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EXISTENCE OF SOLUTIONS TO SINGULAR ELLIPTIC EQUATIONS WITH CONVECTION TERMS VIA THE GALERKIN METHOD

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Abstract. In this article, we use the Galerkin method to show the existence of solutions for the following elliptic equation with convection term

$$-\Delta u = h(x,u) + \lambda g(x,\nabla u) \quad u(x) > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain, $\lambda \geq 0$ is a parameter, h has sublinear and singular terms, and g is a continuous function.

1. Introduction

In this article, we study the existence of solution the problem

$$\begin{gathered} -\Delta u = h(x,u) + \lambda g(x,\nabla u) \quad \text{in } \Omega, \\ u(x) > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$, λ is a positive parameter, the function h has sublinear and singular terms and g is a continuous function. By a solution to (1.1), we mean a u function if $u \in C^2(\Omega) \cap H_0^1(\Omega)$ which is positive and satisfies the equation in the classical sense in Ω.

Nonlinear singular boundary value problem arise in several physical situations such as fluid mechanics, pseudoplastics flow, chemical heterogeneous catalysts, non-Newtonian fluids, biological pattern formation, for more details about this subject, we cite the papers of Fulks and Maybe [15], Callegari and Nashman [7, 8] and the references therein.

When the nonlinearity has not a convection term, that is $g = 0$, there exist a lot of work related to this subject; see for example, Crandall, Rabinowitz and Tartar [9], Dávila and Montenegro [13], Choi and McKenna [6], Coclite and Palmieri [10], Cîrstea, Ghergu and Radulescu [11], Diaz, Morel and Oswald [12], Alves and Corrêa [2], Alves, Corrêa and Gonçalves [3]. The main tools used in the above papers are Sub and Supersolution, Fixed Point Theorems, Bifurcation Theory and Galerkin Method. When the nonlinearity has a convection term, that is $g \neq 0$, we would

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like to cite the papers of Ghergu and Radulescu [17, 18], Zhang [22], Giarrusso and Porru [16], Wood [21] and references therein. In all these papers, the main tools used are again Sub and Supersolution and Fixed Point Theorem. On the other hand, using variational technique, recently, de Figueiredo, Girardi and Matzeu [14] studied elliptic problems, where the nonlinearity depends of the gradient of the solution.

In general the above papers assume that h is monotone and g is homogeneous. Here, we show that the Galerkin method can be used to find solutions to (1.1) for nonlinearities where h is not monotone and g is not homogeneous. This way, we believe that our main results can be see as a complement of the study made in the above papers. Moreover, the method used in the present paper can be used to study singular elliptic equations in $\mathbb{R}^N$ and elliptic systems with convection terms, see examples 3 and 4 below.

The basic hypotheses on functions h and g are the following:

- (H1) The functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ are locally Hölder continuous.
- (H2) There exist constants $b > 0$, $0 < r_i < 1 (i = 1, 2, 3)$ with $r_1 < r_2$, and positive continuous functions $a_i : \overline{\Omega} \to \mathbb{R}$ $(i = 1, 2, 3)$ such that
$$b|\mu|^{r_1} \le h(x, \mu) \le a_1(x) + a_2(x)|\mu|^{r_2} + \frac{a_3(x)}{|\mu|^{r_3}}, \quad \forall (x, \mu) \in \Omega \times \mathbb{R}.$$
- (H3) There exist a constant $0 < r_4 < 1$, and continuous functions a_4 and a_5 such that
$$0 \le g(x, \eta) \le a_5(x) + a_4(x)|\eta|^{r_4}, \quad \forall (x, \eta) \in \Omega \times \mathbb{R}^N.$$

Note that the function h can have a singularity in μ. Thus, our approach consists of associating to problem (1.1) a family of elliptic problems without such singularities. Namely, for each $\epsilon > 0$, we consider the problem
$$\begin{gathered} -\Delta u = h(x, |u| + \epsilon) + \lambda g(x, \nabla u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

Now, problem (1.2) can be treated by Galerkin method. We will show that taking $\epsilon = 1/n$, it is possible to obtain a family of bounded solutions $\{u_n\}$ in $H_0^1(\Omega)$. By passing to the limit in (1.2) with $\epsilon = 1/n$. As $n \to \infty$, we will obtain a solution of (1.1), strictly positive by a result due to Ambrosetti, Brézis and Cerami [5].

The first result of this paper is the following.

Theorem 1.1. *If* (H1)-(H3) *hold, then* (1.1) *has a solution for all* $\lambda \ge 0$*.*

The second result is related with the following hypotheses on g:

- (H4) The conditions of (H2) hold with $r_2 < \min\{\frac{4}{N-2}, 1\}$ when $N \ge 3$ and
$$\frac{a_3}{\phi_1^{r_3}} \in L^p(\Omega) \quad \text{for some } p > \frac{N}{2}$$
where ϕ_1 is a positive eigenfunction corresponding to the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.
- (H5) There exists a local Hölder continuous function $g : [0, +\infty) \to [0, +\infty)$ such that
$$g(x, \eta) = g(|\eta|) \;\; \forall \eta \in \mathbb{R}^N$$

with $g(0) = 0$ and $g(t) > 0$ for all $t > 0$.

(H6) There exists $M^* > 0$ and $\beta \in (0, \frac{2}{N})$ such that

$$\sup_{t\in(0,M)} \frac{g(t)}{t^\beta} = \frac{g(M)}{M^\beta}, \quad \forall M \geq M^*.$$

Theorem 1.2. *Assume that* (H1),(H4), (H5), (H6) *hold. Then there exists* $\lambda^* > 0$ *such that* (1.1) *has a solution for all* $0 \leq \lambda \leq \lambda^*$.

In the sequel, we show some class of problems where our main theorems can be applied to get a positive solution. These problems were not considered in the above references.

Example 1.3. Theorem 1.1 establishes the existence of solutions for the problem

$$\begin{gathered} -\Delta u = h(u) + \lambda g(|\nabla u|) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \end{gathered}$$

for all $\lambda > 0$, where the functions g, f and h are of the type

$$g(t) = \begin{cases} t^{\beta_1}, & 0 \leq t \leq 2 \\ 2^{\beta_1} + \frac{(5^{\beta_2} - 2^{\beta_1})}{5 - 2^{\beta_1}}(t - 2^{\beta_1}), & 2 \leq t \leq 5 \\ t^{\beta_2}, & t \geq 5 \end{cases}$$

$$h(t) = t^{\alpha_1} + \sum_{i=2}^{m} (\cos(it) + 1) t^{\alpha_i} + \sum_{i=1}^{j} t^{-\gamma_i}$$

where $0 < \beta_1 < \beta_2 < 1$, $j, m \in \mathbb{N}$ and $\alpha_i, \gamma_i \in (0, 1)$.

Example 1.4. Theorem 1.2 can be used to prove the existence of solution for the problem

$$\begin{gathered} -\Delta u = h(u) + \lambda g(|\nabla u|) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

has a solution when $\lambda > 0$ is small enough, if g and h are of the form

$$g(t) = \sum_{i=1}^{j} t^{\beta_i} \quad h(t) = t^{\alpha_1} + \sum_{k=2}^{m} (\cos(it) + 1) t^{\alpha_i},$$

for some $j, m \in \mathbb{N}$, $0 < \beta_i$, for $i = 1, \dots, j$, and $\alpha_k \in (0, 1)$ for $k = 1, \dots, m$.

Remark 1.5. Note that we can not apply the Theorem 1.1 to show the existence of solution for the Example 2, because in this example we do not assume that $\beta_i \in (0, 1)$.

The method used in this paper can be applied to prove the existence of solutions for some class of elliptic systems or elliptic problems in whole $\mathbb{R}^N$; for example, Theorem 1.1 can be used to establish the existence of solutions in the following two examples.

Example 1.6.

$$\begin{gathered}-\Delta u=\frac{1}{u^{\alpha}}+h_1(x,u,v)+g_1(x,\nabla u,\nabla v)\quad\text{in }\Omega\\ -\Delta v=\frac{1}{v^{\beta}}+h_2(x,u,v)+g_2(x,\nabla u,\nabla v)\quad\text{in }\Omega\\ u(x),v(x)>0\quad\forall x\in\Omega\\ u=v=0\quad\text{on }\partial\Omega\end{gathered}$$

where h_1 and h_2 are positive functions, which have a subcritical growth at the variables u and v, that is, they are bounded from above by $|u|^{\alpha_1}|v|^{\alpha_2}$ with $\alpha_1,\alpha_2>0$ and $\alpha_1+\alpha_2<1$. Related to functions g_1 and g_2, we can assume that they are positive and bounded from above by $|\nabla u|^{\beta_1}|\nabla v|^{\beta_2}$ with $\beta_1,\beta_2>0$ and $\beta_1+\beta_2<1$.

Example 1.7.

$$-\Delta u=p(x)(h(x,u)+g(x,\nabla u)),\quad \mathbb{R}^N$$

where p is a positive weight satisfying some suitable conditions growth at infinite, while that functions h and g satisfy assumptions of the type (H1)–(H3). Moreover, in this class of problems we use the Sobolev space $D^{1,2}(\mathbb{R}^N)$.

To conclude this introduction, we would like to mention that the class of problems cited in examples 3 and 4 complete the study made in the papers [2], [3] and [18], in the following sense : In [2] and [3] the nonlinearities have not convection term, and in [18] the convection term is homogeneous, while in the present paper this hypothesis is not assumed.

2. Preliminary Results

In this section, we will present some results already known and that will be used in the next section. The lemma below is a consequence of Brouwer's Fixed Point Theorem and its proof can be found in Kesavan [20].

Lemma 2.1. *Let $F:\mathbb{R}^K\to\mathbb{R}^K$ a continuous function with $\langle F(x),x\rangle\geq 0$, for x satisfying $|x|=R>0$, where $\langle x,y\rangle$ is the usual inner product of $\mathbb{R}^K$. So, there exists $z_0\in\overline{B}_R(0)$ such that $F(z_0)=0$.*

Next, we state a result of sub and supersolution due to Ambrosetti, Brézis and Cerami [5]. Consider the following problem

$$\begin{gathered}-\Delta v=f(v),\quad\text{in }\Omega\\ v>0,\quad\text{in }\Omega\\ v(x)=0,\quad\text{on }\partial\Omega\end{gathered}\tag{2.1}$$

We say that $v_1\in C^2(\Omega)\cap C(\overline{\Omega})$ is a subsolution of (2.1) if

$$\begin{gathered}-\Delta v_1\leq f(v_1)\quad\text{in }\Omega\\ v_1>0\quad\text{in }\Omega\\ v_1(x)=0\quad\text{on }\partial\Omega\,.\end{gathered}\tag{2.2}$$

Similarly, $v_2\in C^2(\Omega)\cap C(\overline{\Omega})$ is a supersolution of (2.1) if

$$\begin{gathered}-\Delta v_2\geq f(v_2)\quad\text{in }\Omega\\ v_2>0\quad\text{in }\Omega\\ v_2(x)=0\quad\text{on }\partial\Omega\,.\end{gathered}\tag{2.3}$$

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ such that $t^{-1}f(t)$ is decreasing for $t > 0$. Let v_1 and v_2 satisfying* (2.2) *and* (2.3)*, respectively. So, $v_2 \geq v_1$ in Ω.*

3. Existence of solutions to (1.2)

The first lemma of this section is related with the regularity of the weak solutions to (1.2).

Lemma 3.1 (Regularity)**.** *Assume that* (H1)–(H3) *hold and let $u \in H_0^1(\Omega)$ be a weak solution of* (1.2)*. Then u belongs to $C^2(\Omega) \cap C^1(\overline{\Omega})$.*

Proof. Define $\Phi(x) = h(x, |u| + \epsilon) + \lambda g(x, \nabla u)$. Since $u \in H_0^1(\Omega)$, by (H2)–(H3), $\Phi \in L^{2/r}(\Omega)$, where $r = \max\{r_i, i = 2, 4\}$. Thus, by Agmon [1, Theorem 8.2], all solution of

$$
\begin{gathered}
-\Delta u(x) = \Phi(x) \quad \text{in } \Omega \\
u(x) = 0 \quad \text{on } \partial\Omega
\end{gathered}
$$

belong to $W^{2,s_1}(\Omega)$, where $s_1 = 2/r$ and therefore $u, \nabla u \in L^{s_1}(\Omega)$ and $\Phi \in L^{s_1/r}(\Omega)$. Using again [1], we obtain $u \in W^{2,s_2}(\Omega)$ with $s_2 = 2/r^2$. Since $r \in (0, 1)$, repeating this argument k times, such that $s_k = 2/r^k > N/2$, it follows from the Sobolev embedding that u in $C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Thus, by Schauder regularity theorem and (H1), we conclude that $u \in C^2(\Omega)$. □

To obtain a solution to (1.2), we will apply Galerkin method. Note that, under assumptions (H2)–(H3) and the Maximum Principle, any classical solution u_ϵ of (1.2) is positive; that is, $u_\epsilon(x) > 0$ for all $x \in \Omega$.

Theorem 3.2. *Assume* (H1)–(H3)*. Then* (1.2) *has a solution u_ϵ in $C^2(\Omega) \cap C^1(\overline{\Omega})$.*

Proof. Let $\Sigma = \{e_1, \dots, e_m, \dots\}$ be a orthonormal bases of the Hilbert space $H_0^1(\Omega)$. For each $m \in \mathbb{N}$, define the subspace $V_m = [e_1, \dots, e_m]$; that is, V_m is a $m-$dimensional space generated by the orthonormal set $\{e_1, \dots, e_m\}$. It is well known that $(V_m, \|\cdot\|)$ and $(\mathbb{R}^m, |\cdot|)$ are isomorphic by the natural linear transformation $T : V_m \to \mathbb{R}^m$ given by

$$
v = \sum_{i=1}^{m} \xi_i e_i \to T(v) = \xi = (\xi_1, \dots, \xi_m)
$$

which also satisfies

$$
\|v\| = |T(v)| = |\xi|
$$

where $|\cdot|$ and $\|\cdot\|$ denote the usual norms in $\mathbb{R}^m$ and $H_0^1(\Omega)$, respectively. In the next, we will use the identification

$$
\xi \mapsto \sum_{i=1}^{m} \xi_i e_i = v.
$$

Considering the function $F = (F_1, \dots, F_m) : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$
F_i(\xi) = \int_\Omega \nabla v \nabla e_i dx - \int_\Omega \big(h(x, |v| + \epsilon) + \lambda g(x, \nabla v)\big) e_i dx,
$$

we have

$$\begin{aligned}\langle F(\xi),\xi\rangle &= \int_\Omega |\nabla v|^2 dx - \int_\Omega (h(x,|v|+\epsilon)v + \lambda g(x,\nabla v)v)\,dx\\ &\geq \|v\|^2 - |a_1|_2|v|_2 - |a_2|_{\frac{2}{1-r_2}}|v|_2^{r_2+1} - \epsilon^{r_2}|a_2|_2|v|_2\\ &\quad - \frac{1}{\epsilon^{r_3}}|a_3|_2|v|_2 - \lambda|a_5|_2|v|_2 - \lambda\Big(\int_\Omega a_4^2|\nabla v|^{2r_4}dx\Big)^{1/2}|v|_2 \\ &\geq \|v\|^2 - c_1|a_1|_2\|v\| - c_2|a_2|_{\frac{2}{1-r_2}}\|v\|^{r_2+1} - c_3|a_2|_2\|v\|\\ &\quad - c_4|a_3|_2\|v\| - c_6\lambda|a_5|_2\|v\| - \lambda c_5|a_4|_{\frac{2}{1-r_4}}\|v\|^{r_4+1},\end{aligned} \tag{3.1}$$

where c_3, c_4 depend of ϵ, and c_j are independent of m for $j = 1,\dots,6$. Therefore,

$$\begin{aligned}\langle F(\xi),\xi\rangle &\geq |\xi|^2 - c_1|a_1|_2|\xi| - c_2|a_2|_{\frac{2}{1-r_2}}|\xi|^{r_2+1} - c_3|a_2|_2|\xi| - c_4|a_3|_2|\xi|\\ &\quad - \lambda c_6|a_5|_2|\xi| - \lambda c_5|a_4|_{\frac{2}{1-r_4}}|\xi|^{r_4+1},\end{aligned}$$

from where follows that there exist $\rho, r > 0$, which are independent of m, such that

$$\langle F(\xi),\xi\rangle \geq r > 0 \quad \text{on } |\xi| = \rho.$$

Since F is a continuous functions, from Lemma 2.1, for each $m \in \mathbb{N}$ there exists $\xi_m \in \mathbb{R}^m$ satisfying

$$F(\xi_m) = 0 \quad |\xi_m| \leq \rho. \tag{3.2}$$

Next, we fix $v_m \in H_0^1(\Omega)$ such that $T(v_m) = \xi_m$. Hence, $\|v_m\| \leq \rho$ for all $n \in \mathbb{N}$ and

$$\int_\Omega \nabla v_m \nabla\omega dx = \int_\Omega \big(h(x,|v_m|+\epsilon)\omega + \lambda g(x,\nabla v_m)\omega\big)dx \quad \forall \omega \in V_m.$$

Moreover, passing to a subsequence if necessary, we can assume that there exists $v \in H_0^1(\Omega)$ such that

$$v_m \rightharpoonup v \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad v_m(x) \to v(x) \quad \text{a.e in } \Omega.$$

The next claim is a key point to conclude the proof of the theorem.

Claim 3.3. *The sequence $\{v_m\}$ is strongly convergent to v in $H_0^1(\Omega)$.*

Assuming for a moment the claim, and recalling that for $\omega \in V_k$ and $m \geq k$, we have the equality

$$\int_\Omega \nabla v_m \nabla\omega dx = \int_\Omega \big(h(x,|v_m|+\epsilon)\omega + \lambda g(x,\nabla v_m)\omega\big)dx. \tag{3.3}$$

It follows that

$$\int_\Omega \nabla v \nabla\omega dx = \int_\Omega (h(x,|v|+\epsilon)\omega + \lambda g(x,\nabla v)\omega)\,dx. \tag{3.4}$$

Since, for each $\phi \in H_0^1(\Omega)$, there exist $\{\gamma_i\} \subset \mathbb{R}$ satisfying $\phi = \sum_{i=1}^\infty \gamma_i e_i$, the sequence

$$\phi_k = \sum_{i=1}^k \gamma_i e_i \in V_k,$$

is strongly convergent to ϕ in $H_0^1(\Omega)$. Putting $w = \phi_k$ in (3.4) and taking the limit as $k \to \infty$ we obtain

$$\int_\Omega \nabla v \nabla\phi dx = \int_\Omega (h(x,|v|+\epsilon)\phi + \lambda g(x,\nabla v)\phi)\,dx. \tag{3.5}$$

Thus v is a weak solution of (1.2), and by Lemma 3.1, $v \in C^2(\Omega)\cap C^1(\overline{\Omega})$. Therefore, (1.2) has a classical solution and the proof of Theorem 3.2 is complete. $\square$

Proof of Claim 3.3. Using the weak convergence and the Theorem of the Dominated Convergence, it follows that

$$\int_\Omega \nabla v_m \nabla\omega \to \int_\Omega \nabla v \nabla\omega, \tag{3.6}$$

$$\int_\Omega h(x,|v_m|+\epsilon)\omega \to \int_\Omega h(x,|v|+\epsilon)\omega, \tag{3.7}$$

$$\int_\Omega h(x,|v_m|+\epsilon)(v_m - v) \to 0, \tag{3.8}$$

$$\int_\Omega h(x,|v|+\epsilon)(v_m - v) \to 0. \tag{3.9}$$

From now on, for each $m \in \mathbb{N}$, we consider the function $G_m(x) := g(x, \nabla v_m(x))$. From (H3),

$$|G_m|_{L^{\frac{2N}{(N+2)r_4}}(\Omega)} \le |a_5|_{L^{\frac{2N}{(N+2)r_4}}(\Omega)} + \Big(\int_\Omega a_4(x)^{\frac{2N}{(N+2)r_4}} |\nabla v_m|^{\frac{2N}{N+2}} dx\Big)^{\frac{(N+2)r_4}{2N}}. \tag{3.10}$$

Using (3.2) and Hölder's inequality with exponents $q = \frac{N+2}{N}$ and $p = \frac{N+2}{2}$, we get the estimate

$$|G_m|_{L^{\frac{2N}{(N+2)r_4}}(\Omega)} \le |a_5|_{L^{\frac{2N}{(N+2)r_4}}(\Omega)} + |a_4|_{L^{\frac{N}{r_4}}(\Omega)} |\nabla v_m|^{r_4}_{L^2(\Omega)} \le c_1 + c_2\rho^{r_4}. \tag{3.11}$$

Since $L^{\frac{2N}{(N+2)r_4}}(\Omega)$ is reflexive, up to subsequence, there exists $G \in L^{\frac{2N}{(N+2)r_4}}(\Omega)$ such that $G_m \rightharpoonup G$ in $L^{\frac{2N}{(N+2)r_4}}(\Omega)$; that is,

$$\int_\Omega G_m\varphi dx \to \int_\Omega G\varphi dx \quad \forall \varphi \in L^\theta(\Omega) \tag{3.12}$$

where $\frac{1}{\theta} + \frac{(N+2)r_4}{2N} = 1$. Recalling that the embedding $H_0^1(\Omega) \hookrightarrow L^\theta(\Omega)$ is continuous, (3.3) leads to

$$\int_\Omega |\nabla v|^2 dx - \int_\Omega h(x,|v|+\epsilon)v dx - \lambda \int_\Omega G(x) v dx = 0.$$

On the other hand,

$$\|v_m - v\|^2 = \|v_m\|^2 - \langle v_m, v\rangle + \langle v, v_m - v\rangle = \|v_m\|^2 - \|v\|^2 + o_m(1),$$
$$\int_\Omega h(x,|v_m|+\epsilon)v = \int_\Omega h(x,|v|+\epsilon)v + o_m(1),$$
$$\int_\Omega G(x)v_m dx = \int_\Omega G(x) v dx + o_m(1).$$

From this

$$\|v_m - v\|^2 = \int_\Omega (h(x,|v_m|+\epsilon) - h(x,|v|+\epsilon))v_m dx + \lambda\int_\Omega (G_m(x) - G(x))v_m dx + o_m(1),$$

or equivalently,

$$
\begin{aligned}
&\|v_m - v\|^2 \\
&= \int_\Omega (h(x,|v_m|+\epsilon) - h(x,|v|+\epsilon))(v_m - v)dx + \lambda \int_\Omega (G_m(x) - G(x))(v_m - v)dx \\
&\quad + \int_\Omega (h(x,|v_m|+\epsilon) - h(x,|v|+\epsilon))v dx + \lambda \int_\Omega (G_m(x) - G(x))v dx + o_m(1).
\end{aligned}
$$

Using the weak convergence $v_m \rightharpoonup v$ in $H_0^1(\Omega)$, (3.7)-(3.9) and (3.12), we derive that

$$
\|v_m - v\|^2 \to 0.
$$

This implies that $v_m \to v$ in $H_0^1(\Omega)$, and the proof of Claim 3.3 is complete. □

4. Proof of Theorem 1.1

In this section, we will show firstly the existence of solutions and then its regularity.

Existence. Taking $\epsilon_n = 1/n$ and $u_{\epsilon_n} = u_n$, it follows that

$$
\begin{gathered}
-\Delta u_n = h(x, u_n + 1/n) + \lambda g(x, \nabla u_n) \quad \text{in } \Omega \\
u_n > 0 \quad \text{in } \Omega \\
u_n(x) = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.1}
$$

By the definition of weak solution, taking $\phi = u_n$ as a test function in (4.1), we have

$$
\begin{aligned}
\|u_n\|^2 &\le \Big(\int_\Omega a_1^2 dx\Big)^{1/2}\Big(\int_\Omega u_n^2 dx\Big)^{1/2} + \Big(\int_\Omega a_2^{\frac{2}{1-r_2}} dx\Big)^{\frac{1-r_2}{2}}\Big(\int_\Omega u_n^2 dx\Big)^{\frac{r_2+1}{2}} \\
&\quad + \Big(\int_\Omega a_2^2 dx\Big)^{1/2}\Big(\int_\Omega u_n^2 dx\Big)^{1/2} + \Big(\int_\Omega a_3^{\frac{2}{1+r_3}} dx\Big)^{\frac{1+r_3}{2}}\Big(\int_\Omega u_n^2 dx\Big)^{\frac{1-r_3}{2}} \\
&\quad + \lambda\Big(\int_\Omega a_5^2 dx\Big)^{1/2}\Big(\int_\Omega u_n^2 dx\Big)^{1/2} + \lambda\Big(\int_\Omega a_4^2 |\nabla u_n|^{2r_4} dx\Big)^{1/2}\Big(\int_\Omega u_n^2 dx\Big)^{1/2} \\
&\le c_1 |a_1|_2 \|u_n\| + c_2 |a_2|_{\frac{2}{1-r_2}} \|u_n\|^{r_2+1} + c_3 |a_2|_2 \|u_n\| \\
&\quad + c_4 |a_3|_{\frac{2}{1+r_3}} \|u_n\|^{1-r_3} + \lambda c_6 |a_5|_2 \|u_n\| + \lambda c_5 |a_4|_{\frac{2}{1-r_4}} \|u_n\|^{r_4+1} \\
&\le \widetilde{C}(\|u_n\| + \|u_n\|^{r_2+1} + \|u_n\|^{1-r_3} + \|u_n\|^{r_4+1}).
\end{aligned} \tag{4.2}
$$

So there exists $K > 0$ such that $\|u_n\| \le K$. Up to subsequence, if necessary, we can assume that

$$
u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega), \quad u_n(x) \to u(x) \quad \text{a.e. in } \Omega.
$$

Now, note that by (H2), u_n is a supersolution of

$$
\begin{gathered}
-\Delta v = b\, v^{r_1}, \quad \text{in } \Omega \\
v > 0 \quad \text{in } \Omega \\
v = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.3}
$$

Since this equation has a unique solution in $C^2(\Omega) \cap C(\overline{\Omega})$, which we will denote by Ψ, from Theorem 2.2,

$$
u_n(x) \ge \Psi(x) \quad \forall x \in \Omega,\ \forall n \in \mathbb{N}
$$

and thus

$$u(x) \geq \Psi(x) \quad \text{a.e in } \Omega \tag{4.4}$$

from where it follows that $u(x) > 0$ a.e in Ω.

Claim 4.1. *For each $\omega \in H_0^1(\Omega)$, we have*

$$\int_\Omega h(x, u_n + 1/n)\omega dx \to \int_\Omega h(x, u)\omega dx, \tag{4.5}$$

$$\int_\Omega h(x, u_n + 1/n)(u_n - u) dx \to 0, \tag{4.6}$$

$$\int_\Omega h(x, u)(u_n - u) dx \to 0. \tag{4.7}$$

Proof. From (H2) and (4.4), we have

$$|h(x, u_n + 1/n)w| \leq c_1|w| + c_2|w||u_n|^{r_2} + c_3\frac{|w|}{\Psi^{r_3}},$$

$$|h(x, u_n + 1/n)(u_n - u)| \leq c_1|u_n - u| + c_2|u_n - u||u_n|^{r_2} + c_3\frac{|u_n - u|}{\Psi^{r_3}},$$

$$|h(x, u)(u_n - u)| \leq c_1|u_n - u| + c_2|u_n - u||u|^{r_2} + c_3\frac{|u_n - u|}{\Psi^{r_3}}$$

for some positive constants c_i for $i = 1, 2, 3$. From Ambrosetti, Brézis and Cerami [5], the function Ψ satisfies the following inequality

$$\Psi(x) \geq C\phi_1(x) \quad \forall x \in \Omega,$$

where C is a positive constant and ϕ_1 is a positive eigenfunction related to the first eingenvalue of $(-\Delta, H_0^1(\Omega))$. This way,

$$|h(x, u_n + 1/n)w| \leq c_1|w| + c_2|w||u_n|^{r_2} + c_4\frac{|w|}{\phi_1^{r_3}}, \tag{4.8}$$

$$|h(x, u_n + 1/n)(u_n - u)| \leq c_1|u_n - u| + c_2|u_n - u||u_n|^{r_2} + c_4\frac{|u_n - u|}{\phi_1^{r_3}}, \tag{4.9}$$

$$|h(x, u)(u_n - u)| \leq c_1|u_n - u| + c_2|u_n - u||u|^{r_2} + c_4\frac{|u_n - u|}{\phi_1^{r_3}}. \tag{4.10}$$

From the Hardy-Sobolev inequality found in [4] (see also [19]), we have

$$\frac{|w|}{\phi_1^{r_3}} \in L^1(\Omega), quad\frac{|u_n - u|}{\phi_1^{r_3}} \to 0 \quad \text{in } L^1(\Omega). \tag{4.11}$$

Moreover, by using the compact Sobolev embedding, we derive that

$$\lim_{n\to+\infty} \int_\Omega |u_n - u||u|^{r_2} = \lim_{n\to+\infty} \int_\Omega |u_n - u||u_n|^{r_2} = \lim_{n\to+\infty} \int_\Omega |u_n - u| = 0. \tag{4.12}$$

These limits together with Theorem of the Dominated Convergence imply (4.5)-(4.7). □

Now, the Claim 4.1 combined with arguments explored in the proof of Claim 3.3 imply $u_n \to u$ in $H_0^1(\Omega)$; hence,

$$\int_\Omega \nabla u \nabla \phi dx = \int_\Omega \big(h(x, u)\phi + \lambda g(x, \nabla v)\phi\big) dx \quad \forall \phi \in H_0^1(\Omega).$$

from where it follows that u is a weak solution for (1.1).

Remark 4.2 (Regularity of the solution)**.** From (4.4), it follows that $1/u$ belongs to $L^\infty_{\rm loc}(\Omega)$. Thus, repeating the same arguments explored in the proof of Lemma 3.1, it follows that $u \in C^2(\Omega) \cap H^1_0(\Omega)$.

To conclude this study of regularity, we would like to mention the following fact: If the function a_3 given by (H2) satisfies a hypothesis of the type

$$\frac{a_3}{\phi_1^{r_3}} \in L^\infty(\Omega),$$

using again Regularity Theory, it follows that $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

5. Proof of Theorem 1.2

To prove Theorem 1.2, we will work with an auxiliary problem. For each $M > M^*$, we define the function

$$g_M(t) = \begin{cases} g(t) & \text{if } 0 \le t \le M \\ \frac{g(M)}{M^\beta} t^\beta, & t \ge M \end{cases}$$

and the problem

$$\begin{gathered} -\Delta u = h(x, |u|) + \lambda g_M(|\nabla u|) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.1}$$

Note that if $u_{\lambda,M}$ is a solution of (5.1) satisfying

$$|\nabla u_{\lambda,M}(x)| \le M \quad \forall x \in \Omega, \tag{5.2}$$

then $u_{\lambda,M}$ is a solution for (1.1). From now on, our goal is to prove that there exist $M > 0$ and $\lambda^* = \lambda^*(M) > 0$ such that $u_{\lambda,M}$ satisfies (5.2) for $\lambda \le \lambda^*$.

Since h and g_M satisfy the hypotheses of Theorem 1.1, for all $\lambda > 0$ and $M > M^*$, there exists a solution $u_{\lambda,M}$ of (5.1) satisfying

$$u_{\lambda,M}(x) \ge \Psi(x) \quad \forall x \in \overline{\Omega}.$$

Taking $\lambda^* = \frac{M^\beta}{g(M)}$, (H6) leads to

$$\lambda g_M(|\nabla u_{\lambda,M}(x)|) \le |\nabla u_{\lambda,M}(x)|^\beta \quad \forall x \in \Omega,\ \forall \lambda \le \lambda^*;$$

therefore, there exists $C > 0$ such that

$$\|u_{\lambda,M}\| \le C \quad \forall M > M^*,\ \forall \lambda \le \lambda^* \tag{5.3}$$

where C is independent of λ and M.

Claim 5.1. *There exists $k > 0$ such that $\|u_{\lambda,M}\|_{C^1(\overline{\Omega})} \le k$ for all $M \ge M^*$ and $\lambda \le \lambda^*$.*

Proof. By considering

$$\Phi_{\lambda,M}(x) = h(x, u_{\lambda,M}(x)) + \lambda g_M(|\nabla u_{\lambda,m}(x)|)$$

we have

$$|\Phi_{\lambda,M}(x)| \le a_1(x) + a_2(x)|u_{\lambda,M}(x)|^{r_2} + c_7 \frac{a_3(x)}{\phi_1^{r_3}(x)} + |\nabla u_{\lambda,M}(x)|^\beta \quad \forall \lambda \le \lambda^*.$$

From (H4), (H6) and (5.3), it follows that $\Phi_{\lambda,M} \in L^q(\Omega)$ for $q > \frac{N}{2}$, $q \approx \frac{N}{2}$ and

$$|\Phi_{\lambda,M}|_q \le C_1, \quad \forall \lambda \le \lambda^*,\ M \ge M^*. \tag{5.4}$$

Since $u_{\lambda,M}$ is a solution of the problem

$$-\Delta u(x) = \Phi_{\lambda,M}(x) \quad \text{in } \Omega$$
$$u(x) = 0 \quad \text{on } \partial\Omega$$

by Agmon [1, Theorem 8.2], $u_{\lambda,M} \in W^{2,q}(\Omega)$ and there exists $C_3 > 0$, which is independent of $u_{\lambda,M}$, such that

$$\|u_{\lambda,M}\|_{W^{2,q}(\Omega)} \le C_3 |\Phi_{\lambda,M}|_q.$$

Using the continuous embedding $W^{2,q}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, we obtain

$$\|u_{\lambda,M}\|_{C^1(\overline{\Omega})} \le C_4 |\Phi_{\lambda,M}|_q. \tag{5.5}$$

From (5.4) and (5.5), there exists $k > 0$ such that

$$\|u_{\lambda,M}\|_{C^1(\overline{\Omega})} \le k \quad \forall \lambda \le \lambda^*,\ M \ge M^*.$$

This completes the proof of the claim. □

From Claim 5.1, if $M \ge \max\{M^*, k\}$ and $\lambda \le \lambda^*$, we have

$$|\nabla u_{\lambda,M}(x)| \le M \quad \forall x \in \Omega$$

from where it follows that

$$g_M(|\nabla u_{\lambda,M}(x)|) = g(|\nabla u_{\lambda,M}(x)|) \quad \forall x \in \Omega.$$

Therefore, for $\lambda \le \lambda^*$ and $M \ge \max\{M^*, k\}$ the function $u_{\lambda,M}$ is a solution for (1.1), and the proof of Theorem 1.2 is complete.

References

[1] S. Agmon; *The L^p Approach to the Dirichlet Problem*, Ann. Scuola Norm. Sup. Pisa, 13 (1959), 405-448.

[2] C. O. Alves, F. J. S. A. Corrêa; *On the existence of positive solutions for a class of singular systems involving quasilinear operators*, Appl. Math. and Computation 185 (2007) 727-736.

[3] C. O. Alves, F. J. S. A. Corrêa, J. V. A. Gonçalves; *Existence of solutions for some classes of singular Hamiltonian systems*, Adv. Nonlinear Stud. 5 (2005) 265-278.

[4] A. Alvino, V. Ferone, G. Trombetti; *On the best constant in a HardySobolev inequality*, Applicable Analysis 85 (2006), 171-180.

[5] A. Ambrosetti, H. Brézis, G. Cerami; *Combined Effects of Concave and Convex Nonlinear in Some Elliptic Problems*, Journal of Functional Analysis, 122 (1994), 519- 543.

[6] Y. S. Choi, P. J. McKenna; *A singular Gierer-Meinhardt system of elliptic equations,* Ann. Inst. Henri Poincaré Analyse Non linéare 4 (2000) 503-522.

[7] A. Callegari, A. Nashman; *Some singular nonlinear equation arising in boundary layer theory*, J. Math. Anal. Appl. 64 (1978) 96-105.

[8] A. Callegari, A. Nashman; *A nonlinear singular boundary value problem in the theory of pseudo-plastic fluids*, SIAM J. Appl. Math. 38 (1980) 275-281

[9] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with singular nonlinearity*, Comm. Partial Diff. Equat. 2 (1977) 193-222.

[10] M. M. Coclite, G. Palmieri; *On a singular nonlinear Dirichlet problem*, Comm. Partial Diff. Equat. 14 (1989) 1315-1327.

[11] F. Cîrstea, M. Ghergu, V. Radulescu; *Combined effects of asymptotically linear and singular nonlinearities in bifurcation problem of Lane-Emden-Fowler type*, J. Math. Pures Appl. 84 (2005) 493-508.

[12] J. I. Diaz, J. M. Morel, L. Oswald; *An elliptic equation with singularity nonlinearity*, Comm. Partial Diff. Equat. 12 (1987) 1333-1344.

[13] J. Dávila, M. Montenegro; *Positive versus free boundary solutions to a singular elliptic equation*. J. Anal. Math. 90 (2003), 303–335.

[14] D. G. de Figueiredo, M. Girardi, M. Matzeu; *Semilinear elliptic equations with dependence on the gradient via mountain pass techniques*, Diff. and Integral Eqns. 17 (2004), 119-126.

[15] W. Fulks, J. S. Maybee; *A singular nonlinear equation*, Osaka Math. J. 12 (1960), 1-19.
[16] E. Giarruso, G. Porru; *Problems for elliptic singular equation with a gradient term*, Nonlinear Analysis 65 (2006) 107-128.
[17] M. Ghergu, V. Radulescu; *On a class of sublinear singular elliptic problems with convection term*, J. Math. Anal. Appl. 311 (2005) 635-646.
[18] M. Ghergu, V. Radulescu; *Ground state solutions for the singular Lane-Emden-Fowler equation with sublinear convection term*, J. Math. Anal. Appl. 333 (2007) 265273.
[19] O. Kavian; *Inegalité de Hardy-Sobolev et applications*, Theése de Doctorate de 3eme cycle, Université de Paris VI(1978).
[20] S. Kesavan; *Topics in functional analysis and applications*, John Wiley and Sons, (1989).
[21] A. V. Lair, A. W. Wood; *Large solutions of semilinear elliptic equations with nonlinear gradient terms*, Int. J. Math. Sci., 22 (1999), 869-883.
[22] Z. Zhang; *Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problem with a convection term*, Nonlinear Analysis 8 (1996) 957-961.

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