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DYNAMIC EVOLUTION OF DAMAGE IN ELASTIC-THERMO-VISCOPLASTIC MATERIALS

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Abstract. We consider a mathematical model that describes the dynamic evolution of damage in elastic-thermo-viscoplastic materials with displacement-traction, and Neumann and Fourier boundary conditions. We derive a weak formulation of the system consisting of a motion equation, an energy equation, and an evolution damage inclusion. This system has an integro-differential variational equation for the displacement and the stress fields, and a variational inequality for the damage field. We prove existence and uniqueness of the solution, and the positivity of the temperature.

1. Introduction

The constitutive laws with internal variables has been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the absolute temperature and the damage field, see for examples and details [5, 8, 21, 22, 23, 29, 30] and references therein for the case of hardening, temperature and other internal state variables and the references [12, 13, 14, 22, 25, 27] for the case of damage field.

The aim of this paper is to study the dynamic evolution of damage in elastic-thermo-viscoplastic materials. For this, we consider a rate-type constitutive equation with two internal variables of the form

$$\begin{aligned}\sigma(t) &= \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{E}(\varepsilon(\mathbf{u}(t))) \\ &\quad + \int_0^t \mathcal{G}\Big(\sigma(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))), \varepsilon(\mathbf{u}(s)), \theta(s), \varsigma(s)\Big) ds,\end{aligned} \tag{1.1}$$

in which $\mathbf{u}$, σ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable, θ represents the absolute temperature, ς is the damage field, $\mathcal{A}$ and $\mathcal{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material,

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respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the visco-plastic behavior of the material.

Examples and mechanical interpretation of elastic-viscoplastic can be found in [9, 16]. Dynamic and quasistatic contact problems are the topic of numerous papers, e.g. [1, 2, 4, 11, 26], and the comprehensive references [15, 28]. However, the mathematical problem modelled the quasi-static evolution of damage in thermo-viscoplastic materials has been studied in [22].

The paper is organized as follows. In Section 2 we present the mechanical problem of the dynamic evolution of damage in elastic-thermo-viscoplastic materials. We introduce some notations and preliminaries and we derive the variational formulation of the problem. We prove in Section 3 the existence and uniqueness of the solution as well as the positivity of the temperature.

2. Statement of the Problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with a Lipschitz boundary Γ, partitioned into two disjoint measurable parts Γ_1 and Γ_2 such that $\text{meas}(\Gamma_1) > 0$. We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n, \quad \sigma \cdot \tau = \sigma_{ij} \tau_{ij} \quad \forall \sigma, \tau \in \mathbb{S}_n,$$
$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2} \quad \forall \mathbf{u} \in \mathbb{R}^n, \quad |\sigma| = (\sigma \cdot \sigma)^{1/2} \quad \forall \sigma \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We shall use the notation

$$\begin{gathered} H = L^2(\Omega)^n = \{\mathbf{u} = \{u_i\} : u_i \in L^2(\Omega)\}, \\ \mathcal{H} = \{\sigma = \{\sigma_{ij}\} : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \\ H_1 = \{\mathbf{u} \in H : \varepsilon(\mathbf{u}) \in \mathcal{H}\}, \\ \mathcal{H}_1 = \{\sigma \in \mathcal{H} : \text{Div}(\sigma) \in H\}, \\ V = H^1(\Omega). \end{gathered}$$

Here $\varepsilon : H_1 \to \mathcal{H}$ and $\text{Div} : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{Div}(\sigma) = (\sigma_{ij,j}).$$

The sets H, $\mathcal{H}$, H_1, $\mathcal{H}_1$ and V are real Hilbert spaces endowed with the canonical inner products:

$$\begin{gathered} (\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i dx, \quad (\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} dx, \\ (\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \\ (\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\text{Div}(\sigma), \text{Div}(\tau))_H, \\ (f, g)_V = (f, g)_{L^2(\Omega)} + (f_{x_i}, g_{x_i})_{L^2(\Omega)}\,. \end{gathered}$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}_1}$ and $\|\cdot\|_V$. Since the boundary Γ is Lipschitz continuous, the unit outward normal vector field ν on the boundary is defined a.e. For every vector field $\mathbf{v} \in H_1$ we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on the boundary given by

$$v_\nu = \mathbf{v} \cdot \nu, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu.$$

Let $H_\Gamma = (H^{1/2}(\Gamma))^n$ and $\gamma : H_1 \to H_\Gamma$ be the trace map. We denote by $\mathcal{V}$ the closed subspace of H_1 defined by

$$\mathcal{V} = \{\mathbf{v} \in H_1 : \gamma\mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

We also denote by H'_Γ the dual of H_Γ. Moreover, since $meas(\Gamma_1) > 0$, Korn's inequality holds and thus, there exists a positive constant C_0 depending only on Ω, Γ_1 such that

$$\|\varepsilon(\mathbf{v})\|_{\mathcal{H}} \geq C_0\|\mathbf{v}\|_{H_1} \quad \forall \mathbf{v} \in \mathcal{V}.$$

Furthermore, if $\sigma \in \mathcal{H}_1$ there exists an element $\sigma\nu \in H'_\Gamma$ such that the following Green formula holds

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}(\sigma), \mathbf{v})_H = (\sigma\nu, \gamma\mathbf{v})_{H'_\Gamma \times H_\Gamma} \quad \forall \mathbf{v} \in H_1.$$

In addition, if σ is sufficiently regular (say $\mathcal{C}^1$), then

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}(\sigma), \mathbf{v})_H = \int_\Gamma \sigma\nu \cdot \gamma\mathbf{v} d\gamma \quad \forall \mathbf{v} \in H_1.$$

where $d\gamma$ denotes the surface element. Similarly, for a regular tensor field $\sigma : \Omega \to \mathbb{S}_n$ we define its normal and tangential components on the boundary by

$$\sigma_\nu = \sigma\nu \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu.$$

Moreover, we denote by $\mathcal{V}'$ and V' the dual of the spaces $\mathcal{V}$ and V, respectively. Identifying H, respectively $L^2(\Omega)$, with its own dual, we have the inclusions

$$\mathcal{V} \subset H \subset \mathcal{V}', \quad V \subset L^2(\Omega) \subset V'.$$

We use the notation $\langle\cdot,\cdot\rangle_{\mathcal{V}'\times\mathcal{V}}$, $\langle\cdot,\cdot\rangle_{V'\times V}$ to represent the duality pairing between $\mathcal{V}', \mathcal{V}$ and V', V, respectively.

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem.

The physical setting is the following. An elastic-thermo-viscoplastic body occupies the domain Ω. We assume that the body is clamped on $\Gamma_1 \times (0, T)$, $(T > 0)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_0$ acts on $\Gamma_2 \times (0, T)$ and a volume forces of density $\mathbf{f}$ is applied in $\Omega \times (0, T)$. In addition, we admit a possible external heat source applied in $\Omega \times (0, T)$, given by the function q.

The mechanical problem may be formulated as follows.

Problem (P). Find the displacement field $\mathbf{u} : \Omega \times (0, T) \to \mathbb{R}^n$, the stress field $\sigma : \Omega \times (0, T) \to \mathbb{S}_n$, the temperature $\theta : \Omega \times (0, T) \to \mathbb{R}$ and the damage field $\varsigma : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned}\sigma(t) = \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{E}(\varepsilon(\mathbf{u}(t))) + \int_0^t \mathcal{G}\Big(\sigma(s)\\ - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))), \varepsilon(\mathbf{u}(s)), \theta(s), \varsigma(s)\Big) ds \quad \text{in } \Omega \text{ a.e. } t \in (0, T),\end{aligned} \tag{2.1}$$

$$\rho\ddot{\mathbf{u}} = \mathrm{Div}(\sigma) + \mathbf{f} \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\rho\dot{\theta} - k_0\Delta\theta = \psi(\sigma, \varepsilon(\dot{\mathbf{u}}), \theta, \varsigma) + q \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$\rho\dot{\varsigma} - k_1\Delta\varsigma + \partial_K\varphi(\varsigma) \ni \phi(\sigma, \varepsilon(\mathbf{u}), \theta, \varsigma) \quad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.5}$$

$$\sigma\nu = \mathbf{f}_0 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.6}$$

$$k_0\frac{\partial\theta}{\partial\nu}+\beta\theta=0 \quad \text{on } \Gamma\times(0,T), \tag{2.7}$$

$$\frac{\partial\varsigma}{\partial\nu}=0 \quad \text{on } \Gamma\times(0,T), \tag{2.8}$$

$$\mathbf{u}(0)=\mathbf{u}_0, \quad \dot{\mathbf{u}}(0)=\mathbf{w}, \quad \theta(0)=\theta_0, \quad \varsigma(0)=\varsigma_0 \quad \text{in } \Omega. \tag{2.9}$$

This problem represents the dynamic evolution of damage in elastic-thermo-viscoplastic materials. Equation (2.1) is the elastic-thermo-viscoplastic constitutive law where $\mathcal{A}$ and $\mathcal{E}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function which describes the viscoplastic behavior of the material. (2.2) represents the equation of motion in which the dot above denotes the derivative with respect to the time variable and ρ is the density of mass. Equation (2.3) represents the energy conservation where ψ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and q is a given volume heat source. Inclusion (2.4) describes the evolution of damage field, governed by the source damage function ϕ, where $\partial_K\varphi(\varsigma)$ is the subdifferential of indicator function of the set K of admissible damage functions given by

$$K=\{\xi\in V: 0\le\xi(x)\le 1 \ \text{ a.e. } x\in\Omega\},$$

in such a way that the damage function ς varied between 0 and 1. If $\varsigma=1$ there is no damage in the material, if $\varsigma=0$ the material is completely damaged and if $0<\varsigma<1$ the material is partially damaged.

Equalities (2.5) and (2.6) are the displacement-traction boundary conditions, respectively. (2.7), (2.8) represent, respectively on Γ, a Fourier boundary condition for the temperature and an homogeneous Neumann boundary condition for the damage field on Γ. Finally the functions $\mathbf{u}_0$, $\mathbf{w}$, θ_0 and ς_0 in (2.9) are the initial data.

In the study of the mechanical problem (P), we consider the following hypotheses $\mathcal{A}:\Omega\times\mathbb{S}_n\to\mathbb{S}_n$ satisfies the following properties:

(a) There exists an $L_{\mathcal{G}}>0$ such that $|\mathcal{A}(x,\varepsilon_1)-\mathcal{A}(x,\varepsilon_2)|\le L_{\mathcal{A}}|\varepsilon_1-\varepsilon_2|$ for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}_n$ a.e. $x\in\Omega$;
(b) There exists an $m_{\mathcal{A}}$ such that $(\mathcal{A}(x,\varepsilon_1)-\mathcal{A}(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\ge m_{\mathcal{A}}|\varepsilon_1-\varepsilon_2|^2$ for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}_n$ a.e. $x\in\Omega$; (2.10)
(c) The mapping $x\mapsto\mathcal{A}(x,\varepsilon)$ is Lebesgue measurable on Ω for all $\varepsilon\in\mathbb{S}_n$;
(d) The mapping $x\mapsto\mathcal{A}(x,0)\in\mathcal{H}$.

$\mathcal{E}:\Omega\times\mathbb{S}_n\to\mathbb{S}_n$ satisfies the following properties:

(a) There exists an $L_{\mathcal{E}}>0$ such that $|\mathcal{E}(x,\varepsilon_1)-\mathcal{E}(x,\varepsilon_2)|\le L_{\mathcal{E}}|\varepsilon_1-\varepsilon_2|$ for all $\varepsilon_1,\varepsilon_2\in\mathbb{S}_n$ a.e. $x\in\Omega$;
(b) The mapping $x\mapsto\mathcal{E}(x,\varepsilon)$ is Lebesgue measurable on Ω for all $\varepsilon\in\mathbb{S}_n$; (2.11)
(c) The mapping $x\mapsto\mathcal{E}(x,0)\in\mathcal{H}$.

$\mathcal{G} : \Omega \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}_n$ satisifes the following properties:

(a) There exists an $L_{\mathcal{G}} > 0$ such that $|\mathcal{G}(x, \sigma_1, \varepsilon_1, \theta_1, \varsigma_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2, \theta_2, \varsigma_2)| \leq L_{\mathcal{G}}(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\varsigma_1 - \varsigma_2|)$ for all $\sigma_1, \sigma_2 \in \mathbb{S}_n$, for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}_n$, for all $\theta_1, \theta_2 \in \mathbb{R}$, for all $\varsigma_1, \varsigma_2 \in \mathbb{R}$ a.e. $x \in \Omega$; (2.12)
(b) The mapping $x \to \mathcal{G}(\mathbf{x}, \sigma, \varepsilon, \theta, \varsigma)$ is Lebesgue measurable on Ω for all $\sigma, \varepsilon \in \mathbb{S}_n$, for all $\theta, \varsigma \in \mathbb{R}$;
(c) The mapping $x \to \mathcal{G}(x, 0, 0, 0, 0) \in \mathcal{H}$.

$\psi : \Omega \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following properties:

(a) There exists an $L_\psi > 0$ such that $|\psi(x, \sigma_1, \varepsilon_1, \theta_1, \varsigma_1) - \psi(x, \sigma_2, \varepsilon_2, \theta_2, \varsigma_2)| \leq L_\psi(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\varsigma_1 - \varsigma_2|)$ for all $\sigma_1, \sigma_2 \in \mathbb{S}_n$, for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}_n$, for all $\theta_1, \theta_2 \in \mathbb{R}$, for all $\varsigma_1, \varsigma_2 \in \mathbb{R}$ a.e. $x \in \Omega$; (2.13)
(b) The mapping $x \to \psi(x, \sigma, \varepsilon, \theta, \varsigma)$ is Lebesgue measurable on Ω for all $\sigma, \varepsilon \in \mathbb{S}_n$, for all $\theta, \varsigma \in \mathbb{R}$;
(c) The mapping $x \to \psi(x, 0, 0, 0, 0) \in L^2(\Omega)$.

$\phi : \Omega \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following properties:

(a) There exists an $L_\phi > 0$ such that $|\phi(x, \sigma_1, \varepsilon_1, \theta_1, \varsigma_1) - \phi(x, \sigma_2, \varepsilon_2, \theta_2, \varsigma_2)| \leq L_\phi(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\varsigma_1 - \varsigma_2|)$ for all $\sigma_1, \sigma_2 \in \mathbb{S}_n$, for all $\varepsilon_1, \varepsilon_2 \in \mathbb{S}_n$, for all $\theta_1, \theta_2 \in \mathbb{R}$, for all $\varsigma_1, \varsigma_2 \in \mathbb{R}$ a.e. $x \in \Omega$; (2.14)
(b) The mapping $x \mapsto \phi(x, \sigma, \varepsilon, \theta, \varsigma)$ is Lebesgue measurable on Ω for all $\sigma, \varepsilon \in \mathbb{S}_n$, for all $\theta, \varsigma \in \mathbb{R}$;
(c) The mapping $x \mapsto \phi(x, 0, 0, 0, 0) \in L^2(\Omega)$.

$$\begin{gathered} \rho \in L^\infty(\Omega), \quad \rho \geq \rho^* > 0. \\ \mathbf{f} \in L^2(0, T; H), \quad \mathbf{f}_0 \in L^2(0, T; L^2(\Gamma_2)^n). \\ q \in L^2(0, T; L^2(\Omega)). \end{gathered} \tag{2.15}$$

$$\mathbf{u}_0 \in \mathcal{V}, \quad \mathbf{w}_0 \in H, \quad \theta_0 \in V, \quad \varsigma_0 \in K. \tag{2.16}$$

$$k_i > 0, \quad i = 0, 1. \tag{2.17}$$

We denote by $\mathbf{F}(t) \in \mathcal{V}'$ the following element

$$\langle \mathbf{F}(t), \mathbf{v} \rangle_{\mathcal{V}' \times \mathcal{V}} = (\mathbf{f}(t), \mathbf{v})_H + (\mathbf{f}_0(t), \gamma \mathbf{v})_{L^2(\Gamma_2)^n} \quad \forall \mathbf{v} \in \mathcal{V}, \ \ t \in (0, T). \tag{2.18}$$

The use of (2.15) permits to verify that

$$\mathbf{F} \in L^2(0, T; \mathcal{V}'). \tag{2.19}$$

We introduce the following continuous functionals

$$\mathfrak{a}_0 : V \times V \to \mathbb{R}, \quad \mathfrak{a}_0(\zeta, \xi) = k_0 \int_\Omega \nabla \zeta \cdot \nabla \xi dx + \beta \int_\Gamma \zeta \xi d\gamma, \tag{2.20}$$

$$\mathfrak{a}_1 : V \times V \to \mathbb{R}, \quad \mathfrak{a}_1(\zeta, \xi) = k_1 \int_\Omega \nabla \zeta \cdot \nabla \xi dx. \tag{2.21}$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem (P).

Problem PV. Find the displacement field $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^n$, the stress field $\sigma : \Omega \times (0,T) \to \mathbb{S}_n$, the temperature $\theta : \Omega \times (0,T) \to \mathbb{R}$ and the damage field $\varsigma : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned}\sigma(t) &= \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{E}(\varepsilon(\mathbf{u}(t))) \\ &\quad + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))), \varepsilon(\mathbf{u}(s)), \theta(s), \varsigma(s))ds \quad \text{a.e. } t \in (0,T),\end{aligned} \tag{2.22}$$

$$\langle \rho \ddot{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathcal{V}' \times \mathcal{V}} + (\sigma(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} = \langle \mathbf{F}(t), \mathbf{v} \rangle_{\mathcal{V}' \times \mathcal{V}} \quad \forall \mathbf{v} \in \mathcal{V}, \text{ a.e. } t \in (0,T), \tag{2.23}$$

$$\begin{aligned}&\langle \rho \dot{\theta}(t), \omega \rangle_{V' \times V} + \mathfrak{a}_0(\theta(t), \omega) \\ &= \langle \psi(\sigma(t), \varepsilon(\dot{\mathbf{u}}(t)), \theta(t), \varsigma(t)), \omega \rangle_{V' \times V} + (q(t), \omega)_{L^2(\Omega)} \\ &\quad \forall \omega \in V, \text{ a.e. } t \in (0,T),\end{aligned} \tag{2.24}$$

$$\begin{aligned}&\langle \rho \dot{\varsigma}(t), \xi - \varsigma(t) \rangle_{V' \times V} + \mathfrak{a}_1(\varsigma(t), \xi - \varsigma(t)) \\ &\geq \langle \phi(\sigma(t), \varepsilon(\mathbf{u}(t)), \theta(t), \varsigma(t)), \xi - \varsigma(t) \rangle_{V' \times V} \\ &\quad \forall \xi \in K, \text{ a.e. } t \in (0,T), \varsigma(t) \in K,\end{aligned} \tag{2.25}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{w}, \quad \theta(0) = \theta_0, \quad \varsigma(0) = \varsigma_0 \quad \text{in } \Omega. \tag{2.26}$$

3. Main Results

The main results are stated by the following theorems.

Theorem 3.1 (Existence and uniqueness). *Under assumptions* (2.10)-(2.17)*, there exists a unique solution* $\{\mathbf{u}, \sigma, \theta, \varsigma\}$ *to problem (PV). Moreover, the solution has the regularity*

$$\mathbf{u} \in \mathcal{C}^0(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;H), \tag{3.1}$$

$$\dot{\mathbf{u}} \in L^2(0,T;\mathcal{V}), \tag{3.2}$$

$$\ddot{\mathbf{u}} \in L^2(0,T;\mathcal{V}'), \tag{3.3}$$

$$\sigma \in L^2(0,T;\mathcal{H}), \tag{3.4}$$

$$\theta \in L^2(0,T;V) \cap \mathcal{C}^0(0,T;L^2(\Omega)), \tag{3.5}$$

$$\dot{\theta} \in L^2(0,T;V'), \tag{3.6}$$

$$\varsigma \in L^2(0,T;V) \cap \mathcal{C}^0(0,T;L^2(\Omega)), \tag{3.7}$$

$$\dot{\varsigma} \in L^2(0,T;V'). \tag{3.8}$$

The proof will be done in several steps. Based on classical arguments of functional analysis concerning variational problems, and Banach fixed point theorem.

First step. Take an arbitrary element

$$(\eta, \lambda, \mu) \in \mathbf{L}^2(0,T;\mathcal{V}' \times V' \times V'), \tag{3.9}$$

and consider the auxiliary problem.

Problem PV1$_{(\eta,\lambda,\mu)}$. Find the displacement field $\mathbf{u}_\eta : \Omega \times (0,T) \to \mathbb{R}^n$, the temperature $\theta_\lambda : \Omega \times (0,T) \to \mathbb{R}$ and the damage field $\varsigma_\mu : \Omega \times (0,T) \to \mathbb{R}$ which are solutions of the variational system

$$\begin{aligned}&\langle \rho \ddot{\mathbf{u}}_\eta(t), \mathbf{v} \rangle_{\mathcal{V}' \times \mathcal{V}} + (\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(t))), \varepsilon(\mathbf{v}))_{\mathcal{H}} + \langle \eta(t), \mathbf{v} \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle \mathbf{F}(t), \mathbf{v} \rangle_{\mathcal{V}' \times \mathcal{V}} \\ &\quad \forall \mathbf{v} \in \mathcal{V}, \text{ a.e. } t \in (0,T),\end{aligned} \tag{3.10}$$

$$\begin{gathered}\langle \rho\dot{\theta}_\lambda(t), \omega\rangle_{V'\times V} + \mathfrak{a}_0(\theta_\lambda(t), \omega) = \langle \lambda(t) + q(t), \omega\rangle_{V'\times V} \\ \forall \omega \in V, \text{ a.e. } t \in (0,T),\end{gathered} \tag{3.11}$$

$$\begin{gathered}\langle \rho\dot{\varsigma}_\mu(t), \xi - \varsigma_\mu(t)\rangle_{V'\times V} + \mathfrak{a}_1(\varsigma_\mu(t), \xi - \varsigma_\mu(t)) \\ \geq \langle \mu, \xi - \varsigma_\mu(t)\rangle_{V'\times V} \quad \forall \xi \in K, \text{ a.e. } t \in (0,T), \ \varsigma_\mu(t) \in K,\end{gathered} \tag{3.12}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}_\eta(0) = \mathbf{w}, \quad \theta_\lambda(0) = \theta_0, \quad \varsigma_\mu(0) = \varsigma_0 \quad \text{in } \Omega. \tag{3.13}$$

Lemma 3.2. *For all $(\eta, \lambda, \mu) \in L^2(0,T; \mathcal{V}' \times V' \times V')$, there exists a unique solution $\{\mathbf{u}_\eta, \theta_\lambda, \varsigma_\mu\}$ to the auxiliary problem $PV1_{(\eta,\lambda,\mu)}$ satisfying* (3.1)-(3.3) *and* (3.5)-(3.8).

Proof. Let us introduce the operator $A : \mathcal{V} \to \mathcal{V}'$,

$$\langle A\mathbf{u}, \mathbf{v}\rangle_{\mathcal{V}'\times\mathcal{V}} = (\mathcal{A}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}))_{\mathcal{H}}. \tag{3.14}$$

It follows from hypothesis (2.10) that

$$\|A\mathbf{u} - A\mathbf{v}\|_{\mathcal{V}'} \leq L_{\mathcal{A}}\|\mathbf{u} - \mathbf{v}\|_{\mathcal{V}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

Which proves that A is bounded and hemi-continuous on $\mathcal{V}$.

On the other hand, by (2.10) and Korn's inequality, we find for every $\mathbf{v} \in \mathcal{V}$,

$$\frac{\langle A\mathbf{v}, \mathbf{v}\rangle_{\mathcal{V}'\times\mathcal{V}}}{\|\mathbf{v}\|_{\mathcal{V}}} \geq C_0^2 m_{\mathcal{A}}\|\mathbf{v}\|_{\mathcal{V}}.$$

The passage to the limit in this inequality when $\|\mathbf{v}\|_{\mathcal{V}} \to +\infty$ implies that A is coercive in $\mathcal{V}$.

Next, by definition of A, the use of (2.10) and Korn's inequality permits also to obtain

$$\langle A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v}\rangle_{\mathcal{V}'\times\mathcal{V}} > C_0^2 m_{\mathcal{A}}\|\mathbf{u} - \mathbf{v}\|_{\mathcal{V}} \quad \text{if } \mathbf{u} \neq \mathbf{v}.$$

Then A is strict monotone. Therefore, (3.10) can be rewritten, making use the operator A, as follows

$$\rho\ddot{\mathbf{u}}_\eta(t) + A(\dot{\mathbf{u}}_\eta(t)) = \mathbf{F}_\eta(t) \quad \text{on } \mathcal{V}' \text{ a.e. } t \in (0,T), \tag{3.15}$$

where

$$\mathbf{F}_\eta(t) = \mathbf{F}(t) - \eta(t) \in \mathcal{V}'.$$

We recall that by (2.19) we have $\mathbf{F}_\eta \in L^2(0,T;\mathcal{V}')$. Kipping in mind that the operator A is strict monotone, hemi-continuous, bounded and coercive, then by using classical arguments of functional analysis concerning parabolic equations [7, 19] we can easily prove the existence and uniqueness of $\mathbf{w}_\eta$ satisfying

$$\mathbf{w}_\eta \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}^0(0,T;H), \tag{3.16}$$

$$\dot{\mathbf{w}}_\eta \in L^2(0,T;\mathcal{V}'), \tag{3.17}$$

$$\rho\dot{\mathbf{w}}_\eta(t) + A(\mathbf{w}_\eta(t)) = \mathbf{F}_\eta(t) \ \text{ on } \ \mathcal{V}' \ \text{ a.e. } t \in (0,T), \tag{3.18}$$

$$\mathbf{w}_\eta(0) = \mathbf{w}_0. \tag{3.19}$$

Consider now the function $\mathbf{u}_\eta : (0,T) \to \mathcal{V}$ defined by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{w}_\eta(s)ds + \mathbf{u}_0 \quad \forall t \in (0,T). \tag{3.20}$$

It follows from (3.18) and (3.19) that $\mathbf{u}_\eta$ is a solution of the equation (3.15) and it satisfies (3.1)-(3.3).

Furthermore, by an application of the Poincaré-Friedrichs inequality, we can find a constant $\beta' > 0$ such that

$$\int_\Omega |\nabla\zeta|^2 dx + \frac{\beta}{k_0}\int_\Gamma |\zeta|^2 d\gamma \geq \beta' \int_\Omega |\zeta|^2 dx \quad \forall \zeta \in V.$$

Thus, we obtain

$$\mathfrak{a}_0(\zeta,\zeta) \geq c_1 \|\zeta\|_V^2 \quad \forall \zeta \in V, \tag{3.21}$$

where $c_1 = k_0 \min(1,\beta')/2$, which implies that $\mathfrak{a}_0$ is $V-$elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations, the variational equation (3.11) has a unique solution θ_λ satisfies (3.5)-(3.6).

On the other hand, we know that the form $\mathfrak{a}_1$ is not V-elliptic. To solve this problem we introduce the functions

$$\tilde{\varsigma}_\mu(t) = e^{-k_1 t}\varsigma_\mu(t), \quad \tilde{\xi}(t) = e^{-k_1 t}\xi(t).$$

We remark that if ς_μ, $\xi \in K$ then $\tilde{\varsigma}_\mu$, $\tilde{\xi} \in K$. Consequently, (3.12) is equivalent to the inequality

$$\begin{aligned} &\langle \rho\dot{\tilde{\varsigma}}_\mu(t), \tilde{\xi} - \tilde{\varsigma}_\mu(t)\rangle_{V'\times V} + \mathfrak{a}_1(\tilde{\varsigma}_\mu(t), \tilde{\xi} - \tilde{\varsigma}_\mu(t)) + k_1(\rho\tilde{\varsigma}_\mu, \tilde{\xi} - \tilde{\varsigma}_\mu(t))_{L^2(\Omega)} \\ &\geq \langle e^{-k_1 t}\mu, \tilde{\xi} - \tilde{\varsigma}_\mu(t)\rangle_{V'\times V} \quad \forall \tilde{\xi} \in K, \text{ a.e. } t \in (0,T),\ \tilde{\varsigma}_\mu \in K. \end{aligned} \tag{3.22}$$

The fact that

$$\mathfrak{a}_1(\tilde{\xi},\tilde{\xi}) + k_1(\rho\tilde{\xi},\tilde{\xi})_{L^2(\Omega)} \geq k_1 \min(\rho^*, 1)\|\tilde{\xi}\|_V^2 \quad \forall \tilde{\xi} \in V, \tag{3.23}$$

and using classical arguments of functional analysis concerning parabolic inequalities [7, 10], implies that (3.22) has a unique solution $\tilde{\varsigma}_\mu$ having the regularity (3.7)-(3.8). This completes the proof . □

Let us consider now the auxiliary problem.

Problem PV2$_{(\eta,\lambda,\mu)}$. Find the stress field $\sigma_{\eta,\lambda,\mu} : \Omega \times (0,T) \to \mathbb{S}_n$ which is a solution of the problem

$$\begin{aligned} \sigma_{\eta,\lambda,\mu}(t) = \mathcal{E}(\varepsilon(\mathbf{u}_\eta(t))) + \int_0^t \mathcal{G}\Big(&\sigma_{\eta,\lambda,\mu}(s) \\ &- \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(s))), \varepsilon(\mathbf{u}_\eta(s)), \theta_\lambda(s), \varsigma_\mu(s)\Big) ds \quad \text{a.e. } t \in (0,T), \end{aligned} \tag{3.24}$$

Lemma 3.3. *There exists a unique solution of Problem PV2$_{(\eta,\lambda,\mu)}$ and it satisfies* (3.4). *Moreover, if* $\{\mathbf{u}_\eta, \theta_{\lambda_i}, \varsigma_{\mu_i}\}$ *and* $\sigma_{\eta,\lambda,\mu}$ *represent the solutions of problems PV1$_{(\eta_i,\lambda_i,\mu_i)}$ and PV2$_{(\eta_i,\lambda_i,\mu_i)}$, respectively, for* $i = 1,2$, *then there exists* $c > 0$ *such that*

$$\begin{aligned} &\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}}^2 \\ &\leq c\int_0^t \Big(\|\dot{\mathbf{u}}_{\eta_1}(s) - \dot{\mathbf{u}}_{\eta_2}(s)\|_{\mathcal{V}}^2 + \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_{\mathcal{V}}^2 \\ &\quad + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|_V^2 + \|\varsigma_{\mu_1}(s) - \varsigma_{\mu_2}(s)\|_V^2\Big) ds. \end{aligned} \tag{3.25}$$

Proof. Let $\Sigma_{\eta,\lambda,\mu} : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the mapping given by

$$\begin{aligned} &\Sigma_{\eta,\lambda,\mu}\sigma(t) \\ &= \mathcal{E}(\varepsilon(\mathbf{u}_\eta(t))) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(s))), \varepsilon(\mathbf{u}_\eta(s)), \theta_\lambda(s), \varsigma_\mu(s)) ds. \end{aligned} \tag{3.26}$$

Let $\sigma_i \in L^2(0,T;\mathcal{H})$, $i=1,2$ and $t_1 \in (0,T)$. We find be using hypothesis (2.12) and Hölder's inequality

$$\|\Sigma_{\eta,\lambda,\mu}\sigma_1(t_1) - \Sigma_{\eta,\lambda,\mu}\sigma_2(t_1)\|_{\mathcal{H}}^2 \le L_{\mathcal{G}}^2 T \int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds. \tag{3.27}$$

Integration on the time interval $(0,t_2) \subset (0,T)$, it follows that

$$\int_0^{t_2} \|\Sigma_{\eta,\lambda,\mu}\sigma_1(t_1) - \Sigma_{\eta,\lambda,\mu}\sigma_2(t_1)\|_{\mathcal{H}}^2 dt_1 \le L_{\mathcal{G}}^2 T \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds dt_1.$$

Using again (3.27), it follows that

$$\|\Sigma_{\eta,\lambda,\mu}\sigma_1(t_2) - \Sigma_{\eta,\lambda,\mu}\sigma_2(t_2)\|_{\mathcal{H}}^2 \le L_{\mathcal{G}}^4 T^2 \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds dt_1.$$

For $t_1, t_2, \dots, t_n \in (0,T)$, we generalize the procedure above by recurrence on n. We obtain the inequality

$$\begin{aligned}
&\|\Sigma_{\eta,\lambda,\mu}\sigma_1(t_n) - \Sigma_{\eta,\lambda,\mu}\sigma_2(t_n)\|_{\mathcal{H}}^2 \\
&\le L_{\mathcal{G}}^{2n} T^n \int_0^{tn} \dots \int_0^{t_2}\int_0^{t_1} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 \, ds\, dt_1 \dots dt_{n-1}.
\end{aligned}$$

Which implies

$$\|\Sigma_{\eta,\lambda,\mu}\sigma_1(t_n) - \Sigma_{\eta,\lambda,\mu}\sigma_2(t_n)\|_{\mathcal{H}}^2 \le \frac{L_{\mathcal{G}}^{2n} T^{n+1}}{n!} \int_0^{T} \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\|\Sigma_{\eta,\lambda,\mu}\sigma_1 - \Sigma_{\eta,\lambda,\mu}\sigma_2\|_{L^2(0,T;\mathcal{H})}^2 \le \frac{L_{\mathcal{G}}^{2n} T^{n+2}}{n!} \|\sigma_1 - \sigma_2\|_{L^2(0,T;\mathcal{H})}^2.$$

It follows from this inequality that for n large enough, a power n of the mapping $\Sigma_{\eta,\lambda,\mu}$ is a contraction on the space $L^2(0,T;\mathcal{H})$ and, therefore, from the Banach fixed point theorem, there exists a unique element $\sigma_{\eta,\lambda,\mu} \in L^2(0,T;\mathcal{H})$ such that $\Sigma_{\eta,\lambda,\mu}\sigma_{\eta,\lambda,\mu} = \sigma_{\eta,\lambda,\mu}$, which represents the unique solution of the problem $\mathrm{PV2}_{(\eta,\lambda,\mu)}$. Moreover, if $\{\mathbf{u}_\eta, \theta_{\lambda_i}, \varsigma_{\mu_i}\}$ and $\sigma_{\eta,\lambda,\mu}$ represent the solutions of problem $\mathrm{PV1}_{(\eta_i,\lambda_i,\mu_i)}$ and $\mathrm{PV2}_{(\eta_i,\lambda_i,\mu_i)}$, respectively, for $i=1,2$, then we use (2.10), (2.11), (2.12) and Young's inequality to obtain

$$\begin{aligned}
&\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}}^2 \\
&\le c\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}}^2 + c\int_0^t \Big(\|\dot{\mathbf{u}}_{\eta_1}(s) - \dot{\mathbf{u}}_{\eta_2}(s)\|_{\mathcal{V}}^2 \\
&\quad + \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_{\mathcal{V}}^2 + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|_V^2 + \|\varsigma_{\mu_1}(s) - \varsigma_{\mu_2}(s)\|_V^2\Big) ds.
\end{aligned}$$

Which permits us to obtain, using Gronwall's lemma, the inequality (3.25).
Second step. Let us consider the mapping

$$\Lambda : L^2(0,T;\mathcal{V}' \times V' \times V') \to L^2(0,T;\mathcal{V}' \times V' \times V'),$$

defined by

$$\begin{aligned}
&\Lambda(\eta(t),\lambda(t),\mu(t)) \\
&= \Big(\Lambda_0(\eta(t),\lambda(t),\mu(t)), \psi\big(\sigma_{\eta,\lambda,\mu}(t), \varepsilon(\dot{\mathbf{u}}_\eta(t)), \theta_\lambda(t), \varsigma_\mu(t)\big), \\
&\quad \phi\big(\sigma_{\eta,\lambda,\mu}(t), \varepsilon(\mathbf{u}_\eta(t)), \theta_\lambda(t), \varsigma_\mu(t)\big)\Big),
\end{aligned} \tag{3.28}$$

where the mapping Λ_0 is given by

$$\begin{aligned} &\langle \Lambda_0(\eta(t),\lambda(t),\mu(t)),\mathbf{v}\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &= \Big(\mathcal{E}(\varepsilon(\mathbf{u}_\eta(t))) + \int_0^t \mathcal{G}\Big(\sigma_{\eta,\lambda,\mu}(s) \\ &\quad - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(s))), \varepsilon(\mathbf{u}_\eta(s)), \theta_\lambda(s), \varsigma_\mu(s)\Big) ds, \varepsilon(\mathbf{v})\Big)_{\mathcal{H}}. \end{aligned} \tag{3.29}$$

□

Lemma 3.4. *The mapping* Λ *has a fixed point*

$$(\eta^*,\lambda^*,\mu^*) \in L^2(0,T;\mathcal{V}'\times V'\times V').$$

Proof. Let $t\in(0,T)$ and

$$(\eta_1,\lambda_1,\mu_1),(\eta_2,\lambda_2,\mu_2) \in L^2(0,T;\mathcal{V}'\times V'\times V').$$

Let us start by using hypotheses (2.10), (2.11) and (2.12) to obtain

$$\begin{aligned} &\|\Lambda_0(\eta_1(t),\lambda_1(t),\mu_1(t)) - \Lambda_0(\eta_2(t),\lambda_2(t),\mu_2(t))\|_{\mathcal{V}'} \\ &\le L_{\mathcal{E}}\|\mathbf{u}_{\eta_1}(t)-\mathbf{u}_{\eta_2}(t)\|_{\mathcal{V}} + L_{\mathcal{G}}\int_0^t \Big(\|\sigma_{\eta_1,\lambda_1,\mu_1}(s)-\sigma_{\eta_2,\lambda_2,\mu_2}(s)\|_{\mathcal{H}} \\ &\quad + L_{\mathcal{A}}\|\dot{\mathbf{u}}_{\eta_1}(s)-\dot{\mathbf{u}}_{\eta_2}(s)\|_{\mathcal{V}} + \|\mathbf{u}_{\eta_1}(s)-\mathbf{u}_{\eta_2}(s)\|_{\mathcal{V}} \\ &\quad + \|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|_{L^2(\Omega)} + \|\varsigma_{\mu_1}(s)-\varsigma_{\mu_2}(s)\|_{L^2(\Omega)}\Big) ds \quad \text{a.e. } t\in(0,T). \end{aligned} \tag{3.30}$$

On the other hand, we know that for a.e. $t\in(0,T)$,

$$\|\mathbf{u}_{\eta_1}(t)-\mathbf{u}_{\eta_2}(t)\|_{\mathcal{V}} \le \int_0^t \|\dot{\mathbf{u}}_{\eta_1}(s)-\dot{\mathbf{u}}_{\eta_2}(s)\|_{\mathcal{V}} ds. \tag{3.31}$$

Applying Young's and Hölder's inequalities, (3.30) becomes, via (3.31),

$$\begin{aligned} &\|\Lambda_0(\eta_1(t),\lambda_1(t),\mu_1(t)) - \Lambda_0(\eta_2(t),\lambda_2(t),\mu_2(t))\|^2_{\mathcal{V}'} \\ &\le c\int_0^t \Big(\|\sigma_{\eta_1,\lambda_1,\mu_1}(s)-\sigma_{\eta_2,\lambda_2,\mu_2}(s)\|^2_{\mathcal{H}} + \|\dot{\mathbf{u}}_{\eta_1}(s)-\dot{\mathbf{u}}_{\eta_2}(s)\|^2_{\mathcal{V}} \\ &\quad + \|\mathbf{u}_{\eta_1}(s)-\mathbf{u}_{\eta_2}(s)\|^2_{\mathcal{V}} + \|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|^2_V + \|\varsigma_{\mu_1}(s)-\varsigma_{\mu_2}(s)\|^2_V\Big) ds \end{aligned} \tag{3.32}$$

a.e. $t\in(0,T)$. Furthermore, we find by taking the substitution $\eta=\eta_1$, $\eta=\eta_2$ in (3.10) and choosing $\mathbf{v}=\dot{\mathbf{u}}_{\eta_1}-\dot{\mathbf{u}}_{\eta_2}$ as test function

$$\begin{aligned} &\langle \rho(\ddot{\mathbf{u}}_{\eta_1}(t)-\ddot{\mathbf{u}}_{\eta_2}(t)) + A\dot{\mathbf{u}}_{\eta_1}(t) - A\dot{\mathbf{u}}_{\eta_2}(t), \dot{\mathbf{u}}_{\eta_1}(t)-\dot{\mathbf{u}}_{\eta_2}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &= \langle \eta_2(t)-\eta_1(t), \dot{\mathbf{u}}_{\eta_1}(t)-\dot{\mathbf{u}}_{\eta_2}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} \quad \text{a.e. } t\in(0,T). \end{aligned}$$

By virtue of (2.10), this equation becomes

$$\begin{aligned} &\frac{(\rho^*)^2}{2}\frac{d}{dt}\|\dot{\mathbf{u}}_{\eta_1}(t)-\dot{\mathbf{u}}_{\eta_2}(t)\|^2_H + m_{\mathcal{A}}\|\dot{\mathbf{u}}_{\eta_1}(t)-\dot{\mathbf{u}}_{\eta_2}(t)\|^2_{\mathcal{V}} \\ &\le \|\eta_2(t)-\eta_1(t)\|_{\mathcal{V}'}\|\dot{\mathbf{u}}_{\eta_1}(t)-\dot{\mathbf{u}}_{\eta_2}(t)\|_{\mathcal{V}}. \end{aligned}$$

Integrating this inequality over the interval time variable $(0,t)$, Young inequality leads to

$$(\rho^*)^2\|\dot{\mathbf{u}}_{\eta_1}(t)-\dot{\mathbf{u}}_{\eta_2}(t)\|^2_H + m_{\mathcal{A}}\int_0^t \|\dot{\mathbf{u}}_{\eta_1}(s)-\dot{\mathbf{u}}_{\eta_2}(s)\|^2_{\mathcal{V}} ds$$

$$\le \frac{2}{m_{\mathcal{A}}} \int_0^t \|\eta_1(s) - \eta_2(s)\|^2_{\mathcal{V}'} ds.$$

Consequently,

$$\int_0^t \|\dot{\mathbf{u}}_{\eta_1}(s) - \dot{\mathbf{u}}_{\eta_2}(s)\|^2_{\mathcal{V}} ds \le c \int_0^t \|\eta_1(s) - \eta_2(s)\|^2_{\mathcal{V}'} ds \quad \text{a.e. } t \in (0,T). \tag{3.33}$$

which also implies, using a variant of (3.31), that

$$\|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|^2_{\mathcal{V}} \le c \int_0^t \|\eta_1(s) - \eta_2(s)\|^2_{\mathcal{V}'} ds \quad \text{a.e.} t \in (0,T), \tag{3.34}$$

Moreover, if we take the substitution $\lambda = \lambda_1$, $\lambda = \lambda_2$ in (3.11) and subtracting the two obtained equations, we deduce by choosing $\omega = \theta_{\lambda_1} - \theta_{\lambda_2}$ as test function

$$\begin{aligned} &\frac{(\rho^*)^2}{2} \|\theta_{\lambda_1}(t) - \theta_{\lambda_2}(t)\|^2_{L^2(\Omega)} + c_1 \int_0^t \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|^2_V ds \\ &\le \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{V'} \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|_V ds \quad \text{a.e. } t \in (0,T). \end{aligned}$$

Employing Hölder's and Young's inequalities, we deduce that

$$\begin{aligned} &\|\theta_{\lambda_1}(t) - \theta_{\lambda_2}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|^2_V ds \\ &\le c \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{V'} ds \quad \text{a.e. } t \in (0,T). \end{aligned} \tag{3.35}$$

Substituting now $\{\mu = \mu_1, \xi = \tilde{\varsigma}_{\mu_1}\}$, $\{\mu = \mu_2, \xi = \tilde{\varsigma}_{\mu_2}\}$ in (3.22) and subtracting the two inequalities, we obtain

$$\begin{aligned} &\|\tilde{\varsigma}_{\mu_1}(t) - \tilde{\varsigma}_{\mu_2}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\tilde{\varsigma}_{\mu_1}(t) - \tilde{\varsigma}_{\mu_2}(t)\|^2_V ds \\ &\le c \int_0^t \|e^{-k_1 t}(\mu_1(s) - \mu_2(s))\|^2_{V'} ds \;\; \text{a.e. } t \in (0,T), \end{aligned}$$

from which also follows that

$$\begin{aligned} &\|\varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\varsigma_{\mu_1}(s) - \varsigma_{\mu_2}(s)\|^2_V ds \\ &\le c \int_0^t \|\mu_1(s) - \mu_2(s)\|^2_{V'} ds \;\; \text{a.e. } t \in (0,T), \end{aligned} \tag{3.36}$$

We can infer, using (3.25), (3.32), (3.33), (3.35) and (3.36), that

$$\begin{aligned} &\int_0^t \|\Lambda_0(\eta_1(s), \lambda_1(s), \mu_1(s)) - \Lambda_0(\eta_2(s), \lambda_2(s), \mu_2(s))\|^2_{\mathcal{V}'} ds \\ &\le c \int_0^t \int_0^s \Big(\|\dot{\mathbf{u}}_{\eta_1}(r) - \dot{\mathbf{u}}_{\eta_2}(r)\|^2_{\mathcal{V}} + \|\theta_{\lambda_1}(r) - \theta_{\lambda_2}(r)\|^2_V \\ &\quad + \|\mathbf{u}_{\eta_1}(r) - \mathbf{u}_{\eta_2}(r)\|^2_{\mathcal{V}} + \|\varsigma_{\mu_1}(r) - \varsigma_{\mu_2}(r)\|^2_V \Big) dr\, ds \quad \text{a.e. } t \in (0,T) \\ &\le c \int_0^T \int_0^T \Big(\|\dot{\mathbf{u}}_{\eta_1}(r) - \dot{\mathbf{u}}_{\eta_2}(r)\|^2_{\mathcal{V}} + \|\theta_{\lambda_1}(r) - \theta_{\lambda_2}(r)\|^2_V \\ &\quad + \|\mathbf{u}_{\eta_1}(r) - \mathbf{u}_{\eta_2}(r)\|^2_{\mathcal{V}} + \|\varsigma_{\mu_1}(r) - \varsigma_{\mu_2}(r)\|^2_V \Big) dr\, ds \end{aligned}$$

$$
\begin{aligned}
&\le c\int_0^T \Big(\|\dot{\mathbf{u}}_{\eta_1}(s)-\dot{\mathbf{u}}_{\eta_2}(s)\|_{\mathcal{V}}^2\\
&\quad + \|\mathbf{u}_{\eta_1}(s)-\mathbf{u}_{\eta_2}(s)\|_{\mathcal{V}}^2 + \|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|_V^2 + \|\varsigma_{\mu_1}(s)-\varsigma_{\mu_2}(s)\|_V^2\Big)ds\\
&\le c\int_0^T \Big(\|\eta_1(s)-\eta_2(s)\|_{\mathcal{V}'}^2 + \|\lambda_1(s)-\lambda_2(s)\|_{V'}^2 + \|\mu_1(s)-\mu_2(s)\|_{V'}^2\\
&\quad + \|\mathbf{u}_{\eta_1}(s)-\mathbf{u}_{\eta_2}(s)\|_{\mathcal{V}}^2\Big)ds
\end{aligned}
$$

Thus, by (3.34), we find

$$
\begin{aligned}
&\int_0^T \|\Lambda_0(\eta_1(s),\lambda_1(s),\mu_1(s)) - \Lambda_0(\eta_2(s),\lambda_2(s),\mu_2(s))\|_{\mathcal{V}'}^2 ds\\
&\le c\int_0^T \Big(\|\eta_1(s)-\eta_2(s)\|_{\mathcal{V}'}^2 + \|\lambda_1(s)-\lambda_2(s)\|_{V'}^2 + \|\mu_1(s)-\mu_2(s)\|_{V'}^2\Big)ds\,.
\end{aligned}
\tag{3.37}
$$

Furthermore, hypothesis (2.13) implies

$$
\begin{aligned}
&\int_0^t \|\psi\Big(\sigma_{\eta_1,\lambda_1,\mu_1}(s),\varepsilon(\dot{\mathbf{u}}_{\eta_1}(s)),\theta_{\lambda_1}(s),\varsigma_{\mu_1}(s)\Big)\\
&-\psi\Big(\sigma_{\eta_2,\lambda_2,\mu_2}(s),\varepsilon(\dot{\mathbf{u}}_{\eta_2}(s)),\theta_{\lambda_2}(s),\varsigma_{\mu_2}(s)\Big)\|_{V'}^2 ds\\
&\le 3L_\psi^2\int_0^t \Big(\|\sigma_{\eta_1,\lambda_1,\mu_1}(s)-\sigma_{\eta_2,\lambda_2,\mu_2}(s)\|_{\mathcal{H}}^2 + \|\dot{\mathbf{u}}_{\eta_1}(s)-\dot{\mathbf{u}}_{\eta_2}(s)\|_{\mathcal{V}}^2\\
&\quad + \|\theta_{\lambda_1}(s)-\theta_{\lambda_2}(s)\|_V^2 + \|\varsigma_{\mu_1}(t)-\varsigma_{\mu_2}(t)\|_V^2\Big)ds \quad \text{a.e. } t\in(0,T).
\end{aligned}
$$

This permits us to deduce, via (3.25), (3.33), (3.35) and (3.36), that

$$
\begin{aligned}
&\int_0^T \|\psi\Big(\sigma_{\eta_1,\lambda_1,\mu_1}(s),\varepsilon(\dot{\mathbf{u}}_{\eta_1}(s)),\theta_{\lambda_1}(s),\varsigma_{\mu_1}(s)\Big)\\
&-\psi\Big(\sigma_{\eta_2,\lambda_2,\mu_2}(s),\varepsilon(\dot{\mathbf{u}}_{\eta_2}(s)),\theta_{\lambda_2}(s),\varsigma_{\mu_2}(s)\Big)\|_{V'}^2 ds\\
&\le c\int_0^T \Big(\|\eta_1(s)-\eta_2(s)\|_{\mathcal{V}'}^2 + \|\lambda_1(s)-\lambda_2(s)\|_{V'}^2 + \|\mu_1(s)-\mu_2(s)\|_{V'}^2\Big)ds
\end{aligned}
\tag{3.38}
$$

Similarly, using (3.25), (3.34), (3.35) and (3.36), we obtain the following estimate for ϕ,

$$
\begin{aligned}
&\int_0^T \|\phi\Big(\sigma_{\eta_1,\lambda_1,\mu_1}(s),\varepsilon(\mathbf{u}_{\eta_1}(s)),\theta_{\lambda_1}(s),\varsigma_{\mu_1}(s)\Big)\\
&-\phi\Big(\sigma_{\eta_2,\lambda_2,\mu_2}(s),\varepsilon(\mathbf{u}_{\eta_2}(s)),\theta_{\lambda_2}(s),\varsigma_{\mu_2}(s)\Big)\|_{V'}^2 ds\\
&\le c\int_0^T \Big(\|\eta_1(s)-\eta_2(s)\|_{\mathcal{V}'}^2 + \|\lambda_1(s)-\lambda_2(s)\|_{V'}^2 + \|\mu_1(s)-\mu_2(s)\|_{V'}^2\Big)ds.
\end{aligned}
\tag{3.39}
$$

From (3.37), (3.38) and (3.39), we conclude that there exists a positive constant $C>0$ verifying

$$
\begin{aligned}
&\|\Lambda(\eta_1,\lambda_1,\mu_1)-\Lambda(\eta_2,\lambda_2,\mu_2)\|_{L^2(0,T;\mathcal{V}'\times V'\times V')}\\
&\le C\|(\eta_1-\eta_2,\lambda_1-\lambda_2,\mu_1-\mu_2)\|_{L^2(0,T;\mathcal{V}'\times V'\times V')},
\end{aligned}
\tag{3.40}
$$

and so, by reapplication of mapping $\mathbf{\Lambda}$, yields

$$
\|\Lambda^2(\eta_1,\lambda_1,\mu_1)-\Lambda^2(\eta_2,\lambda_2,\mu_2)\|_{L^2(0,T;\mathcal{V}'\times V'\times V')}
$$

$$\le \frac{C^2}{2!}\|(\eta_1-\eta_2,\lambda_1-\lambda_2,\mu_1-\mu_2)\|_{L^2(0,T;\mathcal{V}'\times V'\times V')}.$$

We generalize this procedure by recurrence on n. Then we obtain the formula

$$\begin{aligned}&\|\Lambda^n(\eta_1,\lambda_1,\mu_1)-\Lambda^n(\eta_2,\lambda_2,\mu_2)\|_{L^2(0,T;\mathcal{V}'\times V'\times V')}\\&\le \frac{C^n}{n!}\|(\eta_1-\eta_2,\lambda_1-\lambda_2,\mu_1-\mu_2)\|_{L^2(0,T;\mathcal{V}'\times V'\times V')}.\end{aligned} \tag{3.41}$$

We know that the sequence $(C^n/n!)_n$ converges to 0. So, for n sufficiently large $\frac{C^n}{n!}<1$. It means that a large power n of the operator Λ is a contraction on $L^2(0,T;\mathcal{V}'\times V'\times V')$. Hence, Banach fixed point theorem shows that Λ admits a unique fixed point $(\eta^*,\lambda^*,\mu^*)\in L^2(0,T;\mathcal{V}'\times V'\times V')$.

We can now prove the existence of a solution to problem (PV). To this aim, it is sufficient to remark that for a.e. $t\in(0,T)$,

$$\begin{gathered}\mathcal{E}(\varepsilon(\mathbf{u}_{\eta^*}(t)))+\int_0^t\mathcal{G}\Big(\sigma_{\eta^*,\lambda^*,\mu^*}(s)-\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_{\eta^*}(s))),\varepsilon(\mathbf{u}_{\eta^*}(s)),\theta_{\lambda^*}(s),\varsigma_{\mu^*}(s)\Big)ds\\=\eta^*(t),\\ \psi(\varepsilon(\mathbf{u}_{\eta^*}(t)),\theta_{\lambda^*}(t),\varsigma_{\mu^*}(t))=\lambda^*(t),\\ \phi(\varepsilon(\mathbf{u}_{\eta^*}(t)),\theta_{\lambda^*}(t),\varsigma_{\mu^*}(t))=\mu^*(t),\end{gathered}$$

which completes the proof. □

Theorem 3.5 (Positivity of the temperature)**.** *Let the hypotheses of Theorem 3.1 hold and suppose in addition that*

$$\psi(\sigma,\varepsilon(\mathbf{u}),\theta,\varsigma)\ge 0 \quad \text{a.e. in } \Omega\times(0,T), \tag{3.42}$$

$$q\ge 0 \quad \text{a.e. in } \Omega\times(0,T), \tag{3.43}$$

$$\theta_0\ge 0 \quad \text{a.e. in } \Omega\times(0,T). \tag{3.44}$$

Then, the solution $\{\mathbf{u},\sigma,\theta,\varsigma\}$ to problem (PV) is such that

$$\theta(x,t)\ge 0 \quad \text{for a.e. } (x,t)\in\Omega\times(0,T). \tag{3.45}$$

Proof. We use a maximum principle argument [5]. Thus, we test the equation (2.24) by the function $-\theta^-$, where f^- denoting the so-called negative part of a function f; i.e., $f^-=\max\{0,-f\}$, and integrate over $(0,T)$. We can infer, using the hypothesis (3.42), (3.43) and (3.44), that

$$\begin{aligned}&\frac{1}{2}(\rho^*)^2\|\theta^-\|^2_{L^\infty(0,T;L^2(\Omega))}+c_1\|\theta^-\|^2_{L^2(0,T;V)}\\&\le-\int_0^t\int_\Omega\psi(\varepsilon(\mathbf{u}(x,s)),\theta(x,s),\varsigma(x,s))\theta^-(x,s)\,dx\,ds\\&\quad-\int_0^t\int_\Omega q(x,s)\theta^-(x,s)\,dx\,ds\le 0\quad \text{a.e. } t\in(0,T).\end{aligned}$$

Consequently

$$\|\theta^-\|_{L^2(0,T;V)\cap L^\infty(0,T;L^2(\Omega))}\le 0,$$

which eventually gives (3.45). □

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