-\tau(s,x(s))), (\Phi x)(s-\alpha(s,x(s))))|\Delta s\\
&\le \lambda r^M \eta^M \omega Q.
\end{aligned}
$$

Finally, for $t \in \mathbb{T}$,

$$
\begin{aligned}
(T_\lambda^i x)^\Delta(t) &= [\lambda \int_{t-\omega}^{t} G_i(t,s) f_i(s,x,\Phi)\Delta s]^\Delta\\
&= [\lambda \int_{t-\omega}^{t} \frac{e_{\ominus a_i}(t,s)}{1-e_{\ominus a_i}(t,t-\omega)} f_i(s,x,\Phi)\Delta s]^\Delta\\
&= \lambda \frac{e_{\ominus a_i}(\sigma(t),t) - e_{\ominus a_i}(\sigma(t),t-\omega)}{1-e_{\ominus a_i}(t,t-\omega)} f_i(t,x,\Phi)\\
&\quad + \lambda \frac{1}{1-e_{\ominus a_i}(t,t-\omega)} \ominus a_i \int_{\tilde a}^{t} e_{\ominus a_i}(t,s) f_i(s,x,\Phi)\Delta s\\
&\quad - \lambda \frac{1}{1-e_{\ominus a_i}(t,t-\omega)} \ominus a_i \int_{\tilde a}^{t-\omega} e_{\ominus a_i}(t,s) f_i(s,x,\Phi)\Delta s\\
&= \ominus a_i (T_\lambda^i x)(t) + \lambda e_{\ominus a_i}(\sigma(t),t) f_i(t,x,\Phi),
\end{aligned}
$$

where $\tilde a \in [0,\omega]_{\mathbb{T}}$ is an arbitrary constant, $i = 1,2,\dots,n$. So we obtain

$$
|(T_\lambda^i x)^\Delta(t)| \le \gamma^i |T_\lambda^i x| + \lambda \kappa^i |f_i(t,x,\Phi)|, \quad i = 1,2,\dots,n.
$$

Then

$$
\begin{aligned}
\|(T_\lambda x)^\Delta(t)\| &= \sum_{i=1}^{n} \max_{t\in[0,\omega]_{\mathbb{T}}} |(T_\lambda^i x)^\Delta(t)|\\
&\le \gamma \sum_{i=1}^{n} \max_{t\in[0,\omega]_{\mathbb{T}}} |T_\lambda^i x| + \lambda\kappa \sum_{i=1}^{n} \max_{t\in[0,\omega]_{\mathbb{T}}} |f_i(t,x,\Phi)|\\
&\le \gamma \|T_\lambda x\| + \lambda\kappa \|f(t,x,\Phi)\|\\
&\le \lambda\gamma r^M \eta^M \omega Q + \lambda\kappa Q.
\end{aligned}
$$

Hence $\{T_\lambda x : x \in K, \|x\| \le \mu\}$ is a family of uniformly bounded and equicontinuous functions on $[0,\omega]_{\mathbb{T}}$. Applying Arzela-Ascoli theorem on time scales [1], the function T_λ is completely continuous. The proof is complete. □

Lemma 2.13. *Existence of nonnegative periodic solutions of* (1.4) *is equivalent to the existence of fixed point problem of T_λ in K.*

The proof of the above lemma is straight forward and we will omit it.

3. Main results

Let

$$f_i^h = \limsup_{\phi_i \to h} \max_{t\in[0,\omega]_{\mathbb{T}}} \frac{|f_i(t,\phi,\Phi)|}{|a_i(t,\phi)|\phi_i}, \quad i = 1,2,\dots,n,$$

where $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T \in C(\mathbb{T}, \mathbb{R}^n)$, $|\phi|_1 = \sum_{i=1}^n \min_{t\in[0,\omega]_{\mathbb{T}}} |\phi_i(t)|$ and $\delta = \frac{r^M\eta^M}{r^l\eta^l}$.

From the definitions of η^M and η^l, it is obvious that $\delta > 1$.

Theorem 3.1. *Assume that* (H1)-(H4) *hold, there are constants* $0 < c_1 < c_2$ *such that the following conditions hold:*

- (H5) $r^M\eta^M > 1$;
- (H6) $f_i^\infty < \omega$, $i = 1,2,\dots,n$;
- (H7) $\int_0^\omega |f(s,x,\Phi)|_0 \Delta s \geq \delta c_2 \omega$ *for* $c_2 \leq |x|_1 \leq \|x\| \leq \delta c_2$;
- (H8) $\int_0^\omega |f(s,x,\Phi)|_0 \Delta s \leq \frac{c_1\omega}{r^M\eta^M}$ *for* $0 \leq |x|_1 \leq \|x\| \leq c_1$

Then (1.4) *has at least three nonnegative ω-periodic solutions for*

$$\frac{1}{r^M\eta^M\omega} < \lambda < \frac{1}{\omega}.$$

Proof. Since $f_i^\infty < \omega$ holds for $1 \leq i \leq n$, there exist $\varepsilon \in (0,\omega)$ and $\theta > 0$ such that $|f_i(t,x,\Phi)| \leq \varepsilon |a_i(t,x)| x_i$ for $x_i \geq \theta$, $t \in [0,\omega]_{\mathbb{T}}$, $i = 1,2,\dots,n$. Let

$$\xi_i = \max_{0\leq x_i\leq\theta, 0\leq t\leq\omega} |f_i(t,x,\Phi)|, \quad i = 1,2,\dots,n, \quad \xi = \sum_{i=1}^n \xi_i.$$

Then $|f_i(t,x,\Phi)| \leq \varepsilon|a_i(t,x)|x_i + \xi_i$ for $x_i \geq 0$, $t \in [0,\omega]_{\mathbb{T}}$, $i = 1,2,\dots,n$. Choose

$$c_4 > \max\Big\{\frac{r^M\eta^M\xi\omega}{\omega-\varepsilon}, \delta c_2\Big\}.$$

Then for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned}
\|T_\lambda x\| &= \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} |T_\lambda^i x| \\
&= \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^t G_i(t,s) f_i(s,x,\Phi)\Delta s \\
&= \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^t |G_i(t,s)||f_i(s,x,\Phi)|\Delta s \\
&\leq \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^t |G_i(t,s)|(\varepsilon|a_i(s,x)|x_i + \xi_i)\Delta s \\
&\leq \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^t |G_i(t,s)|(\varepsilon|a_i(s,x)||x_i| + \xi_i)\Delta s \\
&\leq \lambda[\varepsilon \max_{0\leq i\leq n}\{\int_{t-\omega}^t G_i(t,s)a_i(s,x)\Delta s\} \sum_{i=1}^n \max_{s\in[0,\omega]_{\mathbb{T}}} |x_i(s)|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{n} \max_{t\in[0,\omega]_{\mathbb{T}}} \int_{t-\omega}^{t} |G_i(t,s)|\xi_i\Delta s] \\
&\le \lambda[\varepsilon\|x\| + \sum_{i=1}^{n} r_1^i\eta_1^i\xi_i\omega] \\
&\le \lambda[\varepsilon c_4 + r^M\eta^M\xi\omega] \\
&< \frac{1}{\omega}[\varepsilon c_4 + r^M\eta^M\xi\omega] < c_4.
\end{aligned}
$$

Hence $T_\lambda : \overline{K}_{c_4} \to \overline{K}_{c_4}$.

Next, we define a concave nonnegative continuous function ψ on K by $\psi(x) = \sum_{i=1}^{n} \min_{t\in[0,\omega]_{\mathbb{T}}}$
$|x_i(t)|$, then $\psi(x) \le \|x\|$. Let $c_3 = \delta c_2 = \frac{r^M\eta^M}{r^l\eta^l} c_2$ and $\phi_0(t) = \{\phi_0, 0, \dots, 0\}^T, \phi_0$ is any given number satisfying $c_2 < \phi_0 < c_3$. Then $\phi_0(t) \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset$. Further, for $x \in K(\psi, c_2, c_3)$, by (H7)

$$
\begin{aligned}
\psi(T_\lambda x) &= \sum_{i=1}^{n} \min_{t\in[0,\omega]_{\mathbb{T}}} |(T_\lambda^i x)(t)| \\
&= \sum_{i=1}^{n} \min_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^{t} G_i(t,s) f_i(s,x,\Phi)\Delta s \\
&\ge \lambda r^l\eta^l \sum_{i=1}^{n} \int_0^\omega |f_i(s,x,\Phi)|\Delta s \\
&= \lambda r^l\eta^l \int_0^\omega |f(s,x,\Phi)|_0\Delta s \\
&\ge \lambda r^l\eta^l\delta c_2\omega \\
&= \lambda r^l\eta^l \frac{r^M\eta^M}{r^l\eta^l} c_2\omega \\
&\ge \lambda r^M\eta^M c_2\omega > c_2,
\end{aligned} \tag{3.1}
$$

so condition (i) of Theorem 2.10 holds.

Now, let $x \in \overline{K}_{c_1}$, by (H8)

$$
\begin{aligned}
\|T_\lambda x\| &= \sum_{i=1}^{n} \max_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^{t} G_i(t,s) f_i(s,x,\Phi)\Delta s \\
&\le \lambda r^M\eta^M \sum_{i=1}^{n} \int_0^\omega |f_i(s,x,\Phi)|\Delta s \\
&\le \lambda r^M\eta^M \int_0^\omega |f(s,x,\Phi)|_0\Delta s \\
&< \frac{1}{\omega} r^M\eta^M \frac{c_1\omega}{r^M\eta^M} = c_1,
\end{aligned}
$$

then $T_\lambda x \in \overline{K}_{c_1}$.

Finally, for $x \in K(\psi, c_2, c_4)$ and $\|T_\lambda x\| > c_3$, so

$$
c_3 < \|T_\lambda x\| \le \lambda r^M\eta^M \sum_{i=1}^{n} \max_{t\in[0,\omega]_{\mathbb{T}}} \int_{t-\omega}^{t} |f_i(s,x,\Phi)|\Delta s
$$

$$
\begin{aligned}
&= \lambda r^M \eta^M \sum_{i=1}^n \int_0^\omega |f_i(s,x,\Phi)|\Delta s \\
&= \lambda r^M \eta^M \int_0^\omega |f(s,x,\Phi)|_0 \Delta s,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\psi(T_\lambda x) &= \sum_{i=1}^n \min_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^t G_i(t,s) f_i(s,x,\Phi)\Delta s \\
&\geq \lambda r^l \eta^l \sum_{i=1}^n \int_0^\omega |f_i(s,x,\Phi)|\Delta s \\
&= \lambda r^l \eta^l \int_0^\omega |f(s,x,\Phi)|_0 \Delta s \\
&> \lambda r^l \eta^l \frac{c_3}{\lambda r^M \eta^M} \\
&= \frac{c_3}{\delta} = c_2.
\end{aligned}
\tag{3.2}
$$

So all the conditions of Theorem 2.10 are satisfied. Consequently, (1.4) has at least three nonnegative ω-periodic solutions. This completes the proof. □

Theorem 3.2. *Let $f_i^0 < \omega$, $i = 1, 2, \dots, n$. Assume that there exists a constant $c_2 > 0$ such that* (H1)-(H7) *holds, then* (1.4) *has at least three nonnegative ω-periodic solutions for*

$$
\frac{1}{r^M \eta^M \omega} < \lambda < \frac{1}{\omega}.
$$

Proof. Since $f_i^0 < \omega$ holds for $1 \leq i \leq n$, there exist ρ, ζ, $0 < \rho < \omega$ and $0 < \zeta < c_2$ such that

$$
|f_i(t,x,\Phi)| \leq \rho |a_i(t,x)| x_i, \quad 0 \leq x_i \leq \frac{\zeta}{n}, \quad t \in [0,\omega]_{\mathbb{T}},\ i = 1, 2, \dots, n.
$$

Set $c_1 = \zeta$. For $x \in \overline{K}_{c_1}$,

$$
\begin{aligned}
\|T_\lambda x\| &= \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} \lambda \int_{t-\omega}^t G_i(t,s) f_i(s,x,\Phi)\Delta s \\
&\leq \lambda\rho \sum_{i=1}^n \max_{t\in[0,\omega]_{\mathbb{T}}} \int_{t-\omega}^t G_i(t,s) a_i(s,x) x_i \Delta s \\
&\leq \lambda\rho \max_{0\leq i\leq n} \{\int_{t-\omega}^t G_i(t,s) a_i(s,x)\Delta s\} \sum_{i=1}^n \max_{s\in[0,\omega]_{\mathbb{T}}} |x_i(s)| \\
&\leq \lambda\rho \|x(t)\| \\
&< \frac{1}{\omega}\rho c_1 < c_1.
\end{aligned}
$$

Then condition (ii) of Theorem 2.10 is satisfied. In view of conditions (H6)-(H7), using a similar proof to Theorem 3.1, it can be shown that (3.1) and (3.2) hold. That is, conditions (i) and (iii) of Theorem 2.10 are satisfied. By Theorem 2.10, there exist at least three nonnegative ω-periodic solutions of (1.4). Thus the theorem is proved. □

Theorem 3.3. *Assume that there are constants* $0 < c_1 < c_2$ *such that* (H1)-(H6) *and the following two conditions hold:*

(H9) $\int_0^\omega |f(s,x,\Phi)|_0 \Delta s \geq 2\delta c_2 \omega$ *for* $c_2 \leq |x|_1 \leq \|x\| \leq \delta c_2$;

(H10) $\|f(t,x,\Phi)\| \leq \|x\|$ *for* $0 \leq |x|_1 \leq \|x\| \leq c_1$

Then (1.4) *has at least three nonnegative* ω*-periodic solutions for*

$$\frac{1}{2r^M \eta^M \omega} < \lambda < \frac{1}{r^M \eta^M \omega}.$$

Proof. From (H10), for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}
\|T_\lambda x\| &= \|\lambda \int_{t-\omega}^t G(t,s) f(s,x,\Phi)\Delta s\| \\
&\leq \lambda r^M \eta^M \int_0^\omega \|f(s,x,\Phi)\|\Delta s \\
&\leq \lambda r^M \eta^M c_1 \omega \\
&< \frac{1}{r^M \eta^M \omega} r^M \eta^M c_1 \omega = c_1.
\end{aligned}$$

Then condition (ii) of Theorem 2.10 is satisfied. With (H9) and Theorem 3.1, the proof of conditions (i) and (iii) of Theorem 2.10 is easy and hence we will omit it. This completes the proof. □

4. Examples

Example 4.1. When $\mathbb{T} = \mathbb{R}$, the following system has at least three nonnegative 2π-periodic solutions:

$$\begin{gathered}
\dot{x}_1(t) = -\frac{1}{10\pi}(2+\sin t)x_1(t) + \frac{e^8}{24\pi}[x_1(t+1) + x_2(t)]^2 e^{-x_1(t-1)x_2(t)}|2+\sin u_1(t)|, \\
\dot{u}_1(t) = -(0.85 - 0.05\sin t)u_1(t) + 0.001x_1(t), \\
\dot{x}_2(t) = -\frac{1}{20\pi}(2-\cos t)x_2(t) + \frac{e^8}{36\pi}[x_1(t) + x_2(t+1)]^2 e^{-x_1(t)x_2(t-1)}|2+\cos u_2(t)|, \\
\dot{u}_2(t) = -(0.85 - 0.05\sin t)u_2(t) + 0.001x_2(t)
\end{gathered} \tag{4.1}$$

Proof. Corresponding to system (1.4), we have $a_1(t) = \frac{1}{10\pi}(2+\sin t)$, $a_2(t) = \frac{1}{20\pi}(2-\cos t)$, $b_1(t) = b_2(t) = 0.85 - 0.05\sin t$, $g_i(t) = 0.001x_i(t)$, $i = 1,2$, $f_1(t,x,\Phi) = \frac{e^8}{2\pi}[x_1(t+\theta)+x_2(t)]^2 e^{-x_1(t-\theta)x_2(t)}|2+\sin u_1(t)|$, $f_2(t,x,\Phi) = \frac{e^8}{3\pi}[x_1(t)+x_2(t+\theta)]^2 e^{-x_1(t)x_2(t-\theta)}|2+\cos u_2(t)|$, $\lambda = \frac{1}{12}$ and $\omega = 2\pi$. So we obtain

$$\begin{aligned}
e_{\ominus a_1}(t, t-\omega) &= \exp\Big\{-\int_{t-2\pi}^t \frac{1}{10\pi}(2+\sin s)\,\mathrm{d}s\Big\} = e^{-0.4}, \\
e_{\ominus a_2}(t, t-\omega) &= \exp\Big\{-\int_{t-2\pi}^t \frac{1}{20\pi}(2-\cos t)\,\mathrm{d}s\Big\} = e^{-0.2}, \\
r_1^1 &= \sup_{t\in[0,\omega]_{\mathbb{T}}} \frac{1}{|1 - e_{\ominus a_1}(t,t-\omega)|} = \frac{e^{-0.4}}{e^{-0.4}-1} = 3.033244, \\
r_1^2 &= \sup_{t\in[0,\omega]_{\mathbb{T}}} \frac{1}{|1 - e_{\ominus a_2}(t,t-\omega)|} = \frac{e^{-0.2}}{e^{-0.2}-1} = 5.516650,
\end{aligned}$$

$$r^M = \max\{3.033244, 5.516650\} = 5.516650,$$
$$r^l = \min\{3.033244, 50516650\} = 3.033244,$$
$$\eta_1^1 = \sup_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a_1}(t,u) = \sup_{u\in[t-\omega,t]_{\mathbb{T}}} \exp\Big\{-\int_u^t \frac{1}{10\pi}(2+\sin s)\,\mathrm{d}s\Big\} = 1,$$

$$\eta_1^2 = \sup_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a_2}(t,u) = \sup_{u\in[t-\omega,t]_{\mathbb{T}}} \exp\Big\{-\int_u^t \frac{1}{20\pi}(2-\cos s)\,\mathrm{d}s\Big\} = 1,$$
$$\eta^M = \max\{1,1\} = 1,$$
$$\begin{aligned}\eta_2^1 &= \inf_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a_1}(t,u) = \inf_{u\in[t-\omega,t]_{\mathbb{T}}} \exp\Big\{-\int_u^t \frac{1}{10\pi}(2+\sin s)\,\mathrm{d}s\Big\}\\ &= e^{-0.4} = 0.670320,\end{aligned}$$
$$\begin{aligned}\eta_2^2 &= \inf_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a_2}(t,u) = \inf_{u\in[t-\omega,t]_{\mathbb{T}}} \exp\Big\{-\int_u^t \frac{1}{20\pi}(2-\cos s)\,\mathrm{d}s\Big\}\\ &= e^{-0.2} = 0.818731,\end{aligned}$$
$$\eta^l = \min\{0.670320, 0.818731\} = 0.670320.$$

Then $r^M\eta^M = 5.516650$, $\delta = \frac{5.516650\times 1}{3.033244\times 0.670320} = 2.713226$, it is easy to verify that $\frac{1}{r^M\eta^M\omega} < \lambda < \frac{1}{\omega}$; that is, $\frac{1}{6.132488\pi} < \frac{1}{12} < \frac{1}{2\pi}$. Furthermore, $f_i^\infty < 2\pi$ holds for $i = 1,2$, so conditions (H5) and (H6) of Theorem 3.1 is satisfied. Choose $c_1 = \frac{1}{100000}, c_2 = \frac{1}{2}$, then $c_3 = \delta c_2 = 1.356613$.

For $c_2 \le |x|_1 \le \|x\| \le \delta c_2$, we obtain

$$\begin{aligned}\int_0^\omega |f(s,x,\Phi)|_0 \mathrm{d}s &= \int_0^\omega \frac{e^8}{2\pi}[x_1(t+\theta)+x_2(t)]^2 e^{-x_1(t-\theta)x_2(t)}|2+\sin u_1(t)|\mathrm{d}s\\ &\quad + \int_0^\omega \frac{e^8}{3\pi}[x_1(t)+x_2(t+\theta)]^2 e^{-x_1(t)x_2(t-\theta)}|2+\cos u_2(t)|\mathrm{d}s\\ &\ge \frac{2\omega}{3\pi}e^{8-\delta^2c_2^2}c_2^2\\ &> 24c_2\omega > \delta c_2\omega;\end{aligned}$$

that is, (H7) holds.

For $0 \le |x|_1 \le \|x\| \le c_1$,

$$\begin{aligned}\int_0^\omega |f(s,x,\Phi)|_0 \,\mathrm{d}s &= \int_0^\omega \frac{e^8}{2\pi}[x_1(t+\theta)+x_2(t)]^2 e^{-x_1(t-\theta)x_2(t)}|2+\sin u_1(t)|\mathrm{d}s\\ &\quad + \int_0^\omega \frac{e^8}{3\pi}[x_1(t)+x_2(t+\theta)]^2 e^{-x_1(t)x_2(t-\theta)}|2+\cos u_2(t)|\mathrm{d}s\\ &\le 6e^8c_1^2 < 0.0000006\\ &< \frac{c_1\omega}{r^M\eta^M} = 0.000006066488\pi,\end{aligned}$$

hence (H8) holds, it is obvious that (H1)-(H4) hold. By Theorem 3.1, (4.1) has at least three nonnegative 2π-periodic solutions. □

Example 4.2. When $\mathbb{T}=\mathbb{Z}$, the following system has at least three nonnegative 2-periodic solutions for $1/4<\lambda<1/2$:

$$\begin{gathered}\Delta x(n)=(1-e^{\sin n\pi-\ln\frac{\sqrt{2}}{2}})x(n+1)+\lambda\frac{18x^2(n)(3+\sin(x(n-1))+\cos(u(n)))}{e^{x(n)}},\\ \Delta u(n)=-(0.85-0.05\sin n\pi)u(n)+0.001x(n),\quad n\in\mathbb{Z}\end{gathered}\tag{4.2}$$

Proof. Corresponding to system (1.4), we have $a(n)=e^{\sin n\pi-\log\frac{\sqrt{2}}{2}}-1$, $b(n)=0.85-0.05\sin n\pi$, $g(n)=0.001x(n)$, $f(n,x,\Phi)=\frac{18x^2(n)(3+\sin(x(n-1))+\cos(u(n)))}{e^{x(n)}}$, $\omega=2$. So we obtain

$$\begin{aligned}e_{\ominus a}(n,n-\omega)&=\exp\Big\{\int_{n-\omega}^{n}\operatorname{Log}\Big(1-\frac{a(\tau)}{1+a(\tau)}\Big)\,\Delta\tau\Big\}\\ &=\exp\Big\{\int_{n-2}^{n}\log\Big(\frac{1}{1+a(\tau)}\Big)\,\Delta\tau\Big\}\\ &=\exp\Big\{-\int_{n-2}^{n}\Big(\sin\tau\pi-\log\frac{\sqrt{2}}{2}\Big)\,\Delta\tau\Big\}\\ &=\exp\Big\{-\sum_{\tau=n-2}^{n-1}\Big(\sin\tau\pi-\log\frac{\sqrt{2}}{2}\Big)\Big\}\\ &=\exp\Big\{\log\frac{1}{2}\Big\}=\frac{1}{2}.\end{aligned}$$

Then $r^M=r^l=\frac{1}{1-\frac{1}{2}}=2$. In a similar argument as the above process, it is not difficult to calculate that $\eta^M=1$, $\eta^l=\frac{1}{2}$. Then $r^M\eta^M=2,\delta=\frac{r^M\eta^M}{r^l\eta^l}=\frac{2}{2\times\frac{1}{2}}=2$. Furthermore, $f^\infty=0<2$ and $f^0=0<2\pi$ hold, so conditions (H5) and (H6) are satisfied. Choose $c_2=1$, then $c_3=\delta c_2=2$.

For $c_2\le\|x\|\le 2c_2$, we obtain

$$\begin{aligned}\int_0^\omega|f(s,x,\Phi)|_0\,\Delta s&=\int_0^2\Big|\frac{18x^2(s)(3+\sin(x(s-1))+\cos(u(s)))}{e^{x(s)}}\Big|\,\Delta s\\ &=\sum_{s=0}^{1}\Big|\frac{18x^2(s)(3+\sin(x(s-1))+\cos(u(s)))}{e^{x(s)}}\Big|\\ &\ge\sum_{s=0}^{1}\Big|\frac{18x^2(s)}{e^{x(s)}}\Big|\\ &\ge\sum_{s=0}^{1}\frac{18c_2^2}{e^{2c_2}}=\frac{36}{e^2}\\ &\ge 4=c_2\delta\omega;\end{aligned}$$

that is, (H7) holds.

In addition, it is obvious that (H1)-(H4) hold. By Theorem 3.2, (4.2) has at least three nonnegative 2-periodic solutions. □

Example 4.3. *When $\mathbb{T}=\mathbb{Z}$, the following system has at least three nonnegative 2-periodic solutions for $\frac{1}{4}<\lambda<\frac{1}{2}$:*

$$\begin{gathered}
\Delta x(n)=(1-e^{\cos n\pi-\ln\frac{\sqrt{2}}{2}})x(n+1)+\lambda\frac{18x^2(n)(3+\cos(x(n-1))+\sin^2(u(n)))}{1+x^2(n)},\\
\Delta u(n)=-(0.85-0.05\cos n\pi)u(n)+0.001x(n),\quad n\in\mathbb{Z}\,.
\end{gathered}\tag{4.3}$$

Proof. Corresponding to system (1.4), we have $a(n)=e^{\cos n\pi-\log\frac{\sqrt{2}}{2}}-1, b(n)=0.85-0.05\cos n\pi, g(n)=0.001x(n)$, $f(n,x,\Phi)=\frac{18x^2(n)(3+\cos(x(n-1))+\sin^2(u(n)))}{1+x^2(n)}$, $\omega=2$. So we obtain

$$\begin{aligned}
e_{\ominus a}(n,n-\omega)&=\exp\Big\{\int_{n-\omega}^{n}\operatorname{Log}\Big(1-\frac{a(\tau)}{1+a(\tau)}\Big)\,\Delta\tau\Big\}\\
&=\exp\Big\{\int_{n-2}^{n}\log\Big(\frac{1}{1+a(\tau)}\Big)\,\Delta\tau\Big\}\\
&=\exp\Big\{-\int_{n-2}^{n}\Big(\cos\tau\pi-\log\frac{\sqrt{2}}{2}\Big)\,\Delta\tau\Big\}\\
&=\exp\Big\{-\sum_{\tau=n-2}^{n-1}\Big(\cos\tau\pi-\log\frac{\sqrt{2}}{2}\Big)\Big\}\\
&=\exp\Big\{\log\frac{1}{2}\Big\}=\frac{1}{2}.
\end{aligned}$$

In a similar argument as Example 4.2, it is not difficult to get that $r^M\eta^M=\delta=2$. Furthermore, $f^\infty=0<2$ holds, so conditions (H5) and (H6) are satisfied.

Choose $c_2=1$, then $c_3=\delta c_2=2$. For $c_2\le\|x\|\le 2c_2$, we obtain

$$\begin{aligned}
\int_0^\omega|f(s,x,\Phi)|_0\,\Delta s&=\int_0^2\Big|\frac{18x^2(s)(3+\cos(x(s-1))+\sin^2(u(s)))}{1+x^2(s)}\Big|\,\Delta s\\
&=\sum_{s=0}^{1}\Big|\frac{18x^2(s)(3+\cos(x(s-1))+\sin^2(u(s)))}{1+x^2(s)}\Big|\\
&\ge\sum_{s=0}^{1}\Big|\frac{36}{5}\Big|\\
&\ge 8=2c_2\delta\omega;
\end{aligned}$$

that is, (H9) holds.

Choose $c_1=0.01$. For $0\le\|x\|\le c_1$, we have $\|f(n,x,\Phi)\|\le 90\|x\|^2\le\|x\|$; that is, (H10) holds. In addition, it is obvious that (H1)-(H4) hold. By Theorem 3.3, (4.3) has at least three nonnegative 2-periodic solutions. □

References

[1] R. P. Agarwal, M. Bohner, P. Rehak; *Half-linear dynamic equations, nonlinear analysis and applications*: to V. Lakshmikantham on his 80*th* birthday, vol. 1. Dordrecht: Kluwer Academic Publishers; (2003) 1-57.

[2] D.R. Anderson and J. Hoffacker; *Positive periodic time-scale solutions for functional dynamic equations*, Australian Journal of Mathematical Analysis and Applications 3:1:5 (2006) 1-14.

[3] D. R. Anderson and J. Hoffacker; *Higher-dimensional functional dynamic equations on periodic time scales*, Journal of Difference Equations and Applications, 14:1 (2008) 83-89.

[4] M. Bohner, A. Peterson; *Dynamic equations on time scales, an introduction with applications*, Boston, Birkhauser, 2001.

[5] M. Bohner, A. Peterson; *Advances in Dynamic Equations on Time Scales*, Birkhauser, Boston, 2003.

[6] F. Cao, Z. Han, S. Sun; *Existence of Periodic Solutions for p-Laplacian Equations on Time Scales*, Advances in Difference Equations, 2010 (2010), Article ID 584375, 13 pages.

[7] X. X. Chen, F. D. Chen; *Stable periodic solution of a discrete periodic Lotka-Volterra competition system with a feedback control*, Appl. Math. Comput. 181 (2006) 1446-1454.

[8] S. Hilger; *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Universität Würzburg, Ph.D. thesis (1988) 143 pp.

[9] E. R. Kaufmann, Y. N. Raffoul; *Periodic solutions for a neutral nonlinear dynamical equation on a time scale*, J. Math. Anal. Appl. 319 (2006) 315-325.

[10] R. W. Leggett, L. R. Williams; *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. 28 (1979) 673-688.

[11] W. T. Li, H. F. Huo; *Positive periodic solutions of delay difference equations and applications in population dynamics*, J. Comput. Appl. Math. 176 (2005) 357-369.

[12] Y. K. Li, L. F. Zhu; *Positive periodic solutions for a class of higher-dimensional state-dependent delay functional differential equations with feedback control*, Appl. Math. Comput. 159 (2004) 783-795.

[13] Y. K. Li, L. F. Zhu; *Existence of positive periodic solutions for difference equations with feedback control*, Appl. Math. Lett. 18 (2005) 61-67.

[14] Y. K. Li, L. H. Lu; *Positive periodic solutions of higher-dimensional nonlinear functional difference equations*, J. Math. Anal. Appl. 309 (2005) 284-293.

[15] Y. K. Li, X. Chen, L. Zhao; *Stability and existence of periodic solutions to delayed Cohen Grossberg BAM neural networks with impulses on time scales*, Neurocomputing 72 (2009) 1621-1630.

[16] Y. Li, H. Zhang; *Existence of periodic solutions for a periodic mutualism model on time scales*, J. Math. Anal. Appl. 343 (2008) 818-825.

[17] Y. Li, M. Hu; *Three Positive Periodic Solutions for a Class of Higher-Dimensional Functional Differential Equations with Impulses on Time Scales*, Advances in Difference Equations, 2009 (2009), Article ID 698463, 18 pages.

[18] C. Y. Liu, X. X. Chen; *Almost periodic sequence solutions of a discrete Lotka-Volterra competitive system with feedback control*, Nonlinear Anal.: RWA 10 (2009) 3152-3161.

[19] J. Liu, Y. K. Li, L. L. Zhao; *On a periodic predator-prey system with time delays on time scales*, Commun Nonlinear Sci Numer Simulat 14 (2009) 3432-3438.

[20] Z. Liu; *Existence of Periodic Solutions for a Delayed Ratio-Dependent Three-Species Predator-Prey Diffusion System on Time Scales*, Advances in Difference Equations, 2009 (2009), Article ID 141589, 15 pages.

[21] Z. J. Zeng, Z. C. Zhou; *Multiple positive periodic solutions for a class of state-dependent delay functional differential equations with feedback control*, Appl. Math. Comput. 197 (2008) 306-316.

[22] H. Zhang, Y. Li; *Existence of positive periodic solutions for functional differential equations with impulse effects on time scales*, Commun Nonlinear Sci Numer Simulat 14 (2009) 19-26.

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