CONSTANT INVARIANT SOLUTIONS OF THE POINCARÉ CENTER-FOCUS PROBLEM

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ABSTRACT. We consider the classical Poincaré problem
$$\frac{dx}{dt} = -y - p(x,y), \quad \frac{dy}{dt} = x + q(x,y)$$
where p, q are homogeneous polynomials of degree $n \geq 2$. Associated with this system is an Abel differential equation
$$\frac{d\rho}{d\theta} = \psi_3 \rho^3 + \psi_2 \rho^2$$
in which the coefficients are trigonometric polynomials. We investigate two separate conditions which produce a constant first absolute invariant of this equation. One of these conditions leads to a new class of integrable, center conditions for the Poincaré problem if $n \geq 9$ is an odd integer. We also show that both classes of solutions produce polynomial solutions to the problem.

1. INTRODUCTION

We consider the problem of finding center conditions for the critical point $(0,0)$ for the system
$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y)$$
where P and Q are polynomials satisfying $P(0,0) = Q(0,0) = 0$. That is, we seek to find conditions on P and Q such that all trajectories within a sufficiently small neighborhood of the origin will be closed. In this work we look at the particular system
$$\frac{dx}{dt} = -y - p(x,y), \quad \frac{dy}{dt} = x + q(x,y) \tag{1.1}$$
where p and q are homogeneous polynomials of degree $n \geq 2$. This problem has been the focus of much research since it was first formulated by Poincaré [10].

Corresponding to (1.1) is the first order differential equation
$$\frac{dy}{dx} = -\frac{x + q(x,y)}{y + p(x,y)}. \tag{1.2}$$

Using a polar coordinate transformation, this is transformed into

$$\frac{dr}{d\theta} = \frac{\xi(\theta)\,r(\theta)^n}{1 + \eta(\theta)\,r(\theta)^{n-1}} \tag{1.3}$$

where ξ, η are trigonometric polynomials of degree $n + 1$. This can be further transformed [5] to an Abel equation of the first kind

$$\frac{d\rho}{d\theta} = -(n-1)\xi(\theta)\eta(\theta)\rho^3 + ((n-1)\xi(\theta) - \eta'(\theta))\rho^2. \tag{1.4}$$

Standard center conditions for a general n are given in terms of (1.3), (1.4) by

$$\xi(\theta) = K\eta'(\theta) \tag{1.5}$$

and by $r(\theta_0 - \theta) = r(\theta - \theta_0)$ (the *symmetric or time reversible centers*). In these $K \neq 0$ and θ_0 are constants. In a recent paper [9] the author was able to demonstrate a class of centers of (1.1)–(1.2) by appealing to certain parity properties of the solution of (1.4).

More generally, if (1.2) has a solution expressible in the form $U(x, y) = C$ where U is analytic on a neighborhood of the origin and C is a constant, the origin will be a center. The search for integrability conditions for (1.2), usually in the form of an integrating factor, is one method of determining center conditions.

In this article we consider solutions of (1.3), (1.4). The work is motivated by the fact that if the first absolute invariant I_1 of the Abel equation (1.4) is constant, then the equation is solvable. If the coefficient functions ξ, η satisfy (1.5), it is straightforward to show that the invariant is constant. We mention this case for completeness purposes. Beyond this, we knew from previous work that other constant invariant conditions for (1.4) exist. In particular, the complete symmetric case for $n = 3$, which does not generally satisfy the condition (1.5), is integrable in terms of a constant invariant Abel equation. Also, the symmetric cases for $n = 5, 7, 9, 11$, while not generally having constant invariant I_1, contain subcases that are. It is our purpose here to give a more compete description of these cases, and in so doing, give new, integrable center conditions for (1.1)–(1.2).

In the next section we review the derivation of eqs. (1.3), (1.4) as well as some aspects of the Abel differential equation. In particular, we define the relative invariants of an Abel equation and indicate that if the first absolute invariant I_1 obtained from these is constant, then the equation can be transformed to a separable equation. We mention a particular form for which a constant I_1 can always be obtained and indicate how this would apply to the form obtained from (1.3)–(1.5). We next look at an integrating factor for (1.3) and use the previously obtained results to give a precise form for it.

In section 4 we show that we obtain a constant invariant form for (1.4) which does not satisfy (1.5). This form occurs only when $n \geq 5$ is an odd integer. Using the integrating factor, we obtain the solutions of (1.3) and show that they are analytic on a neighborhood of the origin. These solutions are seen to generalize, to any odd integer, previously given solutions for the symmetric case of $n = 3$. In the next section we develop families of polynomial solutions from each of the constant invariant cases of (1.4) discussed in this paper. In the last section we discuss some of the reasons which led us to assume the particular form of the integrating factor of (1.3).

2. Aspects of the Abel differential equation

If we set $x = r\cos\theta$, $y = r\sin\theta$ in (1.2), we obtain the polar form (1.3) where

$$\xi(\theta) = \sin\theta\, q(\cos\theta, \sin\theta) - \cos\theta\, p(\cos\theta, \sin\theta),$$
$$\eta(\theta) = \sin\theta\, p(\cos\theta, \sin\theta) + \cos\theta\, q(\cos\theta, \sin\theta) \tag{2.1}$$

are homogeneous trigonometric polynomials of degree $n+1$ in $\cos\theta, \sin\theta$. Throughout the remainder of the paper we shall refer to the degree of a homogeneous, trigonometric polynomial as the total degree of $\cos\theta, \sin\theta$ in each term. Following [5], we can further transform (1.3) by setting

$$\rho(\theta) = \frac{r(\theta)^{n-1}}{1 + \eta(\theta)r(\theta)^{n-1}}.$$

This gives (1.4)

$$\frac{d\rho}{d\theta} = -(n-1)\xi(\theta)\eta(\theta)\rho^3 + ((n-1)\xi(\theta) - \eta'(\theta))\rho^2.$$

Now that we have obtained the form of the equation (1.4) being considered, we briefly review certain properties of Abel equations. For a general Abel equation of the first kind $y' = g_3(x)y^3 + g_2(x)y^2 + g_1(x)y + g_0(x)$, it is possible to define recursively an infinite sequence of relative invariants by [4]

$$s_3(x) = g_0(x)g_3^2(x) + \frac{2}{27}g_2^3(x) + \frac{1}{3}(g_3(x)g_2'(x) - g_2(x)g_3'(x) - g_1(x)g_2(x)g_3(x)) \tag{2.2}$$

and

$$s_{2k+1}(x) = g_3(x)s_{2k-1}'(x) + (2k-1)(\frac{1}{3}g_2^2(x) - g_3'(x) - g_1(x)g_3(x))s_{2k-1}(x) \tag{2.3}$$

for $k \geq 2$. From these, a sequence of absolute invariants can be formed. If the first invariant $I_1 = s_5^3/s_3^5$ is constant, the Abel equation can be transformed to a separable equation. This is the only general class of Abel equations which is integrable by quadrature. We note that if $s_3(x) = 0$, the Abel equation is transformable to Bernoulli equation. The values of the parameters for which this will occur will be obvious in what follows, so we do not specifically consider this possibility in this paper.

A sufficient condition that an Abel equation of the form (1.4) has a constant first invariant I_1 is that the coefficient functions satisfy a relation of the form

$$(\frac{g_3(x)}{g_2(x)})' = Cg_2(x) \tag{2.4}$$

where C is a constant. This gives

$$I_1 = \frac{729(1 - 3C)^3}{(9\,C - 2)^2}. \tag{2.5}$$

It is clearly evident that if the coefficient functions of (1.4) satisfy (1.5), the invariant is constant. It should be noted however, that K and η in (1.5) cannot be chosen in an arbitrary fashion. We must have either $K = -1/(n+1)$ (the Hamiltonian condition) or η such that terms of the form $e^{\pm i(n+1)\theta}$ are absent. Otherwise, in the latter case, the functions p, q defined from (2.1) will not be polynomials.

The constant invariant condition associated with (1.5) is well known and the condition itself can be derived in a number of different ways. In the following we seek a more complete characterization of other conditions which produce a constant

invariant I_1 but for which (2.4) is not satisfied. The basis for carrying out this work is prior knowledge of certain conditions of this type.

3. An integrating factor

The constant invariant case defined by (1.5) is valid for any integer $n \geq 2$. For the next case we require that $n \geq 5$ be an odd integer. This is necessitated by the fact that the expressions for ξ, η that we shall obtain involve even powers and this can occur only if n is odd. We begin by considering an integrating factor for the symmetric $n = 3$ case defined by

$$p(x,y) = a_{21}x^2 y + a_{03}y^3, \quad q(x,y) = b_{30}x^3 + b_{12}xy^2.$$

Such an expression for the polar coordinate form (1.3) is given by

$$\mu(r,\theta) = \frac{r}{1 + r^2 f_1(\theta) + r^4 f_2(\theta)} \tag{3.1}$$

where

$$f_1(\theta) = (a_{21} - b_{12})\cos^2\theta + \frac{2a_{03}b_{30} - b_{12}a_{21} + a_{03}b_{12} - b_{12}^2 - a_{03}a_{21}}{a_{03} - a_{21} - b_{12} + b_{30}}$$

and

$$f_2(\theta) = (a_{03}b_{30} - a_{21}b_{12})\Big(\cos^4\theta + \frac{a_{21} + b_{12} - 2a_{03}}{a_{03} - a_{21} - b_{12} + b_{30}}\cos^2\theta$$
$$+ \frac{a_{03}}{a_{03} - a_{21} - b_{12} + b_{30}} \Big).$$

In particular, we find that

$$\eta(\theta) = \frac{a_{03} - a_{21} - b_{12} + b_{30}}{a_{03}b_{30} - a_{21}b_{12}} f_2(\theta) \tag{3.2}$$

where η is given by (2.1).

Based on earlier work involving the constant invariant cases for $n = 5, 7, 9, 11$, we modify the form given by (3.1) to

$$\mu(r,\theta) = \frac{r^{2\alpha+1}}{1 + r^{n-1} f_1(\theta) + r^{2(n-1)} f_2(\theta)} \tag{3.3}$$

where α is a value dependent on n and f_1, f_2 are trigonometric polynomials of degree $n - 1$ and $2(n-1)$ respectively. (As we shall see, the actual degrees of f_1, f_2 are less than these values.) The form (3.3) is not particularly suitable for finding conditions for f_1, f_2 so we set $V = 1/\mu$ in the general condition for exactness

$$\frac{\partial}{\partial r}(\mu r^n \xi) = -\frac{\partial}{\partial \theta}(\mu(1 + r^{n-1}\eta))$$

to obtain

$$r^n \xi \frac{\partial V}{\partial r} + (1 + r^{n-1}\eta)\frac{\partial V}{\partial \theta} - r^{n-1}(n\xi + \eta')V = 0.$$

Substituting the appropriate form of (3.3) in this and collecting powers of r yields the system of equations

$$(n + 2\alpha + 1)\xi + \eta' - f_1' = 0 \tag{3.4}$$

$$2(\alpha + 1)\xi f_1 - f_2' - \eta f_1' + \eta' f_1 = 0 \tag{3.5}$$

$$(n - 2\alpha - 3)\xi f_2 + \eta f_2' - \eta' f_2 = 0 \tag{3.6}$$

which the functions ξ, η, f_1, f_2 must satisfy.

In the process of solving (3.4)–(3.6), we shall introduce two assumptions which will produce sufficient conditions for a solution to be found. They are both based on the known forms of (3.3) for previously obtained integrating factors. The first of these is to assume that we can write

$$\eta' = \lambda f_1', \qquad \xi = \psi f_1' \tag{3.7}$$

where λ, ψ are trigonometric polynomials to be determined. The coefficient functions in (3.1) and ξ, η for the symmetric $n = 3$ case also satisfy (3.7).

The conditions (3.7) may seem rather artificial, however we have found by studying other integrating factors for integrable forms of (1.2)–(1.3) that they frequently satisfy relations of a similar type. For example, the functions $f_1(\theta) = 2a(\cos\theta)^{n-1} - 2b(\sin\theta)^{n-1}$ and η for the system

$$p(x, y) = ax^{n-1}y - 2bx^2y^{n-2} + by^n, \quad q(x, y) = ax^n - 2ax^{n-2}y^2 + bxy^{n-1}$$

obtained in [1] satisfy a relation of the form $\eta = \lambda f_1$ where $\lambda = (1/2)\cos 2\theta$. An integrating factor for this system is given by

$$\mu(x, y) = (1 + 2(ax^{n-1} + by^{n-1}) + (ax^{n-1} - by^{n-1})^2)^{-(n+3)/(2n-2)}.$$

We believe that this or similar structure is inherent in most (if not all) integrating factors and that it can be used to facilitate the development other integrable forms of (1.2).

If (3.7) is satisfied then (3.4) reduces to the algebraic condition

$$(n + 2\alpha + 1)\psi + \lambda = 1. \tag{3.8}$$

The second condition is to note that

$$\alpha = \frac{1}{2}(n - 3) \tag{3.9}$$

for these integrating factors. With this (3.6) becomes elementary and furnishes the relation

$$f_2 = A\eta \tag{3.10}$$

which corresponds to (3.2). Here $A \neq 0$ is a constant. Now substituting (3.3), (3.7) and (3.8) in (3.5) we obtain the relation

$$\eta = ((n - 1)\psi + \lambda)f_1 - A\lambda.$$

Differentiating this and using (3.7), (3.8) with (3.9) gives a differential equation for f_1

$$(1 - \lambda)f_1' + \lambda' f_1 = 2A\lambda'.$$

This can be solved to give

$$f_1 = 2B(\lambda - 1) + 2A \tag{3.11}$$

where $B \neq 0$ (the 2 is added for later convenience) is a constant.

From the preceding we can see that all functions can be expressed in terms of λ which can be chosen in an arbitrary fashion. Let

$$N = \left[\frac{1}{4}(n - 1)\right] \tag{3.12}$$

where $[\dots]$ is the greatest integer function and define

$$\lambda(\theta) = a_0 + \sum_{k=1}^{N}(a_{2k}\cos 2k\theta + b_{2k}\sin 2k\theta) \tag{3.13}$$

where a_0 and a_{2k}, b_{2k} are arbitrary real numbers such that λ is not constant. In terms of this function we find that

$$\xi = \frac{B}{n-1}(1-\lambda)\lambda', \quad \eta = B\lambda^2 + A - B. \tag{3.14}$$

As we noted earlier the degree of f_1 given by (3.11) is only $2N$ rather than the maximal value $n - 1$ that it could have in more general circumstances.

4. The Abel equation and related integrals

With ξ, η given by (3.14) we can now show that the trigonometric form (1.4) has constant invariant. Using λ as independent variable, it becomes

$$\frac{d\rho}{d\lambda} = B(\lambda - 1)(B\lambda^2 + A - B)\rho^3 + B(1 - 3\lambda)\rho^2. \tag{4.1}$$

This equation is quite simple and is rather unusual in that both relative invariants s_3, s_5 defined by (2.2), (2.3) are constant. Denoting the coefficients of ρ^2, ρ^3 in (4.1) by g_2, g_3 respectively, we find

$$s_3 = \frac{2}{27}g_2^3 + \frac{1}{3}(g_3 g_2' - g_2 g_3') = \frac{2}{27}B^2(9A - 8B)$$

and

$$s_5 = (g_2^2 - 3g_3')s_3 = \frac{2}{27}B^3(4B - 3A)(9A - 8B).$$

where the primes denote differentiation with respect to λ. From these we have

$$I_1 = \frac{s_5^3}{s_3^5} = \frac{729}{4}\frac{(4B - 3A)^3}{B(9A - 8B)^2}. \tag{4.2}$$

Now that we have established that (1.4) is solvable if (3.14) is satisfied, we proceed to show that these conditions do produce new center conditions for (1.1)–(1.2). To do this we shall obtain the solution of (1.3) using the integrating factor (3.3). Since the cartesian form of μ is not positive on a neighborhood of the origin, we will need to establish that the solutions themselves are analytic there.

Substituting for f_1, f_2 in terms of λ and again using λ as independent variable, we have

$$\begin{aligned}
U(r, \theta) &= \int \mu(r, \theta)\xi(\theta)r^n \, d\theta + F(r) \\
&= \int \frac{B(1-\lambda)r^{2n-2} \, d\lambda}{1 + 2(B(\lambda - 1) + A)r^{n-1} + A(B\lambda^2 + A - B)r^{2n-2}} + F(r) \\
&= -B \int \frac{r^{2n-2}x \, dx}{ABr^{2n-2}x^2 + 2B(Ar^{n-1} + 1)r^{n-1}x + (Ar^{n-1} + 1)^2} + F(r)
\end{aligned}$$

where $x = \lambda - 1$ and F is a suitably chosen function of r. The discriminant of the denominator is given by

$$\Delta = 4B(B - A)(Ar^{n-1} + 1)^2 r^{2n-2}.$$

Within the context of the center–focus problem, we require that a solution of (1.3) satisfy $1 + r^{n-1}\eta > 0$. Based on this, we can assume that the sign of Δ is determined by the sign of $B(B - A)$. This leads to three separate forms for the solution $U(r, \theta) = C$.

We have

$$U(r,\theta) = (B - K)\ln|AB\lambda r^{n-1} - KAr^{n-1} + B - K|$$
$$- (B + K)\ln|AB\lambda r^{n-1} + KAr^{n-1} + B + K| \quad \text{if } \Delta > 0, \tag{4.3}$$

where $K = \sqrt{B(B - A)}$;

$$U(r,\theta) = \ln\Big(ABr^{2n-2}\lambda^2 + 2Br^{n-1}\lambda + A(A - B)r^{2n-2}$$
$$+ 2(A - B)r^{n-1} + 1\Big) - 2K\tan^{-1}(K\frac{Ar^{n-1}\lambda + 1}{Ar^{n-1} + 1}) \quad \text{if } \Delta < 0, \tag{4.4}$$

where $K = \sqrt{B/(A - B)}$; and

$$U(r,\theta) = \ln(Ar^{n-1}\lambda + 1) + \frac{Ar^{n-1} + 1}{Ar^{n-1}\lambda + 1} \quad \text{if } \Delta = 0. \tag{4.5}$$

We can easily see that each of these solutions is analytic at $r = 0$, so its corresponding cartesian form is analytic on a neighborhood of the critical point $(0,0)$ of (1.1). This means that the functions ξ, η defined in (3.14) do produce centers of (1.1)–(1.2) for any odd integer $n \geq 5$. Also, these solutions generalize to any odd value of n the solutions for the symmetric case for $n = 3$ given in [8] for the cartesian case and in [2] for the polar case.

The centers defined for the cases $n = 5, 7$ ($N = 1$) are symmetric centers because there always exists a transformation $\theta \to \theta + \theta_0$ for some constant θ_0 such that λ will become an even function. The functions ξ, η in (3.14) will then be respectively odd, even which are exactly the conditions for the symmetric case. For $n \geq 9$ it is no longer generally possible to transform λ in this fashion, so the center conditions defined in this way are of a type not previously recorded.

We note for future reference that the $\Delta = 0$ case of (4.5) is satisfied by the parameter choice $A = B$. Substituting this in the invariant (4.2), we find that $I_1 = 729/4$. From this it is easy to see that (4.3), (4.4) give $I_1 > 729/4$, $I_1 < 729/4$ respectively.

We summarize the results of the last 2 sections in the following proposition.

Proposition 4.1. *Let ξ, η be given by (3.14) where λ is defined by (3.12), (3.13). Then the trigonometric Abel differential equation (1.4) has constant first absolute invariant $I_1 = s_5^3/s_3^5$ given by (4.2).*

Corollary 4.2. *Let ξ, η be given by (3.14) where λ is defined by (3.12), (3.13). Then the system (1.1)–(1.2) has a center at the origin for odd integers $n \geq 5$. For $n = 5, 7$ the centers are symmetric centers and for $n \geq 9$ the centers are new. Each of these center conditions is integrable and integrals for them are given in polar form by (4.3)-(4.5).*

5. Polynomial solutions

The usual Poincaré solution of (1.2) can be expressed in the form

$$U(x,y) = U_2(x,y) + \sum_{k=3}^{\infty} U_k(x,y) = C, \tag{5.1}$$

where $U_2(x,y) = \frac{1}{2}(x^2 + y^2)$ and $U_k(x,y)$ are homogeneous polynomials of degree $k(n-1) + 2$. If the series terminates for some positive integer $k = M \geq 2$, the solution is a polynomial $\mathcal{P}(x,y)$. We can further note that any integral power

$N_1 \geq 2$ of $\mathcal{P}$ will again be a polynomial solution, but $\mathcal{P}^{N_1}$ will not satisfy a relation of the basic form (5.1). Polynomial solutions have been discussed by other authors [3, 6], but they have restricted their discussion to the case for which $N_1 = 1$. However, in the course of this and other related work, we have found a number of families of solutions whose series solutions terminate if we take some power $U^{N_1}, N_1 \geq 2$ of the form (5.1), but do not if the requirement $N_1 = 1$ is imposed. Both constant invariant forms of (1.4) which are discussed in this paper include polynomial solutions of this type.

Suppose ξ, η satisfy (1.5). Then (1.3) has an elementary integrating factor of the form $\mu(r) = r^\alpha$ where

$$\alpha = -\frac{1}{K}(1 + nK). \tag{5.2}$$

The solution is easily given by

$$r^{\alpha+1}\Big[1 + \frac{1+\alpha}{n+\alpha}\eta(\theta)r^{n-1}\Big] = C \tag{5.3}$$

where C is a constant. If terms of degree $n + 1$ are absent in η, the term in the square bracket apart from the 1 is a homogeneous polynomial of degree $n - 1$ in x, y. So if $\alpha + 1$ is a positive rational number, (5.3) is a polynomial in x, y or is easily converted to one by taking a suitable power of the expression. In the case $\alpha = 1$ $(K = -1/(n+1))$, η can be chosen to have degree $n + 1$ and (5.3) gives the Hamiltonian solution. An integral of the type (5.3) was given in [7].

The second class of polynomial solutions can be obtained directly from (4.3). The maximal degree of λ in the arguments of the logarithmic terms is $2N$ where N is given by (3.12). So we can write $r^{n-1}\lambda = r^{n-1-2N}(r^{2N}\lambda)$. From this it is obvious that these terms, apart from the constants $(B \pm K)$, are homogeneous polynomials of degree $n - 1$ in x, y. If the coefficients of the logarithmic terms are negative rational numbers, $-R_1, -R_2$, the solution can be transformed to a polynomial. Setting $B - K = -R_1, -(B + K) = -R_2$ where $K = \sqrt{B(B - A)}$ and solving gives

$$A = \frac{2R_1 R_2}{R_1 - R_2}, \quad B = \frac{1}{2}(R_2 - R_1), \quad K = \frac{1}{2}(R_1 + R_2). \tag{5.4}$$

If $R_1 = P_1/Q_1, R_2 = P_2/Q_2$ for positive integers $P_1, \ldots, Q_2$ are given in reduced form, the polynomial solution is

$$U(r,\theta) = |AB\lambda r^{n-1} - KAr^{n-1} + B - K|^{N_1}|AB\lambda r^{n-1} + KAr^{n-1} + B + K|^{N_2} \tag{5.5}$$

where A, B, K are given by (5.4). In this $N_1 = P_1 Q_2/M, N_2 = P_2 Q_1/M$ where M is the greatest common divisor of $P_1 Q_2, P_2 Q_1$.

We now give without proof a cartesian form of the polynomials given by (5.5). This second form was derived using a somewhat different approach (a different integrating factor is involved), but the two sets are equivalent in that they both can be shown to produce an integrating factor of the form (3.3). We believe that this form is more useful than that given by (5.5) and it is also valid for $n = 3$. All parameters in this new form are unrelated to those previously used except that $n \geq 3$ is still an odd integer.

Let $m = (n-1)/2$ and define

$$\beta = \begin{cases} \frac{1}{2}m & \text{if } m \text{ is even} \\ \frac{1}{2}(m+1) & \text{if } m \text{ is odd and } n \neq 3 \\ 0 & \text{if } n = 3. \end{cases}$$

Let

$$r(x,y) = \sum_{k=0}^{n-1-2\beta} a_k x^{n-1-2\beta-k} y^k$$

be a homogeneous polynomial of degree $n - 1 - 2\beta$. The polynomial solution is given by

$$\mathcal{P}(x,y) = ((x^2 + y^2)^\beta r(x,y) + 1)^a (K(x^2 + y^2)^m - \alpha(x^2 + y^2)^\beta r(x,y) + 1)^b \quad (5.6)$$

where $\alpha = a/b$ for positive integers a, b and $K \neq 0$ is a constant. Differentiating (5.6), we find that $\mathcal{P}(x,y)$ satisfies (1.2) where

$$
\begin{aligned}
p(x,y) &= \frac{\alpha\beta + m}{m} y(x^2 + y^2)^\beta r(x,y) + \frac{\alpha}{2m}(x^2 + y^2)^{\beta+1} \frac{\partial r}{\partial y} \\
&\quad - \frac{\alpha(\alpha+1)}{2mK}(x^2 + y^2)^{2\beta-m}((x^2 + y^2)\frac{\partial r}{\partial y} + 2\beta\, y\, r(x,y))r(x,y)
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
q(x,y) &= \frac{\alpha\beta + m}{m} x(x^2 + y^2)^\beta r(x,y) + \frac{\alpha}{2m}(x^2 + y^2)^{\beta+1} \frac{\partial r}{\partial x} \\
&\quad - \frac{\alpha(\alpha+1)}{2mK}(x^2 + y^2)^{2\beta-m}((x^2 + y^2)\frac{\partial r}{\partial x} + 2\beta\, x\, r(x,y))r(x,y)
\end{aligned}
\tag{5.8}
$$

are homogeneous polynomials of degree n. We note that these forms remain valid for the case $n = 3$ ($\beta = 0, m = 1$).

Polynomials of the types discussed here have appeared previously. Case(IV) of [6, Theorem 7] (when suitably corrected) and Case (I) of [6, Theorem 6] are examples of the constant invariant case defined by (5.3). For $n = 3$ the polynomial solution (5.6) is

$$\mathcal{P}(x,y) = (a_0 x^2 + a_1 xy + a_2 y^2 + 1)^a (K(x^2 + y^2) - \alpha(a_0 x^2 + a_1 xy + a_2 y^2) + 1)^b. \quad (5.9)$$

We believe that the expression given in the Appendix of [3] should be of this type.

6. Discussion

In this section we briefly discuss the motivation which led to the results given in this paper. It has primarily to do with the values which the first absolute invariant I_1 of a constant invariant Abel equation can take on. If an Abel equation is of type constant invariant, the solution must have a particular structure: essentially that given by (4.3)-(4.5).

We start with the constant invariant form given by (1.5),(5.3). This leads to a form of (1.4) for which the value C in (2.4) is given by $C = -(n-1)K/(Kn-K-1)^2$. Solving (5.2) for K and substituting in the value for C, we find that

$$I_1 - \frac{729}{4} = \frac{19683}{4} \frac{(n-1)^2(\alpha+1)^2(\alpha+n)^2}{(2n-1+\alpha)^2(n+1+2\alpha)^2(n-\alpha-2)^2}$$

which is obviously nonnegative ($\alpha = n - 2$ leads to a Bernoulli equation) for the range of parameter values, $n \geq 2, \alpha > -1$, necessary for a polynomial solution.

Moreover, since $\alpha = -1, -n$ are not allowed values in the solution (5.3) this expression is always positive. So, within the constraints of the Poincaré problem defined by (1.3)–(1.5), it is not possible to obtain solutions of the form (4.4), (4.5). We conclude the discussion of these centers by mentioning (although not central to the theme of this paper) that many of the solutions (5.3) for the Bernoulli form $\alpha = n-2$ are *isochronic*. That is, all trajectories of (1.1) within a sufficiently small neighborhood of the origin have the same minimal period.

The second class of constant invariant centers arose by extending conditions for $n = 3$ symmetric centers. We had learned that this case is solvable in terms of a constant invariant Abel equation whose invariant can take on any value. This led us to search for other constant invariant conditions of this type. By using Maple to solve an extensive system of nonlinear equations, we found constant invariant conditions for $n = 5, 7, 9, 11$. In these, even though restricted to the symmetric case, the invariant could take any value, unlike the case of the previous class of centers. It was the solution of these cases along with the integrating factor (3.1) and the solutions (5.6) which produced the form of the integrating factor (3.3), the conditions (3.7) and the value of α (3.9).

The polynomial solutions (5.6) defined by (5.7),(5.8) lead to an Abel equation whose first invariant I_1 satisfies

$$I_1 - \frac{729}{4} = \frac{19683}{4} \frac{\alpha^2(\alpha+1)^2}{(\alpha-1)^2(2\alpha+1)^2(\alpha+2)^2}.$$

This is similar to the case of the polynomials defined by (1.5). Furthermore, these solutions do not generally belong to the symmetric class of centers for $n \geq 9$. So, because of this and the discussion in previous paragraph, we felt that there were likely constant invariant forms of a non–symmetric type for the odd integers which would produce invariant values $I_1 \leq 729/4$.

The results of this paper give sufficient conditions for (1.4) to have a constant first invariant. We do not know if there are other constant invariant conditions, but we think it unlikely. We do believe, however, that it is probable that the conditions defined by (3.14) are special cases of more general, integrable center conditions. We hope to consider this and other applications of conditions like those in (3.7) in upcoming work.

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