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A BOUNDARY VALUE PROBLEM OF FRACTIONAL ORDER AT RESONANCE

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Abstract. We establish solvability of a boundary value problem for a nonlinear differential equation of fractional order by means of the coincidence degree theory.

1. Introduction

This article is a study of the boundary value problem of fractional order with non-local conditions

$$\mathcal{D}^{\alpha}u(t) = f(t, u(t), u'(t)), \quad \text{a. e. } t \in (0,1),$$
$$\mathcal{D}_{0+}^{\alpha-2}u(0) = 0, \quad \eta u(\xi) = u(1),$$

where $1 < \alpha < 2$, $0 < \xi < 1$ and $\eta\xi^{\alpha-1} = 1$. It will be shown that, with the present choice of boundary conditions, the boundary value problem is at resonance. We apply a well-known degree theory theorem for coincidences due to Mawhin [16].

The monographs [10, 20, 21, 22] are commonly cited for the theory of fractional derivatives and integrals and applications to differential equations of fractional order. Contributions to the theory of initial and boundary value problems for nonlinear differential equations of fractional order have been made by several authors including a recent monograph [13] and the papers [1, 2, 9, 15, 24]. Although an application of the coincidence degree theory to a fractional order problem is not known to the author, we can account for several results that have been devoted to both theoretical developments [5, 17, 19] and applications [23] to various types of boundary and initial value problems. A broad range of scenarios of resonant problems were studied in the framework of ordinary differential and difference equations [17] (more generally, dynamic equations on time scales [3, 11]) on bounded and unbounded [12] domains with periodic [18], non-local boundary conditions [4, 6, 7, 8, 23] as well as boundary value problems with impulses [14].

2. Technical preliminaries

We start out by introducing the reader to the fundamental tools of fractional calculus and the coincidence degree theory.

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The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^p[0,1]$, $1 \le p < \infty$, is the integral

$$\mathcal{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds. \tag{2.1}$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n = [\alpha] + 1$, is defined by

$$\mathcal{D}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds. \tag{2.2}$$

Let $AC[0,1]$ denote the space of absolutely continuous functions on the interval $[0,1]$ and $AC^n[0,1] = \{u \in AC[0,1] : u^{(n)} \in AC[0,1]\}$, $n = 0,1,2,\dots$. We make use of several relationships between (2.1) and (2.2) that are stated in the next two theorems (see [10, 20, 22]).

Theorem 2.1. (a) *The equality $\mathcal{D}^\alpha \mathcal{I}^\alpha g = g$ holds for every $g \in L^1[0,1]$;*
(b) *For $u \in L^1[0,1]$, $n = [\alpha] + 1$, $\beta > 0$, if $\mathcal{I}^{n-\alpha} u \in AC^{n-1}[0,1]$, then*

$$\mathcal{I}^\beta \mathcal{D}^\alpha u(t) = \mathcal{D}^{\alpha-\beta} u(t) - \sum_{k=0}^{n-1} \frac{t^{\beta-k-1}}{\Gamma(\beta-k)} \Big(\frac{d^{n-k-1}}{dt^{n-k-1}} \mathcal{I}^{n-\alpha} u \Big)(0).$$

For $\alpha < 0$, we introduce the notation $\mathcal{I}^\alpha = \mathcal{D}^{-\alpha}$.

Theorem 2.2. *If $\beta, \alpha + \beta > 0$ and $g \in L^1[0,1]$, then the equality*

$$\mathcal{I}^\alpha \mathcal{I}^\beta g \,. = \mathcal{I}^{\alpha+\beta} g$$

Definition 2.3. Let X and Z be real normed spaces. A linear mapping $L : \operatorname{dom} L \subset X \to Z$ is called a Fredholm mapping if the following two conditions hold:

(i) $\ker L$ has a finite dimension, and
(ii) $\operatorname{Im} L$ is closed and has a finite codimension.

If L is a Fredholm mapping, its (Fredholm) *index* is the integer $\operatorname{Ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L$.

In this note we are concerned with a Fredholm mapping of index zero. From Definition 2.3 it follows that there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} Q$$

and that the mapping

$$L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is one-to-one and onto. The inverse of $L|_{\operatorname{dom} L \cap \ker P}$ we denote by $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$.

If L is a Fredholm mapping of index zero, then, for every isomorphism $J : \operatorname{Im} Q \to \ker L$, the mapping $JQ + K_{P,Q} : Z \to \operatorname{dom} L$ is an isomorphism and, for every $u \in \operatorname{dom} L$,

$$(JQ + K_{P,Q})^{-1} u = (L + J^{-1} P) u.$$

Definition 2.4. Let $L : \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N : E \to Z$ be a mapping. We say that N is L-compact on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are continuous and compact on E. In addition, we say, that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

The existence of a solution of the equation $Lu = Nu$ will be shown using [16, Theorem IV.13].

Theorem 2.5. *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

- (i) *$Lu \neq \lambda Nu$ for every $(u, \lambda) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times (0, 1)$;*
- (ii) *$Nu \notin \operatorname{Im} L$ for every $u \in \ker L \cap \partial\Omega$;*
- (iii) *$\deg(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$, with $Q : Z \to Z$ a continuous projector such that $\ker Q = \operatorname{Im} L$ and $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.*

Then the equation $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Suppose now that the function f satisfies the Carathéodory conditions with respect to $L^p[0, 1]$, $p \geq 1$; that is, the following conditions hold:

- (C1) for each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t, z)$ is Lebesgue measurable;
- (C2) for a. e. $t \in [0, 1]$, the mapping $z \mapsto f(t, z)$ is continuous on $\mathbb{R}^n$;
- (C3) for each $r > 0$, there exists a nonnegative $\phi_r \in L^p[0, 1]$ such that, for a. e. $t \in [0, 1]$ and every z such that $|z| \leq r$, we have $|f(t, z)| \leq \phi_r(t)$.

3. Main results

Consider the differential equation

$$\mathcal{D}^\alpha u(t) = f(t, u(t), u'(t)), \quad \text{a. e. } t \in (0, 1), \tag{3.1}$$

of fractional order $1 < \alpha < 2$, subject to the boundary conditions

$$\mathcal{D}^{\alpha-2} u(0) = 0, \tag{3.2}$$

$$\eta u(\xi) = u(1), \tag{3.3}$$

where $0 < \xi < 1$ and

$$\eta \xi^{\alpha-1} = 1. \tag{3.4}$$

We let the following assumption stand throughout this article:

- (P) $p > \frac{1}{\alpha-1}$ and $q = \frac{p}{p-1}$.

Let $AC_{\text{loc}}(0, 1]$ be the space consisting of functions that are absolutely continuous on every interval $[a, 1] \subset (0, 1]$. We introduce the space

$$X_0 = \{u : u \in AC[0, 1],\ u' \in AC_{\text{loc}}(0, 1],\ \mathcal{D}^\alpha u \in L^p[0, 1]\}.$$

Let

$$X = \{u \in C[0, 1] \cap C^1(0, 1] : \lim_{t \to 0^+} t^{2-\alpha} u'(t) \text{ exists}\}$$

with the weighted norm $\|u\| = \max\{\|u\|_0, \|t^{2-\alpha}u'\|_0\}$, where $\|\cdot\|_0$ is the max-norm and $\|t^{2-\alpha}v\|_0 = \sup_{t \in (0,1]} |t^{2-\alpha}v(t)|$. Let $Z = L^p[0, 1]$ with the usual norm $\|\cdot\|_p$, where p satisfies (P). Define the mapping $L : \operatorname{dom} L \subset X \to Z$ with

$$\operatorname{dom} L = \{u \in X_0 : u \text{ satisfies (3.2) and (3.3)}\}$$

and $Lu(t) = \mathcal{D}^\alpha u(t)$.

Define the mapping $N : X \to Z$ by
$$Nu(t) = f(t, u(t), u'(t)).$$

Lemma 3.1. *The mapping $L : \operatorname{dom} L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. It is easy to see that $\ker L = \{ct^{\alpha-1} : c \in \mathbb{R}\}$. We claim that
$$\operatorname{Im} L = \{g \in Z : \eta \mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)\}.$$
Let $g \in Z$ and
$$u(t) = \mathcal{I}^\alpha g(t) + ct^{\alpha-1}, \quad c \in \mathbb{R}.$$
Then $\mathcal{D}^\alpha u(t) = g(t)$, a. e. in $(0,1)$. By Theorem 2.2,
$$\begin{aligned}\mathcal{D}^{\alpha-2}u(t) &= \mathcal{I}^{2-\alpha}u(t)\\ &= \mathcal{I}^{2-\alpha}\mathcal{I}^\alpha g(t) + c\mathcal{I}^{2-\alpha}(t^{\alpha-1})\\ &= \mathcal{I}^2 g(t) + c\Gamma(\alpha)t,\end{aligned}$$
so that $\mathcal{D}^{\alpha-2}u(0) = 0$. One can readily verify that, in view of (3.4), u satisfies (3.3) provided $\eta\mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)$. It is obvious that $u \in AC[0,1]$. Then u' exists, for a. e. $t \in (0,1]$, and, by Theorem 2.2,
$$u'(t) = \mathcal{I}^{\alpha-1}g(t) + c(\alpha-1)t^{\alpha-2}.$$
Moreover,
$$\lim_{t\to 0^+} t^{2-\alpha}u'(t) = c(\alpha-1)$$
since
$$\lim_{t\to 0^+} t^{2-\alpha}|\mathcal{I}^{\alpha-1}g(t)| \le \lim_{t\to 0^+} \frac{t^{1/q}\|g\|_p}{\Gamma(\alpha-1)((\alpha-2)q+1)^{1/q}} = 0.$$
Let $t_1, t_2 \in (0,1)$ and $t_1 < t_2$. Then
$$\begin{aligned}&|\mathcal{I}^{\alpha-1}g(t_2) - \mathcal{I}^{\alpha-1}g(t_1)|\\ &= \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_2}(t_2-s)^{\alpha-2}g(s)\,ds - \int_0^{t_1}(t_1-s)^{\alpha-2}g(s)\,ds\Big|\\ &= \frac{1}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}g(s)\,ds + \int_0^{t_1}\big((t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}\big)g(s)\,ds\Big|\\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}|g(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big((t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}\big)|g(s)|\,ds\\ &\le C_1(t_2-t_1)^{\alpha-2+\frac{1}{q}}\|g\|_p + C_1\Big[\int_0^{t_1}\big((t_1-s)^{\alpha-2} - (t_2-s)^{\alpha-2}\big)^q\,ds\Big]^{1/q}\|g\|_p\\ &\le C_1(t_2-t_1)^{\alpha-2+\frac{1}{q}}\|g\|_p + C_1\Big[\int_0^{t_1}\Big((t_1-s)^{(\alpha-2)q} - (t_2-s)^{(\alpha-2)q}\Big)\,ds\Big]^{1/q}\|g\|_p\\ &\le C_1(t_2-t_1)^{\alpha-2+\frac{1}{q}}\|g\|_p\\ &\quad + C_1\Big(t_1^{(\alpha-2)q+1} - t_2^{(\alpha-2)q+1} + (t_2-t_1)^{(\alpha-2)q+1}\Big)^{1/q}\|g\|_p,\end{aligned}$$
where C_1 is a generic constant that depends only on α and p. Thus, $u' \in AC_{\text{loc}}(0,1]$. Combining the preceding observations, we obtain that $u \in \operatorname{dom} L$. So, $\{g \in Z : \eta\mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)\} \subseteq \operatorname{Im} L$.

Let $u \in \operatorname{dom} L$. Then, for $\mathcal{D}^\alpha u \in \operatorname{Im} L$, we have, by Theorem 2.1(b) and (3.2),

$$\mathcal{I}^\alpha \mathcal{D}^\alpha u(t) = u(t) - \frac{\mathcal{D}^{\alpha-1}u(0)}{\Gamma(\alpha)} t^{\alpha-1} - \frac{\mathcal{D}^{\alpha-2}u(0)}{\Gamma(\alpha-1)} t^{\alpha-2} = u(t) - \frac{\mathcal{D}^{\alpha-1}u(0)}{\Gamma(\alpha)} t^{\alpha-1},$$

which, due to the boundary conditions (3.2), (3.3) together with (3.4), implies that $\mathcal{D}^\alpha u$ satisfies $\eta \mathcal{I}^\alpha \mathcal{D}^\alpha u(\xi) = \mathcal{I}^\alpha \mathcal{D}^\alpha u(1)$. Hence, $\operatorname{Im} L \subseteq \{g \in Z : \eta \mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)\}$. Therefore, $\operatorname{Im} L = \{g \in Z : \eta \mathcal{I}^\alpha g(\xi) = \mathcal{I}^\alpha g(1)\}$.

Define $Q : Z \to Z$ by

$$Qg(t) = \kappa \left(\eta \mathcal{I}^\alpha g(\xi) - \mathcal{I}^\alpha g(1)\right) t^{\alpha-1},$$

where

$$\kappa = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)(\xi^\alpha - 1)}.$$

Then

$$\begin{aligned} Q^2 g(t) &= \kappa \left(\eta \mathcal{I}^\alpha Qg(\xi) - \mathcal{I}^\alpha Qg(1)\right) t^{\alpha-1} \\ &= \kappa\Big(\frac{\eta}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} Qg(s)\, ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} Qg(s)\, ds\Big) t^{\alpha-1} \\ &= \kappa\Big(\frac{\eta}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} s^{\alpha-1}\, ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{\alpha-1}\, ds\Big) Qg(t) \\ &= \kappa \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (\eta \xi^{2\alpha-1} - 1) Qg(t) \\ &= Qg(t) \end{aligned}$$

in view of (3.4). Therefore, $Q : Z \to Z$ is a continuous linear projector with $\operatorname{Ker} Q = \operatorname{Im} L$.

Let $g \in Z$ be written as $g = (g - Qg) + Qg$ with $g - Qg \in \operatorname{Ker} Q = \operatorname{Im} L$ and $Qg \in \operatorname{Im} Q$. Hence, $Z = \operatorname{Im} L + \operatorname{Im} Q$. Let $g \in \operatorname{Im} L \cap \operatorname{Im} Q$ and set $g(t) = ct^{\alpha-1}$ to obtain that

$$0 = \gamma \mathcal{I}^\alpha g(\xi) - \mathcal{I}^\alpha g(1) = \frac{c\Gamma(\alpha)}{\Gamma(2\alpha)} (\eta \xi^{2\alpha-1} - 1) = \frac{c}{\kappa},$$

which implies that $c = 0$. Hence $\{0\} = \operatorname{Im} L \cap \operatorname{Im} Q$ and so $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Note that $\operatorname{Ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L = 0$; that is, L is a Fredholm mapping of index zero. □

Define $P : X \to X$ by

$$Pu(t) = \frac{1}{\Gamma(\alpha)} \mathcal{D}^{\alpha-1} u(0) t^{\alpha-1}.$$

Since $0 < \alpha - 1 < 1$,

$$\mathcal{D}^{\alpha-1} u(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{1-\alpha} u(s)\, ds.$$

Then

$$\begin{aligned} P^2 u(t) &= \frac{1}{\Gamma(\alpha)} \mathcal{D}^{\alpha-1} (Pu)(0) t^{\alpha-1} \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{1}{\Gamma(\alpha)} \Big(\frac{d}{dt} \int_0^t (t-s)^{1-\alpha} s^{\alpha-1}\, ds\Big)\Big|_{t=0} Pu(t) \\ &= Pu(t). \end{aligned}$$

We have that $P : X \to X$ is a continuous linear projector. Note that $\ker P = \{u \in X : \mathcal{D}^{\alpha-1}u(0) = 0\}$. For $u \in X$,

$$\|Pu\|_0 = \frac{1}{\Gamma(\alpha)}|\mathcal{D}^{\alpha-1}u(0)|$$

and

$$\|t^{2-\alpha}(Pu)'\|_0 = \frac{1}{\Gamma(\alpha-1)}|\mathcal{D}^{\alpha-1}u(0)|.$$

Hence,

$$\|Pu\| = \frac{1}{\Gamma(\alpha)}|\mathcal{D}^{\alpha-1}u(0)|. \tag{3.5}$$

Define $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ by

$$K_P g(t) = \mathcal{I}^\alpha g(t), \quad t \in (0,1).$$

For $g \in \operatorname{Im} L$,

$$LK_P g(t) = \mathcal{D}^\alpha \mathcal{I}^\alpha g(t) = g(t)$$

by Theorem 2.1(a). For $u \in \operatorname{dom} L \cap \ker P$, we have $\mathcal{D}^{\alpha-2}u(0) = 0$ and $\mathcal{D}^{\alpha-1}u(0) = 0$. Hence, by Theorem 2.1(b),

$$\begin{aligned} K_P Lu(t) &= \mathcal{I}^\alpha \mathcal{D}^\alpha u(t) \\ &= u(t) - \frac{\mathcal{D}^{\alpha-1}u(0)}{\Gamma(\alpha)}t^{\alpha-1} - \frac{\mathcal{D}^{\alpha-2}u(0)}{\Gamma(\alpha-1)}t^{\alpha-2} \\ &= u(t). \end{aligned}$$

Thus,

$$K_P = \Big(L|_{\operatorname{dom} L \cap \ker P}\Big)^{-1}.$$

Furthermore, using (P), we have

$$\begin{aligned} \|t^{2-\alpha}(K_P g)'\|_0 &= \max_{t\in(0,1]} \left|t^{2-\alpha}(K_P g)'(t)\right| \\ &\le \max_{t\in(0,1]} \frac{t^{2-\alpha}}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}|g(s)|\,ds \\ &\le \max_{t\in(0,1]} \frac{t^{2-\alpha}}{\Gamma(\alpha-1)} \Big(\int_0^t (t-s)^{(\alpha-2)q}\,ds\Big)^{1/q}\|g\|_p \\ &= \frac{\alpha-1}{\Gamma(\alpha)} \frac{1}{((\alpha-2)q+1)^{1/q}}\|g\|_p. \end{aligned}$$

Similarly,

$$\begin{aligned} \|K_P g\|_0 &= \max_{t\in[0,1]} |K_P g(t)| \\ &\le \max_{t\in[0,1]} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|g(s)|\,ds \\ &\le \max_{t\in[0,1]} \frac{1}{\Gamma(\alpha)} \Big(\int_0^t (t-s)^{(\alpha-1)q}\,ds\Big)^{1/q}\|g\|_p \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{((\alpha-1)q+1)^{1/q}}\|g\|_p. \end{aligned}$$

Hence

$$\|K_P g\| \le \Lambda\|g\|_p, \tag{3.6}$$

where

$$\Lambda = \frac{1}{\Gamma(\alpha)} \max\Big\{\frac{1}{((\alpha-1)q+1)^{1/q}}, \frac{\alpha-1}{((\alpha-2)q+1)^{1/q}}\Big\}. \tag{3.7}$$

We introduce

$$\begin{aligned} QNu(t) &= \kappa(\eta \mathcal{I}^\alpha Nu(\xi) - \mathcal{I}^\alpha Nu(1)) t^{\alpha-1} \\ &= \frac{\kappa}{\Gamma(\alpha)}\Big(\eta \int_0^\xi (\xi-s)^{\alpha-1} f(s,u(s),u'(s))\,ds \\ &\quad - \int_0^1 (1-s)^{\alpha-1} f(s,u(s),u'(s))\,ds\Big) t^{\alpha-1} \end{aligned}$$

and

$$K_{P,Q}Nu(t) = K_P(I-Q)Nu(t) = \frac{\kappa}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(Nu(s) - QNu(s))\,ds.$$

Now we are in position to prove the existence results. We impose the condiitions

(H1) there exists a positive constant K such that $u \in \operatorname{dom} L \setminus \operatorname{Ker} L$ with $\min_{t\in[0,1]} |\mathcal{D}^{\alpha-1}(t)| > K$ implies $QNu(t) \neq 0$ on $(0,1]$;

(H2) there exist $\delta, \beta, t^{\alpha-2}\gamma, \rho \in L^p[0,1]$ and a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ and $x_0 > 0$ with the properties:

(a)

$$\|\beta\|_p + \|t^{\alpha-2}\gamma\|_p < \frac{\Gamma(\alpha)}{1+\Gamma(\alpha)\Lambda};$$

(b) for all $x \geq x_0$

$$\begin{aligned} x &\geq \frac{K + (1+\Gamma(\alpha)\Lambda)\|\delta\|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)(\|\beta\|_p + \|t^{\alpha-2}\gamma\|_p)} \\ &\quad + \frac{(1+\Gamma(\alpha)\Lambda)\|\rho\|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)(\|\beta\|_p + \|t^{\alpha-2}\gamma\|_p)}\phi(x); \end{aligned} \tag{3.8}$$

(c) $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ satisfies

$$|f(t,x,y)| \leq \delta(t) + \beta(t)|x| + \gamma(t)|y| + \rho(t)\phi(|x|);$$

(H3) there exists a constant $B > 0$ such that, for every $c \in \mathbb{R}$ satisfying $|c| > B$ we have

$$\operatorname{sgn}\,[c(\eta \mathcal{I} u_c(\xi) - \mathcal{I} u_c(1))] \neq 0,$$

where $u_c(t) = ct^{\alpha-1}$.

Theorem 3.2. *If the hypotheses* (P), (H1)-(H3) *are satisfied, then the boundary value problem* (3.1)-(3.4) *has a solution.*

Proof. Let $\Omega_1 = \{u \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lu = \lambda Nu \text{for some } \lambda \in (0,1)\}$. Applying (H1), $QNu(t) = 0$ for all $t \in [0,1]$. Hence there exists $t_0 \in (0,1]$ such that $|\mathcal{D}^{\alpha-1}(t_0)| \leq K$. By Theorem 2.1 with $\beta = 1$,

$$\begin{aligned} \mathcal{I}\mathcal{D}^\alpha u(t_0) &= \mathcal{D}^{\alpha-1}u(t_0) - \mathcal{D}^{\alpha-1}u(0) - \mathcal{D}^{\alpha-2}u(0)t_0^{-1} \\ &= \mathcal{D}^{\alpha-1}u(t_0) - \mathcal{D}^{\alpha-1}u(0) \end{aligned}$$

since $u \in \operatorname{dom} L$. That is,

$$\mathcal{D}^{\alpha-1}u(0) = \mathcal{D}^{\alpha-1}u(t_0) - \int_0^{t_0} \mathcal{D}^\alpha u(s)\,ds,$$

which implies

$$\begin{aligned}|\mathcal{D}^{\alpha-1}u(0)| &\le |\mathcal{D}^{\alpha-1}u(t_0)| + \int_0^{t_0} |\mathcal{D}^{\alpha}u(s)|\,ds \\ &\le K + \|Lu\| \\ &< K + \|Nu\|_p.\end{aligned}$$

By (3.5),

$$\|Pu\| = \frac{1}{\Gamma(\alpha)}|\mathcal{D}^{\alpha-1}u(0)| < \frac{1}{\Gamma(\alpha)}(K + \|Nu\|_p).$$

Since $(I-P)u \in \operatorname{dom} L \cap \operatorname{Ker} P = \operatorname{Im} K_P$, for $u \in \Omega_1$, $\|(I-P)u\| < \Lambda\|Nu\|_p$ by (3.6) and (3.7). Also $Pu \in \operatorname{Im} P = \operatorname{Ker} L \subset \operatorname{dom} L$ and, therefore,

$$\|u\| \le \|Pu\| + \|(I-P)u\| < \frac{K}{\Gamma(\alpha)} + \Big(\frac{1}{\Gamma(\alpha)} + \Lambda\Big)\|Nu\|_p.$$

From (H2) and the previous inequality, it follows that

$$\begin{aligned}\|t^{2-\alpha}u'\|_0 < \frac{K}{\Gamma(\alpha)} + \Big(\frac{1}{\Gamma(\alpha)} + \Lambda\Big)\Big(&\|\delta\|_p + \|\beta\|_p\|\|u\|_0 \\ &+ \|t^{\alpha-2}\gamma\|_p\|t^{2-\alpha}u'\|_0 + \|\rho\|_p\phi(\|u\|_0)\Big)\end{aligned}$$

or

$$\begin{aligned}\|t^{2-\alpha}u'\|_0 &< \frac{K + (1+\Gamma(\alpha)\Lambda)\|\delta\|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)\|t^{\alpha-2}\gamma\|_p} + \frac{(1+\Gamma(\alpha)\Lambda)\|\beta\|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)\|t^{\alpha-2}\gamma\|_p}\|u\|_0 \\ &\quad + \frac{(1+\Gamma(\alpha)\Lambda)\|\rho\|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)\|t^{\alpha-2}\gamma\|_p}\phi(\|u\|_0).\end{aligned} \tag{3.9}$$

Combining the above inequality with

$$\begin{aligned}\|u\|_0 < \frac{K}{\Gamma(\alpha)} + \Big(\frac{1}{\Gamma(\alpha)} + \Lambda\Big)\Big(&\|\delta\|_p + \|\beta\|_p\|\|u\|_0 \\ &+ \|t^{\alpha-2}\gamma\|_p\|t^{2-\alpha}u'\|_0 + \|\rho\|_p\phi(\|u\|_0)\Big)\end{aligned}$$

we obtain

$$\begin{aligned}\|u\|_0 &< \frac{K + (1+\Gamma(\alpha)\Lambda)\|\delta\|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)(\|\beta\|_p + \|t^{\alpha-2}\gamma\|_p)} \\ &\quad + \frac{(1+\Gamma(\alpha)\Lambda)\|\rho\|_p}{\Gamma(\alpha) - (1+\Gamma(\alpha)\Lambda)(\|\beta\|_p + \|t^{\alpha-2}\gamma\|_p)}\phi(\|u\|_0),\end{aligned}$$

for all $u \in \Omega_1$. Suppose that Ω_1 is unbounded. If $\{\|t^{2-\alpha}u'\|_0 : u \in \Omega_1\}$ is unbounded, then, by (3.9), so is $\{\|u\|_0 : u \in \Omega_1\}$. So, it suffices to consider the case that $\{\|u\|_0 : u \in \Omega_1\}$ is unbounded. Then, in view of (3.8), we arrive at a contradiction. Therefore, Ω_1 is bounded.

Set $\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\}$. Hence $u_c \in \ker L$ is given by $u_c(t) = ct^{\alpha-1}$, $c \in \mathbb{R}$. Then $(QN)(ct^{\alpha-1}) = 0$, since $Nu \in \operatorname{Im} L = \ker Q$. It follows from (H3) that $\|u_c\| = \max\{\|u_c\|_0, \|t^{2-\alpha}u_c'\|_0\} = \max\{|c|, (\alpha-1)|c|\} = |c| \le B$; that is, Ω_2 is bounded.

Define the isomorphism $J : \operatorname{Im} Q \to \ker L$ by $Ju_c = u_c$, $u_c(t) = ct^{\alpha-1}$ for $c \in \mathbb{R}$. Let $\Omega_3 = \{u \in \ker L : -\lambda J^{-1}u + (1-\lambda)QNu = 0,\ \lambda \in [0,1]\}$, if $\operatorname{sgn}[c(\eta\mathcal{I}u_c(\xi) - \mathcal{I}u_c(1))] = -1$. Then $u \in \Omega_3$ implies $\lambda c = (1-\lambda)\left(\eta\mathcal{I}u_c(\xi) - \mathcal{I}u_c(1)\right)$. If $\lambda = 1$, then

$c=0$ and, if $\lambda\in[0,1)$ and $|c|>B$, then $0<\lambda c^2=(1-\lambda)c\,(\eta\mathcal{I}u_c(\xi)-\mathcal{I}u_c(1))<0$, which is a contradiction. Let $\Omega_3=\{u\in\ker L:\lambda J^{-1}u+(1-\lambda)QNu=0,\ \lambda\in[0,1]\}$ if $\operatorname{sgn}[c\,(\eta\mathcal{I}u_c(\xi)-\mathcal{I}u_c(1))]=1$, and we arrive at a contradiction, again. Thus, $\|u_c\|\le B$, for all $u_c\in\Omega_3$.

Let Ω be open and bounded such that $\cup_{i=1}^3\overline{\Omega}_i\subset\Omega$. Then the assumptions (i) and (ii) of Theorem 2.5 are fulfilled. It is a straightforward exercise to show that the mapping N is L-compact on $\overline{\Omega}$. Lemma 3.1 establishes that L is a Fredholm mapping of index zero.

Define

$$H(u,\lambda)=\pm\lambda\operatorname{Id}u+(1-\lambda)JQNu.$$

By the degree property of invariance under a homotopy, if $u\in\ker L\cap\partial\Omega$, then

$$\begin{aligned}\deg(JQN|_{\ker L\cap\partial\Omega},\Omega\cap\ker L,0)&=\deg(H(\cdot,0),\Omega\cap\ker L,0)\\&=\deg(H(\cdot,1),\Omega\cap\ker L,0)\\&=\deg(\pm\operatorname{Id},\Omega\cap\ker L,0)\neq0.\end{aligned}$$

Therefore, the assumption (iii) of Theorem 2.5 is fulfilled and the proof is completed. $\square$

Suppose that the hypothesis (H2) is replaced by

(H2') there exist $\delta,\beta,t^{\alpha-2}\gamma,t^{\alpha-2}\rho\in L^p[0,1]$ and a continuous nondecreasing function $\phi:[0,\infty)\to[0,\infty)$ and $y_0>0$ with the properties:

(a)
$$\|\beta\|_p+\|t^{\alpha-2}\gamma\|_p<\frac{\Gamma(\alpha)}{1+\Gamma(\alpha)\Lambda};$$

(b) for all $y\in[0,\infty)$ and $t\in[0,1]$,
$$t^{2-\alpha}\phi(y)\le\phi(t^{2-\alpha}y);$$

(c) for all $y\ge y_0$,
$$\begin{aligned}y\ge&\frac{K+(1+\Gamma(\alpha)\Lambda)\|\delta\|_p}{\Gamma(\alpha)-(1+\Gamma(\alpha)\Lambda)(\|\beta\|_p+\|t^{\alpha-2}\gamma\|_p)}\\&+\frac{(1+\Gamma(\alpha)\Lambda)\|t^{\alpha-2}\rho\|_p}{\Gamma(\alpha)-(1+\Gamma(\alpha)\Lambda)(\|\beta\|_p+\|t^{\alpha-2}\gamma\|_p)}\phi(y);\end{aligned}$$

(d) $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ satisfies
$$|f(t,x,y)|\le\delta(t)+\beta(t)|x|+\gamma(t)|y|+\rho(t)\phi(|y|).$$

Then we have the following existence criterion whose proof is analogous to that of Theorem 3.2.

Theorem 3.3. *If the hypotheses* (P), (H1), (H2'), (H3) *are satisfied, then the boundary value problem* (3.1)-(3.4) *has a solution.*

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