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POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN

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Abstract. In this article, we study a class of boundary value problems with p-Laplacian. By using a "Green-like" functional and applying the fixed point index theory, we obtain eigenvalue criteria for the existence of positive solutions. Several explicit conditions are derived as consequences, and further results are established for the multiplicity and nonexistence of positive solutions. Extensions are also given to partial differential BVPs with p-Laplacian defined on annular domains.

1. Introduction

In this article, we study the following boundary value problem (BVP) that includes the equation with p-Laplacian

$$-(\phi(q(t)u'))' = w(t)f(t,u), \quad 0 < t < 1, \tag{1.1}$$

and the boundary condition (BC)

$$(qu')(0) = 0, \quad u(1) + a(qu')(1) = 0, \tag{1.2}$$

where $\phi(u) = |u|^{p-1}u$ with $p > 0$, $a > 0$, $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $q \in L[0,1]$ with $q(t) \geq \delta > 0$ on $[0,1]$, and $w \in L[0,1]$ with $w(t) \geq 0$ a.e. on $[0,1]$, and $\int_0^1 w(t)dt > 0$.

BVPs with p-Laplacian have been investigated for decades. Results are obtained for the existence of positive solutions for different BCs. To name a few, see [1, 12] for Dirichlet BCs, [6, 31] for periodic BCs, and [32] for the general separated BCs. For the work on m-point p-Laplacian BVPs, see [10, 11, 13, 28] and the references therein. As a special case with $p = 1$, the BVPs consisting of (1.1) and various BCs have been extensively studied. We refer to the reader [2, 3, 8, 15, 19, 20, 21, 22, 24, 25, 26, 30] and references therein.

Among various criteria for the existence of positive solutions, some were established using the first eigenvalue of an associated Sturm-Liouville problem (SLP), see, for example [8, 16, 23, 24, 26, 29]. Such eigenvalue criteria are usually sharper than criteria obtained in some other ways especially when they involve the behavior of f as u near 0 and ∞. Therefore, a natural question arises: Are there parallel eigenvalue criteria for the p-Laplacian BVP (1.1), (1.2) using the first eigenvalue of

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an associated half-linear SLP? To the best knowledge of the authors, no answers can be found in the literature although the spectral theory for half-linear SLPs has been well developed, see [4, 7, 17, 18, 27]. The main difficulties for the extension lie in the facts that no Green's functions can be found for equations with p-Laplacian since the solutions of half-linear equations do not form a linear space and the important Lagrange bracket property for linear SLPs is not satisfied by the half-linear SLPs.

In this paper, by constructing a "Green-like" functional and applying a different fixed point index theory, we obtain eigenvalue criteria for the p-Laplacian BVP (1.1), (1.2). More specifically, we show that BVP (1.1), (1.2) has at least one positive solution if the first eigenvalue of an associated half-linear SLP satisfies certain relations with the behavior of the function f as u near 0 and ∞. Some explicit conditions are derived as consequences, and further results are also given for the multiplicity and nonexistence of positive solutions. Our work is new and improves most existing results on BVPs with p-Laplacian when restricted to problem (1.1), (1.2).

Finally, we extend our results to partial differential BVPs with p-Laplacian on annular domains and hence obtain criteria for the existence, multiplicity, and nonexistence of positive radial solutions.

This paper is organized as follows: after this introduction, we state our main results in Section 2. The proofs are given in Section 3. Extensions to p-Laplacian partial differential equations are given in Section 4. Several examples are presented in Section 5 as applications.

2. Main Results

For the function f given in (1.1), define

$$\begin{aligned} f_0 &= \liminf_{u\to 0^+} \min_{t\in[0,1]} f(t,u)/u^p, \quad & f^0 &= \limsup_{u\to 0^+} \max_{t\in[0,1]} f(t,u)/u^p, \\ f_\infty &= \liminf_{u\to \infty} \min_{t\in[0,1]} f(t,u)/u^p, \quad & f^\infty &= \limsup_{u\to \infty} \max_{t\in[0,1]} f(t,u)/u^p. \end{aligned} \tag{2.1}$$

Consider the half-linear SLP consisting of the equation

$$-\left(\phi(q(t)u')\right)' = \lambda w(t)\phi(u),\ 0<t<1, \tag{2.2}$$

and BC (1.2). SLP (2.2), (1.2) is called the SLP associated with BVP (1.1), (1.2). It is well known that SLP (2.2), (1.2) has infinite number of real eigenvalues $\{\lambda_n\}_{n=0}^\infty$ satisfying

$$-\infty < \lambda_0 < \lambda_1 < \dots < \lambda_n < \cdots, \quad \text{and } \lambda_n \to \infty;$$

and the eigenfunction v_n associated with λ_n has exactly n zeros in $(0,1)$. We refer to the reader Binding and Drábek [4] and Kong and Kong [17] for the details. Moreover, we have the following result.

Lemma 2.1. *The first eigenvalue λ_0 of SLP (2.2), (1.2) is positive.*

Our major result below is on the existence of positive solutions of BVP (1.1), (1.2) using the relationships among λ_0, f_0, and f_∞.

Theorem 2.2. *Let λ_0 be the first eigenvalue of SLP (2.2), (1.2). Then BVP (1.1), (1.2) has at least one positive solution if either $f^0 < \lambda_0 < f_\infty$ or $f^\infty < \lambda_0 < f_0$.*

Let

$$q^* = \sup_{t\in[0,1]} 1/q(t), \quad \alpha = a/(a+q^*), \quad \beta = \phi^{-1}\Big(\int_0^1 w(\tau)d\tau\Big), \tag{2.3}$$

where ϕ^{-1} is the inverse function of ϕ. It is easy to see that $0 < \alpha < 1$. The following corollary, which gives explicit conditions without using λ_0, follows directly from Theorem 2.2.

Corollary 2.3. *BVP* (1.1), (1.2) *has at least one positive solution if either of the following holds:*

(a) $f^0 < ((a+q^*)\beta)^{-p}$ *and* $f_\infty > (a\alpha\beta)^{-p}$*;*
(b) $f_0 > (a\alpha\beta)^{-p}$ *and* $f^\infty < ((a+q^*)\beta)^{-p}$.

Next, we derive criteria for the existence of positive solutions based on the behavior of $f(t,u)$ for u in two disjoint closed intervals. Below we use the notation $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Theorem 2.4. *Assume there exist* $0 < l_1 < l_2$ *(respectively,* $0 < l_2 < l_1$*), such that*

$$f(t,u) \le l_1^p((a+q^*)\beta)^{-p} \quad \text{for all } (t,u) \in [0,1]\times[\alpha l_1, l_1], \tag{2.4}$$

$$f(t,u) \ge l_2^p(a\beta)^{-p} \quad \text{for all } (t,u) \in [0,1]\times[\alpha l_2, l_2]. \tag{2.5}$$

Then BVP (1.1), (1.2) *has at least one positive solution* u *with* $l_1 \le \|u\| \le l_2$ *(respectively,* $l_2 \le \|u\| \le l_1$*).*

As extensions of Theorems 2.2 and 2.4, we have the following results.

Theorem 2.5. *Assume there exists* $l_1 > 0$ *such that* (2.4) *holds. Then*

(a) *BVP* (1.1), (1.2) *has at least one positive solution* u *with* $\|u\| \le l_1$ *if* $f_0 > \lambda_0$*;*
(b) *BVP* (1.1), (1.2) *has at least one positive solution* u *with* $\|u\| \ge l_1$ *if* $f_\infty > \lambda_0$.

Theorem 2.6. *Assume there exists* $l_2 > 0$ *such that* (2.5) *holds. Then*

(a) *BVP* (1.1), (1.2) *has at least one positive solution* u *with* $\|u\| \le l_2$ *if* $f^0 < \lambda_0$*;*
(b) *BVP* (1.1), (1.2) *has at least one positive solution* u *with* $\|u\| \ge l_2$ *if* $f^\infty < \lambda_0$.

Combining Theorems 2.5 and 2.6 we obtain a result on the existence of at least two positive solutions.

Theorem 2.7. *Assume either*

(a) $f_0 > \lambda_0$ *and* $f_\infty > \lambda_0$*, and there exists* $l > 0$ *such that*

$$f(t,u) < l^p((a+q^*)\beta)^{-p} \text{ for all } (t,u) \in [0,1]\times[\alpha l, l]; \text{or} \tag{2.6}$$

(b) $f^0 < \lambda_0$ *and* $f^\infty < \lambda_0$*, and there exists* $l > 0$ *such that*

$$f(t,u) > l^p(a\beta)^{-p} \text{ for all } (t,u) \in [0,1]\times[\alpha l, l]. \tag{2.7}$$

Then BVP (1.1), (1.2) *has at least two positive solutions* u_1 *and* u_2 *with* $\|u_1\| < l < \|u_2\|$.

Note that in Theorem 2.7, the inequalities in (2.6) and (2.7) are strict and hence are different from those in (2.4) and (2.5) in Theorem 2.4. This is to guarantee that the two solutions u_1 and u_2 are different.

By applying Theorem 2.4 repeatedly, we obtain criteria for the existence of multiple positive solutions.

Theorem 2.8. *Let $\{l_i\}_{i=1}^N \subset \mathbb{R}$ such that $0 < l_1 < l_2 < \dots < l_N$. Assume either*

(a) *f satisfies (2.6) with $r = l_i$ when i is odd, and satisfies (2.7) with $r = l_i$ when i is even; or*
(b) *f satisfies (2.6) with $r = l_i$ when i is even, and satisfies (2.7) with $r = l_i$ when i is odd.*

Then BVP (1.1), (1.2) has at least $N-1$ positive solutions u_i with $l_i < \|u_i\| < l_{i+1}$, $i = 1, 2, \dots, N-1$.

Theorem 2.9. *Let $\{l_i\}_{i=1}^\infty \subset \mathbb{R}$ such that $0 < l_1 < l_2 < \dots$ and $\lim_{i\to\infty} l_i = \infty$. Assume either*

(a) *f satisfies (2.4) with $l_1 = l_i$ when i is odd, and satisfies (2.5) with $l_2 = l_i$ when i is even; or*
(b) *f satisfies (2.4) with $l_1 = l_i$ when i is even, and satisfies (2.5) with $l_2 = l_i$ when i is odd.*

Then BVP (1.1), (1.2) has an infinite number of positive solutions.

The following is an immediate consequence of Theorem 2.9.

Corollary 2.10. *Let $\{l_i\}_{i=1}^\infty \subset \mathbb{R}$ such that $0 < l_1 < l_2 < \dots$ and $\lim_{i\to\infty} l_i = \infty$. Let $E_1 = \cup_{i=1}^\infty [\alpha l_{2i-1}, l_{2i-1}]$ and $E_2 = \cup_{i=1}^\infty [\alpha l_{2i}, l_{2i}]$. Assume*

$$\limsup_{E_1 \ni u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u^p} < ((a+q^*)\beta)^{-p}, \quad \liminf_{E_2 \ni u\to\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u^p} > (a\alpha\beta)^{-p}.$$

Then BVP (1.1), (1.2) has an infinite number of positive solutions.

Finally, we present a result on the nonexistence of positive solutions of BVP (1.1), (1.2).

Theorem 2.11. *BVP (1.1), (1.2) has no positive solutions if*

(a) *$f(t,u)/u^p < ((a+q^*)\beta)^{-p}$ for all $(t,u) \in [0,1] \times (0,\infty)$, or*
(b) *$f(t,u)/u^p > (a\alpha\beta)^{-p}$ for all $(t,u) \in [0,1] \times (0,\infty)$.*

3. Proofs

Proof of Lemma 2.1. To prove this lemma, we need to normalize BC (1.2) using the generalized sine and cosine functions established by Elbert, see [7] for the detail.

It can be shown that (1.2) is equivalent to the BC

$$(qu')(0) = 0, \quad C(\theta^*)u(1) - S(\theta^*)(qu')(1) = 0, \tag{3.1}$$

where $C(\theta)$ and $S(\theta)$ are the generalized sine and cosine functions, $\theta^* \in (\pi_p/2, \pi_p)$ with $\pi_p = 2\pi((p+1)\sin(\pi/(p+1)))^{-1}$ such that $S(\theta^*)/C(\theta^*) = -a$.

Now we treat (3.1) as a function of θ and let $\lambda_0(\theta)$ be the first eigenvalue of SLP (2.2), (3.1) for $\theta \in [\pi_p/2, \pi_p)$. By [17, Corollary 3.9], λ_0 is strictly increasing. Note that (3.1) with $\theta = \pi_p/2$ becomes

$$(qu')(0) = 0, \quad (qu')(1) = 0. \tag{3.2}$$

In this case, $\lambda_0(\pi_p/2) = 0$ is the first eigenvalue of SLP (2.2), (3.2) with an associated eigenfunction $v_0 \equiv 1$. As a result, $\lambda_0(\theta) > 0$ for $\theta \in (\pi_p/2, \pi_p)$. In particular, $\lambda_0(\theta^*) > 0$, i.e., the first eigenvalue of SLP (2.2), (1.2) is positive. $\square$

With $\|u\| = \max_{t\in[0,1]} |u(t)|$, it is clear that $(C[0,1], \|\cdot\|)$ is a Banach space. Let $C_+[0,1] = \{u \in C[0,1] \mid u \geq 0 \text{ on } [0,1]\}$. Define $\Gamma: C_+[0,1] \to C[0,1]$ by

$$(\Gamma u)(t) = \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)ds, \quad t \in (0,1), \tag{3.3}$$

where ϕ^{-1} is the inverse function of ϕ, and

$$G_u(t,s) = \begin{cases} a, & 0 \leq s \leq t, \\ a + \frac{1}{q(s)}\phi^{-1}\Big(\frac{\int_0^s w(\tau)f(\tau,u(\tau))d\tau}{\int_0^1 w(\tau)f(\tau,u(\tau))d\tau}\Big), & t \leq s \leq 1. \end{cases} \tag{3.4}$$

Remark 3.1. We observe that the operator Γ defined by (3.3) is the same as

$$\begin{aligned}(\Gamma u)(t) = &\int_t^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ a\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big).\end{aligned} \tag{3.5}$$

Remark 3.2. It is easy to see that for any $u \in C_+[0,1]$

$$a \leq G_u(t,s) \leq a + q^* \text{ on } [0,1]\times[0,1], \tag{3.6}$$

where q^* is defined by (2.3).

Lemma 3.3. *A function $u(t)$ is a solution of* (1.1), (1.2) *if and only if u is a fixed point of Γ.*

Proof. Assume $u(t)$ is a solution of BVP (1.1), (1.2). From (1.1) and the first BC in (1.2) we see that for any $t \in (0,1)$

$$(qu')(t) = -\phi^{-1}\Big(\int_0^t w(\tau)f(\tau,u(\tau))d\tau\Big),$$

$$u(t) = u(0) - \int_0^t \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)f(\tau,u(\tau))d\tau\Big)ds. \tag{3.7}$$

Then by the second BC in (1.2), we have

$$u(0) = \int_0^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)f(\tau,u(\tau))d\tau\Big)ds + a\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big).$$

By (3.7) and (3.5),

$$\begin{aligned}u(t) = &\int_t^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &+ a\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big) = (\Gamma u)(t).\end{aligned}$$

Thus, u is a fixed point of the operator Γ. The opposite direction can be verified by reversing the argument. We omit the details. $\square$

Let
$$K = \{u \in C[0,1] \mid \min_{t\in[0,1]} u(t) \ge \alpha\|u\|\}, \tag{3.8}$$
where α is defined by (2.3). Clearly, K is a cone contained in $C_+[0,1]$. For $l > 0$, define
$$K_l = \{u \in K | \|u\| < l\}, \quad \partial K_l = \{u \in K | \|u\| = l\}, \tag{3.9}$$
and let $\mathrm{i}(\Gamma, K_l, K)$ be the fixed point index of Γ on K_l with respect to K.

Lemma 3.4. *Γ is completely continuous and $\Gamma K \subset K$.*

Proof. By (3.5), it is easy to see that Γ is completely continuous on $C_+[0,1]$. For any $u \in K$, by (2.3), (3.3), and (3.6)
$$\begin{aligned}
\min_{t\in[0,1]} (\Gamma u)(t) &= \min_{t\in[0,1]} \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\ge \int_0^1 a\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&= \alpha\int_0^1 (a+q^*)\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&\ge \alpha \max_{t\in[0,1]} \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\
&= \alpha\|\Gamma u\|.
\end{aligned} \tag{3.10}$$
Therefore, $\Gamma K \subset K$. □

Our proofs for the existence of positive solutions are based on the following fixed point index theorem, see [33, page 529, A2, A3] for the detail.

Lemma 3.5. *Let $0 < l_1 < l_2$ satisfy*
$$\mathrm{i}(\Gamma, K_{l_1}, K) = 0 \quad \textit{and} \quad \mathrm{i}(\Gamma, K_{l_2}, K) = 1;$$
or
$$\mathrm{i}(\Gamma, K_{l_1}, K) = 1 \quad \textit{and} \quad \mathrm{i}(\Gamma, K_{l_2}, K) = 0.$$
Then Γ has a fixed point in $K_{l_2} \setminus \overline{K}_{l_1}$.

To prove Theorem 2.2, we also need the lemma below, see [14, Lemma 2.3.1 and Corollary 2.3.1] for details.

Lemma 3.6. *Let $l > 0$. Then*
- (a) $\mathrm{i}(\Gamma, K_l, K) = 1$ *if $u \ne \mu\Gamma u$ for all $u \in \partial K_l$ and $\mu \in [0,1]$;*
- (b) $\mathrm{i}(\Gamma, K_l, K) = 0$ *if there exists $v_0 \in K \setminus \{0\}$ such that $u - \Gamma u \ne \mu v_0$ for all $u \in \partial K_l$ and $\mu \ge 0$.*

Proof of Theorem 2.2. Assume $f^0 < \lambda_0 < f_\infty$. Let λ_0 be the first eigenvalue of SLP (2.2), (1.2) with an associated positive eigenfunction v_0. Define $\Gamma_1 : C_+[0,1] \to C_+[0,1]$ as
$$\begin{aligned}
&(\Gamma_1 u)(t) \\
&= \int_t^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)\phi(u(\tau))d\tau\Big)ds + a\phi^{-1}\Big(\int_0^1 w(s)\phi(u(s))ds\Big).
\end{aligned} \tag{3.11}$$
It is easy to verify that $\lambda_0^{-1/p}$ is an eigenvalue of Γ_1 with v_0 as an associated eigenfunction, i.e., $\Gamma_1 v_0 = \lambda_0^{-1/p} v_0$. Hence $v_0 = \lambda_0^{1/p}\Gamma_1 v_0$.

Since $f^0 < \lambda_0$, there exists $\underline{l} > 0$ such that $f(t,u) < \lambda_0 u^p = \lambda_0\phi(u)$ for any $(t,u) \in [0,1]\times[0,\underline{l}]$. For any $u \in \partial K_{\underline{l}}$, $\alpha\underline{l} \le u(t) \le \underline{l}$ on $[0,1]$. By (3.5) and (3.11), for $t\in[0,1]$

$$\begin{aligned}
&(\Gamma u)(t)\\
&= \int_t^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)f(\tau,u(\tau))d\tau\Big)ds + a\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)\\
&< \lambda_0^{1/p}\Big[\int_t^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)\phi(u(\tau))d\tau\Big)ds + a\phi^{-1}\Big(\int_0^1 w(\tau)\phi(u(\tau))d\tau\Big)\Big]\\
&= \lambda_0^{1/p}(\Gamma_1 u)(t).
\end{aligned} \tag{3.12}$$

Without loss of generality, we assume that Γu has no fixed point on $\partial K_{\underline{l}}$. For otherwise, the proof is done. We show that $u \ne \mu\Gamma u$ for all $u\in\partial K_{\underline{l}}$ and $\mu\in[0,1]$. Obviously, it is true for $\mu = 0,1$. So we only consider $\mu\in(0,1)$. Assume the contrary, i.e., there exist $u_0\in\partial K_{\underline{l}}$ and $\mu_0\in(0,1)$ such that $u_0(t) = \mu_0(\Gamma u_0)(t)$. By (3.12), for $t\in[0,1]$

$$u_0(t) = \mu_0(\Gamma u_0)(t) < \mu_0\lambda_0^{1/p}(\Gamma_1 u_0)(t). \tag{3.13}$$

In view of the fact that $u_0(t) > 0$ and $v_0(t) > 0$ on $[0,1]$, the set $\{\mu \mid u_0(t) \le \mu v_0(t) \text{ for } t\in[0,1]\}$ is not empty. Define $\mu_1 = \min\{\mu \mid u_0(t)\le \mu v_0(t) \text{ for } t\in[0,1]\}$. Then $\mu_1 > 0$, and from (3.11) and by the nondecreasing property of Γ_1 we have that for $t\in[0,1]$

$$\lambda_0^{1/p}(\Gamma_1 u_0)(t) \le \lambda_0^{1/p}(\Gamma_1(\mu_1 v_0))(t) = \mu_1\lambda_0^{1/p}(\Gamma_1 v_0)(t) = \mu_1 v_0(t).$$

Thus by (3.13) $u_0(t) < \mu_0\mu_1 v_0(t) < \mu_1 v_0(t)$ on $[0,1]$, which contradicts the definition of μ_1. Therefore, $u\ne\mu\Gamma u$ for all $u\in\partial K_{\underline{l}}$ and $\mu\in[0,1]$. By Lemma 3.6 (a), $\mathfrak{i}(\Gamma, K_{\underline{l}}, K) = 1$.

Since $f_\infty > \lambda_0$, there exists $\tilde{l} > \underline{l}$ such that $f(t,u) > \lambda_0 u^p = \lambda_0\phi(u)$ for all $(t,u)\in[0,1]\times(\tilde{l},\infty)$. Choose $\bar{l} \ge \alpha^{-1}\tilde{l}$. Then for any $u\in\partial K_{\bar{l}}$, $u(t)\ge\alpha\bar{l} = \tilde{l}$ on $[0,1]$. By (3.5) and (3.11), for $t\in[0,1]$

$$\begin{aligned}
&(\Gamma u)(t)\\
&= \int_t^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)f(\tau,u(\tau))d\tau\Big)ds + a\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)\\
&> \lambda_0^{1/p}\Big[\int_t^1 \frac{1}{q(s)}\phi^{-1}\Big(\int_0^s w(\tau)\phi(u(\tau))d\tau\Big)ds + a\phi^{-1}\Big(\int_0^1 w(\tau)\phi(u(\tau))d\tau\Big)\Big]\\
&= \lambda_0^{1/p}(\Gamma_1 u)(t).
\end{aligned} \tag{3.14}$$

Without loss of generality, we assume that Γu has no fixed point on $\partial K_{\bar{l}}$. For otherwise, the proof is done. We show that $u - \Gamma u \ne \mu v_0$ for any $u\in\partial K_{\bar{l}}$ and $\mu\ge 0$. Obviously, it is true for $\mu = 0$. so we only consider $\mu > 0$. Assume the contrary, i.e., there exist $u^0\in\partial K_{\bar{l}}$ and $\mu^0 > 0$ such that $u^0 - \Gamma u^0 = \mu^0 v_0$. Then on [0,1] we have

$$u^0(t) = (\Gamma u^0)(t) + \mu^0 v_0(t) > \mu^0 v_0(t).$$

Define $\mu_2 = \max\{\mu \mid u^0(t) \ge \mu v_0(t)$ for $t \in [0,1]\}$. Then $\mu_2 \ge \mu^0$ and $u^0(t) \ge \mu_2 v_0(t)$ on [0,1]. From (3.14) we see that for $t \in [0,1]$

$$\begin{aligned} u^0(t) &= \Gamma u^0(t) + \mu^0 v_0(t) > \lambda_0^{1/p}(\Gamma_1 u^0)(t) + \mu^0 v_0(t) \\ &\ge \lambda_0^{1/p}(\Gamma_1 \mu_2 v_0)(t) + \mu^0 v_0(t) = \mu_2 \lambda_0^{1/p}(\Gamma_1 v_0)(t) + \mu^0 v_0(t) \\ &= \mu_2 v_0(t) + \mu^0 v_0(t) = (\mu_2 + \mu^0) v_0(t), \end{aligned}$$

which contradicts the definition of μ_2. Therefore, $u - \Gamma u \ne \mu v_0$ for any $u \in \partial K_{\bar{l}}$ and $\mu \ge 0$. By Lemma 3.6 (b), $\mathrm{i}(\Gamma, K_{\bar{l}}, K) = 0$. By Lemma 3.5, BVP (1.1), (1.2) has at least one positive solution.

The case for $f^\infty < \lambda_0 < f_0$ can be proved similarly. We omit the details. □

Proof of Corollary 2.3. It suffices to show that

$$((a+q^*)\beta)^{-p} \le \lambda_0 \le (a\alpha\beta)^{-p},$$

and then the conclusion follows from Theorem 2.2. Let λ_0 be the first eigenvalue of SLP (2.2), (1.2) with an associated positive eigenfunction v_0. Let Γ_1 be defined by (3.11). Then as shown in the proof of Theorem 2.2, we have $v_0 = \lambda_0^{1/p}\Gamma_1 v_0$. Moreover, for $t \in [0,1]$,

$$(\Gamma_1 v_0)(t) = \int_0^1 G_1(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)\phi(v_0(\tau))d\tau\Big)ds, \tag{3.15}$$

where

$$G_1(t,s) = \begin{cases} a, & 0 \le s \le t, \\ a + \frac{1}{q(s)}\phi^{-1}\Big(\frac{\int_0^s w(\tau)\phi(v_0(\tau))d\tau}{\int_0^1 w(\tau)\phi(v_0(\tau))d\tau}\Big), & t \le s \le 1. \end{cases}$$

Clearly, $a \le G_1(t,s) \le a + q^*$. By (2.3)

$$\begin{aligned} \|v_0\| &= \max_{t\in[0,1]} v_0(t) = \max_{t\in[0,1]} \lambda_0^{1/p}(\Gamma_1 v_0)(t) \\ &= \max_{t\in[0,1]} \lambda_0^{1/p}\int_0^1 G_1(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)\phi(v_0(\tau))d\tau\Big)ds \\ &\le \lambda_0^{1/p}\int_0^1 (a+q^*)\phi^{-1}\Big(\int_0^s w(\tau)d\tau\Big)\|v_0\|ds \\ &= \lambda_0^{1/p}(a+q^*)\beta\|v_0\|. \end{aligned}$$

Therefore, $\lambda_0 \ge ((a+q^*)\beta)^{-p}$.

Similar to (3.10) we have that $v_0(t) \ge \alpha\|v_0\|$ for $t \in [0,1]$. Thus

$$\begin{aligned} \|v_0\| &\ge \lambda_0^{1/p}(\Gamma_1 v_0)(t) = \lambda_0^{1/p}\int_0^1 G_1(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)\phi(v_0(\tau))d\tau\Big)ds \\ &\ge \lambda_0^{1/p}\int_0^1 a\phi^{-1}\Big(\int_0^1 w(\tau)d\tau\Big)\alpha\|v_0\|ds \\ &= \lambda_0^{1/p}a\alpha\beta\|v_0\|; \end{aligned}$$

i.e., $\lambda_0 \le (a\alpha\beta)^{-p}$. This completes the proof. □

To prove Theorem 2.4 we need the following well-known lemma on fixed point indices. See [5, 14] for details.

Lemma 3.7. *Let $l > 0$ and assume $\Gamma u \ne u$ for $u \in \partial K_l$. Then*

(a) $\mathfrak{i}(\Gamma, K_l, K) = 1$ *if* $\|\Gamma u\| \le \|u\|$ *for* $u \in \partial K_l$.
(b) $\mathfrak{i}(\Gamma, K_l, K) = 0$ *if* $\|\Gamma u\| \ge \|u\|$ *for* $u \in \partial K_l$.

Proof of Theorem 2.4. Without loss of generality, we assume $\Gamma u \ne u$ on $\partial K_{l_1} \cup \partial K_{l_2}$. For otherwise, Γ has a positive fixed point.

For any $u \in \partial K_{l_1}$, $\alpha l_1 \le u(t) \le l_1$ on $[0,1]$. From (2.4), $f(t,u(t)) \le l_1^p((a+q^*)\beta)^{-p}$ on $[0,1]$. Then by (2.3) and (3.6),

$$\begin{aligned} \|\Gamma u\| &= \max_{t\in[0,1]} \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau) f(\tau,u(\tau))d\tau\Big)ds \\ &\le \max_{t\in[0,1]} \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)d\tau\Big) l_1((a+q^*)\beta)^{-1} ds \le l_1. \end{aligned}$$

Thus $\|\Gamma u\| \le \|u\|$. By Lemma 3.7 (a), $\mathfrak{i}(\Gamma, K_{l_1}, K) = 1$.

For any $u \in K_{l_2}$, $\alpha l_2 \le u(t) \le l_2$ on $[0,1]$. From (2.5), $f(t,u(t)) \ge l_2^p(a\beta)^{-p}$ on $[0,1]$. Then by (2.3) and (3.6)

$$\begin{aligned} \|\Gamma u\| &\ge \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau) f(\tau,u(\tau))d\tau\Big)ds \\ &\ge \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)d\tau\Big) l_2(a\beta)^{-1} ds \ge l_2. \end{aligned}$$

Thus $\|\Gamma u\| \ge \|u\|$. By Lemma 3.7 (b), $\mathfrak{i}(\Gamma, K_{l_2}, K) = 0$.

By Lemma 3.5, Γ has a fixed point $u \in K_{l_2} \setminus \overline{K}_{l_1}$ if $l_1 < l_2$, and Γ has a fixed point $u \in K_{l_1} \setminus \overline{K}_{l_2}$ if $l_1 > l_2$. In each case, u is a positive function with $\min\{l_1, l_2\} \le \|u\| \le \max\{l_1, l_2\}$. □

The proofs of Theorems 2.5 and 2.6 are in the same way and hence we only give the proof of Theorem 2.5.

Proof of Theorem 2.5. (a) If there exists $l_1 > 0$ such that (2.4) holds, then by the proof of Theorem 2.4, $\mathfrak{i}(\Gamma, K_{l_1}, K) = 1$. By the proof of Theorem 2.2, $f_0 > \lambda_0$ implies there exists $0 < l_2 < l_1$ with $\mathfrak{i}(\Gamma, K_{l_2}, K) = 0$. Then the conclusion follows from Lemma 3.5. Part (b) can be proved similarly. □

Proof of Theorem 2.7. (a) Assume there exists $l > 0$ such that (2.6) holds. Then there exist l_1 and l_2 such that $l_1 < l < l_2$ and $f(t,u) < (l_i^p((a+q^*)\beta)^{-p}$ on $[0,1] \times [\alpha l_i, l_i]$, $i = 1,2$. By Theorem 2.5 (a) and (b), BVP (1.1), (1.2) has two positive solutions u_1 and u_2 satisfying $\|u_1\| \le l_1$ and $\|u_2\| \ge l_2$.

Part (b) can be proved similarly. □

Theorems 2.8 and 2.9 can be obtained by applying Theorem 2.4 repeatedly. We omit the details.

Proof of Corollary 2.10. From the assumption we see that for sufficiently large i

$$\frac{f(t,u)}{u^p} < ((a+q^*)\beta)^{-p} \quad \text{for all } (t,u) \in [0,1] \times [\alpha l_{2i-1}, l_{2i-1}]$$

and

$$\frac{f(t,u)}{u^p} > (a\alpha\beta)^{-p} \quad \text{for all } (t,u) \in [0,1] \times [\alpha l_{2i}, l_{2i}].$$

This shows that for sufficiently large i,

$$f(t,u) < u^p((a+q^*)\beta)^{-p} \le l_{2i-1}^p((a+q^*)\beta)^{-p} \quad \text{on } [0,1] \times [\alpha l_{2i-1}, l_{2i-1}]$$

and

$$f(t,u) > u^p(a\alpha\beta)^{-p} \geq (\alpha l_{2i})^p(a\alpha\beta)^{-p} = l_{2i}^p(a\beta)^{-p} \quad \text{on } [0,1]\times[\alpha l_{2i}, l_{2i}].$$

Therefore, the conclusion follows from Theorem 2.9. □

Proof of Theorem 2.11. (a) Assume BVP (1.1), (1.2) has a positive solution u with $\|u\| = l$ for some $l > 0$. Then u is a fixed point of the operator Γ defined by (3.3). From the assumption, for any $t \in [0,1]$, $f(t,u(t)) < u^p(t)((a+q^*)\beta)^{-p} \leq l^p((a+q^*)\beta)^{-p}$. Hence

$$\begin{aligned} u(t) = (\Gamma u)(t) &= \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &< l/((a+q^*)\beta)\int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)d\tau\Big)ds \leq l, \end{aligned}$$

which contradicts $\|u\| = l$. Therefore, BVP (1.1), (1.2) has no positive solutions.

(b) Assume BVP (1.1), (1.2) has a positive solution u with $\|u\| = l$ for some $l > 0$. Then $\alpha l \leq u(t) \leq l$ on $[0,1]$. From the assumption, for any $t \in [0,1]$, $f(t,u(t)) > u^p(t)(a\alpha\beta)^{-p} \geq l^p(a\beta)^{-p}$. Hence

$$\begin{aligned} u(t) = (\Gamma u)(t) &= \int_0^1 G_u(t,s)\phi^{-1}\Big(\int_0^1 w(\tau)f(\tau,u(\tau))d\tau\Big)ds \\ &> l/(a\beta)\int_0^1 G_u(t_1,s)\phi^{-1}\Big(\int_0^1 w(\tau)d\tau\Big)ds \geq l, \end{aligned}$$

which contradicts $\|u\| = l$. Therefore, BVP (1.1), (1.2) has no positive solutions. □

4. Partial BVPs with p-Laplacian

In this section, we extend our results in Section 2 to BVPs for partial differential equations with p-Laplacian defined on annular domains. Let $0 < r_1 < r_2$, $n \in \mathbb{N}$, and denote $\Omega = B(0,r_2) \setminus \overline{B(0,r_1)}$, where $B(0,r)$ is the ball in $\mathbb{R}^n$ centered at 0 with radius r. Consider the scalar BVP

$$-\operatorname{div}(|\nabla v|^{p-1}\nabla v) = h(|x|)f(v) \quad \text{in } \Omega, \tag{4.1}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B(0,r_1), \qquad v + b\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B(0,r_2), \tag{4.2}$$

where $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $\operatorname{div}(y)$ is the divergence of $y : \mathbb{R}^n \to \mathbb{R}$, $b \in \mathbb{R}$, ∇v is the gradient of v, and $\partial v/\partial\nu$ is the outward normal derivative of v along $\partial B(0,r_i)$, $i = 1,2$. We assume that $h \in L[r_1,r_2]$, $h \geq 0$ a.e. on (r_1,r_2), and $\int_{r_1}^{r_2} h(s)ds > 0$.

The next lemma shows the relation between the partial BVP (4.1), (4.2) and the ordinary BVP (1.1), (1.2).

Lemma 4.1. *Let $r = |x|$, $t = t(r) := \int_{r_1}^{r} s^{(1-n)/p}ds/\int_{r_1}^{r_2} s^{(1-n)/p}ds$, and $r = r(t)$ be its inverse function. Then BVP (4.1), (4.2) has a positive radial solution $v(|x|)$ if and only if BVP (1.1), (1.2) with $q \equiv 1$,*

$$a = \frac{b r_2^{\frac{1-n}{p}}}{\int_{r_1}^{r_2} s^{\frac{1-n}{p}}ds}, \quad \text{and} \quad w(t) = h(r(t))r^{\frac{(p+1)(n-1)}{p}}(t)\Big(\int_{r_1}^{r_2} s^{\frac{1-n}{p}}ds\Big)^{p+1} \tag{4.3}$$

has a positive solution $u(t)$.

Proof. We first claim that the existence of a positive radial solution of BVP (4.1), (4.2) is equivalent to the existence of positive solution of BVP consisting of the equation

$$-\frac{d}{dr}(r^{n-1}\phi(\frac{d\tilde{v}}{dr})) = r^{n-1}h(r)f(\tilde{v}), \quad r_1 < r < r_2, \tag{4.4}$$

and the BC

$$\frac{d\tilde{v}}{dr}(r_1) = 0, \quad \tilde{v}(r_2) + b\frac{d\tilde{v}}{dr}(r_2) = 0, \tag{4.5}$$

where $\tilde{v}(r) = v(|x|)$. In fact, the proof for the case when $p = 1$ is given in many books such as [9] which can be easily extended to the general case.

Let $\tilde{v}(r)$ be a positive solution of BVP (4.4), (4.5) and $u(t) = \tilde{v}(r(t))$. Then $\frac{d\tilde{v}}{dr} = \frac{du}{dt}\frac{dt}{dr}$. We note from the definition of $t(r)$ that

$$\frac{dt}{dr} = \frac{r^{\frac{1-n}{p}}}{\int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds}.$$

By (4.4),

$$\begin{aligned}
r^{n-1}h(r)f(\tilde{v}) &= -\frac{d}{dr}\Big(r^{n-1}\phi(\frac{d\tilde{v}}{dr})\Big) = -\frac{d}{dt}\Big(r^{n-1}\phi(\frac{du}{dt}\frac{dt}{dr})\Big)\frac{dt}{dr}\\
&= -\frac{d}{dt}\Big(\Big(\int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds\Big)^{-p}\phi(\frac{du}{dt})\Big)\frac{dt}{dr}\\
&= -\Big(\int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds\Big)^{-p}\frac{d}{dt}\Big(\phi(\frac{du}{dt})\Big)\frac{dt}{dr}\\
&= -r^{\frac{1-n}{p}}\Big(\int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds\Big)^{-p-1}\frac{d}{dt}\left(\phi(\frac{du}{dt})\right).
\end{aligned}$$

Therefore,

$$-\frac{d}{dt}\left(\phi(\frac{du}{dt})\right) = h(r(t))r^{\frac{(p+1)(n-1)}{p}}(t)\Big(\int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds\Big)^{p+1}f(u) = w(t)f(u),$$

which means that $u(t)$ is a positive solution of (1.1) under (4.3). It is also easy to see that $u(t)$ satisfies BC (1.2).

The opposite direction can be verified by reversing the argument. We omit the details. $\square$

Clearly, all the assumptions of BVP (1.1), (1.2) are guaranteed by (4.3). In this case, since $q \equiv 1$, from (2.3) we have

$$\alpha = \frac{br_2^{\frac{1-n}{p}}}{br_2^{\frac{1-n}{p}} + \int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds}, \tag{4.6}$$

$$\begin{aligned}
\beta &= \phi^{-1}\Big(\int_0^1 h(r(\tau))r^{\frac{(p+1)(n-1)}{p}}(\tau)\Big(\int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds\Big)^{p+1}d\tau\Big)\\
&= \phi^{-1}\Big(\int_{r_1}^{r_2} h(r)r^{n-1}\Big(\int_{r_1}^{r_2} s^{\frac{1-n}{p}}\,ds\Big)^{p}dr\Big).
\end{aligned}\tag{4.7}$$

Let $f_0, f^0, f_\infty, f^\infty$ be defined by (2.1), λ_0 the first eigenvalue of SLP (2.2), (1.2) associated with BVP (1.1), (1.2) with $q \equiv 1$, and a and w defined by (4.3). Denote $\|v\| = \max_{x\in\Omega}|v(x)|$ for $v \in C(\Omega, \mathbb{R})$.

Now we apply the results in Section 2 to derive criteria for the existence, multiplicity, and nonexistence of positive radial solutions of BVP (4.1), (4.2). In the theorems below, a, α, and β are defined by (4.3), (4.6), and (4.7), respectively.

Theorem 4.2. *BVP* (4.1), (4.2) *has at least one positive radial solution if either* $f^0 < \lambda_0 < f_\infty$ *or* $f^\infty < \lambda_0 < f_0$.

Corollary 4.3. *BVP* (4.1), (4.2) *has at least one positive radial solution if either*

(a) $f^0 < ((a+1)\beta)^{-p}$ *and* $f_\infty > (a\alpha\beta)^{-p}$; *or*
(b) $f_0 > (a\alpha\beta)^{-p}$ *and* $f^\infty < ((a+1)\beta)^{-p}$.

Theorem 4.4. *Assume there exist* $0 < l_1 < l_2$ *(respectively,* $0 < l_2 < l_1$*) such that*

$$f(v) \le l_1^p((a+1)\beta)^{-p} \text{ for all } v \in [\alpha l_1, l_1], \tag{4.8}$$

$$f(v) \ge l_2^p(a\beta)^{-p} \text{ for all } v \in [\alpha l_2, l_2]. \tag{4.9}$$

Then BVP (4.1), (4.2) *has at least one positive radial solution v with* $l_1 \le \|v\| \le l_2$ *(respectively,* $l_2 \le \|v\| \le l_1$*).*

Theorem 4.5. *Assume there exists* $l_1 > 0$ *such that* (4.8) *holds. Then*

(a) *BVP* (4.1), (4.2) *has at least one positive radial solution v with* $\|v\| \le l_1$ *if* $f_0 > \lambda_0$;
(b) *BVP* (4.1), (4.2) *has at least one positive radial solution v with* $\|v\| \ge l_1$ *if* $f_\infty > \lambda_0$.

Theorem 4.6. *Assume there exists* $l_2 > 0$ *such that* (4.9) *holds. Then*

(a) *BVP* (4.1), (4.2) *has at least one positive radial solution v with* $\|v\| \le l_2$ *if* $f^0 < \lambda_0$;
(b) *BVP* (4.1), (4.2) *has at least one positive radial solution v with* $\|v\| \ge l_2$ *if* $f^\infty < \lambda_0$.

Theorem 4.7. *Assume either*

(a) $f_0 > \lambda_0$, $f_\infty > \lambda_0$, *and there exists* $l > 0$ *such that*
$$f(v) < l^p((a+1)\beta)^{-p} \text{ for all } v \in [\alpha l, l]; \text{or} \tag{4.10}$$
(b) $f^0 < \lambda_0$, $f^\infty < \lambda_0$, *and there exists* $l > 0$ *such that*
$$f(v) \ge l^p(a\beta)^{-p} \text{ for all } v \in [\alpha l, l]. \tag{4.11}$$

Then BVP (4.1), (4.2) *has at least two positive radial solutions* v_1 *and* v_2 *with* $\|v_1\| < l < \|v_2\|$.

Theorem 4.8. *Let* $\{l_i\}_{i=1}^N \subset \mathbb{R}$ *such that* $0 < l_1 < l_2 < \dots < l_N$. *Assume either*

(a) f *satisfies* (4.10) *with* $r = l_i$ *when i is odd, and satisfies* (4.11) *with* $r = l_i$ *when i is even; or*
(b) f *satisfies* (4.10) *with* $r = l_i$ *when i is even, and satisfies* (4.11) *with* $r = l_i$ *when i is odd.*

Then BVP (4.1), (4.2) *has at least* $N-1$ *positive radial solutions* v_i *with* $l_i < \|v_i\| < l_{i+1}$, $i = 1, 2, \dots, N-1$.

Theorem 4.9. *Let* $\{l_i\}_{i=1}^\infty \subset \mathbb{R}$ *such that* $0 < l_1 < l_2 < \dots$ *and* $\lim_{i\to\infty} l_i = \infty$. *Assume either*

(a) f *satisfies* (4.8) *with* $l_1 = l_i$ *when i is odd, and satisfies* (4.9) *with* $l_2 = l_i$ *when i is even; or*

(b) *f satisfies* (4.8) *with $l_1 = l_i$ when i is even, and satisfies* (4.9) *with $l_2 = l_i$ when i is odd.*

Then BVP (4.1), (4.2) *has an infinite number of positive radial solutions.*

Corollary 4.10. *Let $\{l_i\}_{i=1}^{\infty} \subset \mathbb{R}$ such that $0 < l_1 < l_2 < \dots$ and $\lim_{i\to\infty} l_i = \infty$. Let $E_1 = \cup_{i=1}^{\infty}[\alpha l_{2i-1}, l_{2i-1}]$ and $E_2 = \cup_{i=1}^{\infty}[\alpha l_{2i}, l_{2i}]$. Assume*

$$\limsup_{E_1 \ni v\to\infty} \frac{f(v)}{v^p} < ((a+1)\beta)^{-p} \quad \textit{and} \quad \liminf_{E_2 \ni v\to\infty} \frac{f(v)}{v^p} > (a\alpha\beta)^{-p}.$$

Then BVP (4.1), (4.2) *has an infinite number of positive radial solutions.*

Theorem 4.11. *BVP* (4.1), (4.2) *has no positive radial solutions if*

(a) *$f(v)/v^p < ((a+1)\beta)^{-p}$ for all $v \in (0,\infty)$, or*
(b) *$f(v)/v^p > (a\alpha\beta)^{-p}$ for all $v \in (0,\infty)$.*

Remark 4.12. Note that when $r_1 \to 0+$, the annulus Ω for the domain of (4.1) approaches a disk centered at the origin with radius r_2, and the first BC in (4.2) reduces to $\frac{\partial v}{\partial \nu}|_{x=0} = 0$ which is automatically satisfied by radial solutions. Hence, the p-Laplacian partial BVP defined on the disk

$$-\operatorname{div}(\phi(\nabla v)) = h(|x|)f(v) \quad \text{in } B(0,r_2), \tag{4.12}$$

$$v + b\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B(0,r_2). \tag{4.13}$$

can be treated as the limiting problem of BVP (4.1), (4.2) as $r \to 0+$. Therefore, the results for BVP (4.1), (4.2) can be extended to BVP (4.12), (4.13) with the modification $r_1 = 0$. The only problem in this extension is that the integral $\int_{r_1}^{r_2} s^{(1-n)/p} ds$ may become divergent as $r_1 \to 0+$. However, this does not occur under the additional assumption that $p + 1 - n > 0$.

5. Examples

In this section, we give several examples as applications of our results.

Example 5.1. Let $S(\theta)$ denote the general sine function and let $\theta^* \in (\pi_p/2, \pi_p)$ be a solution of $S(\theta) + S'(\theta) = 0$. Consider the BVP

$$\begin{gathered} -(\phi(u'))' = f(u),\ 0 < t < 1, \\ u'(0) = 0,\ u(1) + (\theta^* - \pi_p/2)^{-1}u'(1) = 0, \end{gathered} \tag{5.1}$$

where $f(u) = [p(\theta^* - \pi_p/2)^{p+1} + c(\tan^{-1}(u) - \pi/4)]u^p$ with $0 < |c| < p(\theta^* - \pi_p/2)^{p+1}4/\pi$. Then BVP (5.1) has at least one positive solution.

In fact, $S(\theta)$ is the unique solution of the initial value problem

$$\begin{gathered} -(\phi(u'))' = p\phi(u), \\ u(0) = 0, \quad u'(0) = 1. \end{gathered}$$

Note that $S'(\pi_p/2) = 0$. Hence p is the first eigenvalue of the SLP

$$\begin{gathered} -(\phi(u'))' = \lambda\phi(u), \\ u'(\pi_p/2) = 0,\ u(\theta^*) + u'(\theta^*) = 0, \end{gathered}$$

with the associated eigenfunction $S(\theta)$.

Make the transformation $t = (\theta - \pi_p/2)/(\theta^* - \pi_p/2)$ in the above problem. Then similar to the proof of Lemma 4.1, we find that $p(\theta^* - \pi_p/2)^{p+1}$ is the first eigenvalue of the SLP

$$-\frac{d}{dt}(\phi(\frac{du}{dt})) = \lambda\phi(u),$$
$$\frac{du}{dt}(0) = 0, \quad u(1) + (\theta^* - \pi_p/2)^{-1}\frac{du}{dt}(1) = 0.$$

Note that

$$f_0 = f^0 = p(\theta^* - \pi_p/2)^{p+1} - c\pi/4, \quad f_\infty = f^\infty = p(\theta^* - \pi_p/2)^{p+1} + c\pi/4.$$

Then the conclusion follows from Theorem 2.2. Note that in this example, c can be arbitrarily close to 0.

Example 5.2. Consider the ordinary BVP

$$\begin{gathered} -(\phi(u'))' = k(u^{p/2} + u^{2p}), \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) + au'(1) = 0. \end{gathered} \tag{5.2}$$

Let $l = 4^{-1/(3p)}$, $\gamma_1 = (al)^{p/2}((a+1)^{3p/2} + (al)^{3p/2})^{-1}$, and $\gamma_2 = l^{p/2}(a+1)^p a^{-2p}(1+l^{3p/2})^{-1}$. Then

(a) BVP (5.2) has at least one positive solution when $k = \gamma_1$;
(b) BVP (5.2) has at least two positive solutions u_1 and u_2 with $\|u_1\| < l < \|u_2\|$ when $0 < k < \gamma_1$;
(c) BVP (5.2) has no positive solutions when $k > \gamma_2$.

In fact, the equation in (5.2) is of the form of (1.1) with $w(t) \equiv 1$ and $f(u) = k(u^{p/2} + u^{2p})$. Clearly, $f_0 = f_\infty = \infty$, $f(u)/u^p$ is decreasing on $(0, l]$, increasing on $[l, \infty)$, and hence reaches minimum value at l. By (2.3), $\alpha = a/(a+1)$, $\beta = 1$. When $k = \gamma_1$, $f(\alpha l)/(\alpha l)^p = (a+1)^{-p}$. Hence for $u \in [\alpha l, l]$, $f(u)/u^p \le f(\alpha l)/(\alpha l)^p = (a+1)^{-p}$, which follows that $f(u) \le u^p(a+1)^{-p} \le l^p(a+1)^{-p}$. Therefore, by Theorem 2.5 (a), BVP (5.2) has a positive solution u_1 with $\|u_1\| \le l$. Similarly, by Theorem 2.5 (b) we can also show that BVP (5.2) has a positive solution u_2 with $\|u_2\| \ge l$. However, u_1 and u_2 may be the same when $\|u_1\| = \|u_2\| = l$.

When $0 < k < \gamma_1$, by the similar argument as above and applying Theorem 2.5 (a), we obtain the conclusion.

When $k > \gamma_2$, $f(u)/u^p > (a\alpha\beta)^{-p} = (a+1)^p(a^2)^{-p}$ on $(0, \infty)$. Then the conclusion follows from Theorem 2.11 (b).

Example 5.3. Consider the BVP

$$\begin{gathered} -\operatorname{div}(|\nabla v|^{p-1}\nabla v) = k(v^{-p/2} + v^{-2p})^{-1} \quad \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B(0, r_1), \quad v + b\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B(0, r_2), \end{gathered} \tag{5.3}$$

where $0 < r_1 < r_2$. Let $l = 4^{1/(3p)}$ and a, α, β be defined by (4.3), (4.6), (4.7) with $h \equiv 1$. Denote

$$\gamma_3 = (a+1)^{p/2}((al)^{3p/2} + (a+1)^{3p/2})a^{-3p}l^{-p}\beta^{-p},$$
$$\gamma_4 = (l^{3p/2} + 1)l^{-p}(a+1)^{-p}\beta^{-p}.$$

Then

(a) BVP (5.3) has at least one positive solution when $k = \gamma_3$;
(b) BVP (5.3) has at least two positive solutions when $k > \gamma_3$;

(c) BVP (5.3) has no positive solutions when $0 < k < \gamma_4$.

In fact, the equation in (5.3) is of the form of (4.1) with $w(t) \equiv 1$ and $f(v) = k(v^{-p/2} + v^{-2p})^{-1}$. It is clear that $f^0 = f^\infty = 0$ and $f(v)/v^p$ reaches maximum at l. By a similar argument to Example 5.2 and applying Theorems 4.6, 4.7, and 4.11, we can prove the results. We omit the details.

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