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SOLVABILITY OF A THREE-POINT NONLINEAR BOUNDARY-VALUE PROBLEM

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Abstract. Using the Leray Schauder nonlinear alternative, we prove the existence of a nontrivial solution for the three-point boundary-value problem

$$u'' + f(t,u) = 0, \quad 0 < t < 1$$
$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta),$$

where $\eta \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$. Some examples are given to illustrate the results obtained.

1. Introduction and preliminaries

During the previous years, many authors have studied three-point boundary-value problems (BVP) for second order differential equations. Such problems have potential applications in physics, biology, chemistry, etc. For example, a second-order three-point (BVP) is used as a model for the membrane response of a spherical cap [10] in nonlinear diffusion generated by nonlinear sources and in chemical reactor theory.

In this article, we investigate the existence of a nontrivial solution for the second-order three-point boundary-value problem

$$u'' + f(t,u) = 0, \quad 0 < t < 1, \tag{1.1}$$
$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta) \tag{1.2}$$

where $\eta \in (0,1)$, α, $\beta \in \mathbb{R}$, $f \in C([0,1]\times\mathbb{R},\mathbb{R})$. We do not assume any monotonicity condition on the nonlinearity f, and the parameters α and β belong to $\mathbb{R}$, so our conditions are more general than the conditions found in the literature. Such problem arises in the study of the equilibrium states of a heated bar. In this situation two controllers at $t = 0$ and $t = \eta$ alter the heat according to the temperatures detected by a sensor at $t = 1$.

This study is motivated by Il'in and Moiseev's results [5], on similar boundary value problems for certain linear ordinary differential equations. Many of the results involving nonlocal boundary value problems are studied in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In [12] the author used the Leray-Shauder nonlinear alternative to

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establish some results on the existence of solutions for the equation (1.1) subject to the conditions

$$u(0) = 0, \quad u(1) = \alpha u(\eta)$$

Gupta [3] studied certain three-point boundary-value problem with the above nonlocal conditions. Similar boundary value problem with the conditions

$$u(0) = \alpha u'(0), \quad u(1) = \beta u(\eta)$$

is considered in [11]. Under some conditions on the nonlinearity of f and by using Leray Schauder nonlinear alternative, we establish the existence of nontrivial solution of the BVP (1.1)-(1.2). Some examples illustrate our results.

This article is organized as follows. First, we list some preliminary material to be used later. Then in Section 2, we present and prove our main results which consist in existence theorems and corollaries. We end our work with some illustrating examples.

Let $E = C[0,1]$, with the supremum norm $\|y(t)\| = \sup_{t\in[0,1]} |y(t)|$, for all $y \in E$. Now we state two preliminary results.

Lemma 1.1. *Let $y \in E$. If $\beta(\eta + \alpha) \neq \alpha + 1$, then the three-point BVP*

$$u''(t) + y(t) = 0, \quad 0 < t < 1$$
$$u(0) = \alpha u'(0), \quad u(1) = \beta u'(\eta)$$

has a unique solution

$$u(t) = -\int_0^t (t-s)y(s)ds + \frac{t+\alpha}{1+\alpha-\beta}\int_0^1 (1-s)y(s)ds - \beta\frac{t+\alpha}{1+\alpha-\beta}\int_0^\eta y(s)ds.$$

Proof. Rewriting the differential equation as $u''(t) = -y(t)$, and integrating twice, we obtain

$$u(t) = -\int_0^t (t-s)y(s)ds + C_1 t + C. \tag{1.3}$$

Then $u'(0) = C_1$,

$$u'(\eta) = -\int_0^\eta y(s)ds + C_1,$$
$$u(1) = -\int_0^1 (1-s)y(s)ds + C_1 + C_2,$$
$$u(0) = C_2 = \alpha C_1.$$

Combining these with the second equality in condition (1.2), we obtain

$$-\int_0^1 (1-s)y(s)ds + C_1(1+\alpha) = -\beta\int_0^\eta y(s)ds + C_1\beta,$$

which is equivalent to

$$C_1(1+\alpha-\beta) = \int_0^1 (1-s)y(s)ds - \beta\int_0^\eta y(s)ds.$$

Since $1 + \alpha - \beta \neq 0$,

$$C_1 = \frac{1}{1+\alpha-\beta}(\int_0^1 (1-s)y(s)ds - \beta\int_0^\eta y(s)ds).$$
$$C_2 = \alpha C_1.$$

Substituting C_1 and C_2 by their values in (1.3), we obtain the solution in the statement of the lemma. This completes the proof. $\square$

We define the integral operator $T : E \to E$, by

$$\begin{aligned} Tu(t) = &-\int_0^t (t-s)f(s,u(s))ds + \frac{t+\alpha}{1+\alpha-\beta}\int_0^1 (1-s)f(s,u(s))ds \\ &- \beta\frac{t+\alpha}{1+\alpha-\beta}\int_0^\eta f(s,u(s))ds \end{aligned} \tag{1.4}$$

By Lemma 1.1, BVP (1.1)-(1.2) has a solution if and only if the operator T has a fixed point in E. By Ascoli Arzela theorem we prove that T is a completely continuous operator. Now we cite the Leray Schauder nonlinear alternative.

Lemma 1.2 ([1]). *Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

2. Main Results

In this section, we present and prove our main results.

Theorem 2.1. *We assume that $f(t,0) \neq 0$, $1+\alpha \neq \beta$ and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|f(t,x)| \le k(t)|x| + h(t), \quad a.e.\ (t,x) \in [0,1]\times\mathbb{R}, \tag{2.1}$$

$$(1+\frac{|1+\alpha|}{|1+\alpha-\beta|})\int_0^1 (1-s)k(s)ds + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta k(s)ds < 1 \tag{2.2}$$

Then BVP (1.1)-(1.2) has at least one nontrivial solution $u^* \in C[0,1]$.

Proof. Setting

$$M = (1+\frac{|1+\alpha|}{|1+\alpha-\beta|})\int_0^1 (1-s)k(s)ds + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta k(s)ds,$$

$$N = (1+\frac{|1+\alpha|}{|1+\alpha-\beta|})\int_0^1 (1-s)h(s)ds + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta (\eta-s)h(s)ds.$$

By (2.2), we have $M < 1$. Since $f(t,0) \neq 0$, then there exists an interval $[\sigma,\tau] \subset [0,1]$ such that $\min_{\sigma\le t\le\tau}|f(t,0)| > 0$ and as $h(t) \ge |f(t,0)|$, for all $t \in [0,1]$ then $N > 0$. Let $m = \frac{N}{(1-M)}$, $\Omega = \{u \in C[0,1] : \|u\| < m\}$. Assume that $u \in \partial\Omega$, $\lambda > 1$ such $Tu = \lambda u$, then

$$\begin{aligned} \lambda m = \lambda\|u\| = \|Tu\| &= \max_{0\le t\le 1}|(Tu)(t)| \\ &\le \|u\|\Big[(1+\frac{|1+\alpha|}{|1+\alpha-\beta|})\int_0^1 (1-s)k(s)ds + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta k(s)ds \\ &\quad + (1+\frac{|1+\alpha|}{|1+\alpha-\beta|})\int_0^1 (1-s)h(s)ds + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta h(s)ds\Big] \\ &= M\|u\| + N \end{aligned}$$

From this we obtain

$$\lambda \le M + \frac{N}{m} = M + \frac{N}{N(1-M)^{-1}} = M + (1-M) = 1,$$

this contradicts $\lambda > 1$. By Lemma 1.2 we conclude that operator T has a fixed point $u^* \in \overline{\Omega}$ and then BVP (1.1)-(1.2) has a nontrivial solution $u^* \in C[0,1]$. $\square$

Theorem 2.2. *Assume* (2.1) *and one of the following four conditions:*

(1) *There exists a constant* $p > 1$ *such that*

$$\int_0^1 k^p(s)ds < \Big[\frac{(1+q)^{1/q}}{1+|\frac{1+\alpha}{1+\alpha-\beta}| + (|\beta\frac{1+\alpha}{1+\alpha-\beta}|)(\eta(1+q))^{1/q}}\Big]^p \quad (\frac{1}{p}+\frac{1}{q}=1); \tag{2.3}$$

(2) *There exists constant* $\mu > -1$ *such that*

$$k(s) < \frac{(\mu+1)(\mu+2)}{1+|\frac{1+\alpha}{1+\alpha-\beta}| + (|\beta\frac{1+\alpha}{1+\alpha-\beta}|)(\mu+2)\eta^{\mu+1}} s^\mu \tag{2.4}$$

and

$$\operatorname{meas}\big\{s\in[0,1] : k(s) < \frac{(\mu+1)(\mu+2)}{1+|\frac{1+\alpha}{1+\alpha-\beta}| + |\beta\frac{1+\alpha}{1+\alpha-\beta}|(\mu+2)\eta^{\mu+1}} s^\mu\big\} > 0;$$

(3) *The function* $k(s)$ *satisfies*

$$k(s) < \frac{|1+\alpha-\beta|}{|1+\alpha|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta} \tag{2.5}$$

and

$$\operatorname{meas}\big\{s\in[0,1] : k(s) < \frac{|1+\alpha-\beta|}{|1+\alpha|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta}\big\} > 0;$$

(4) *The function* $f(t,x)$ *satisfies*

$$\omega = \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| < \frac{1}{2}(\frac{|1+\alpha-\beta|}{|1+\alpha|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta}). \tag{2.6}$$

Then BVP (1.1)-(1.2) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. Let M be defined as in the proof of Theorem 2.1. To prove Theorem 2.2, we only need to prove that $M < 1$.

(1) By using Hölder inequality, we obtain

$$\begin{aligned} M &\le \big(1+|\frac{1+\alpha}{1+\alpha-\beta}|\big)\Big(\int_0^1 k^p(s)ds\Big)^{1/p}\Big(\int_0^1 (1-s)^q ds\Big)^{1/q} \\ &\quad + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\Big(\int_0^\eta k^p(s)ds\Big)^{1/p}\Big(\int_0^\eta ds\Big)^{1/q}. \end{aligned}$$

Then

$$\begin{aligned} M &\le \Big(\int_0^1 k^p(s)ds\Big)^{1/p}\Big[\big(1+|\frac{1+\alpha}{1+\alpha-\beta}|\big)\Big(\int_0^1 (1-s)^q ds\Big)^{1/q} \\ &\quad + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\Big(\int_0^\eta ds\Big)^{1/q}\Big]. \end{aligned}$$

Integrating, it yields

$$\begin{aligned} M &< \frac{(1+q)^{1/q}}{(1+|\frac{1+\alpha}{1+\alpha-\beta}|)+|\beta\frac{1+\alpha}{1+\alpha-\beta}|(\eta(1+q))^{1/q}} \\ &\quad \times\Big[(1+\frac{|1+\alpha|}{|1+\alpha-\beta|})(\frac{1}{1+q})^{1/q} + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\eta^{1/q}\Big] = 1. \end{aligned}$$

(2) Using the same reasoning as in the proof of the first statement we obtain

$$
\begin{aligned}
M &< \frac{(\mu+1)(\mu+2)}{1+|\frac{1+\alpha}{1+\alpha-\beta}|+(|\beta\frac{1+\alpha}{1+\alpha-\beta}|)(\mu+2)\eta^{\mu+1}} \\
&\quad\times\Big[\big(1+|\frac{1+\alpha}{1+\alpha-\beta}|\big)\int_0^1(1-s)s^\mu ds+|\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta s^\mu ds\Big] \\
&= \frac{(\mu+1)(\mu+2)}{1+|\frac{1+\alpha}{1+\alpha-\beta}|+|\beta\frac{1+\alpha}{1+\alpha-\beta}|(\mu+2)\eta^{\mu+1}}\big(1+|\frac{1+\alpha}{1+\alpha-\beta}|\big)\big(\frac{1}{(\mu+1)(\mu+2)}\big) \\
&\quad+|\beta\frac{1+\alpha}{1+\alpha-\beta}|\big(\frac{\eta^{\mu+1}}{\mu+1}\big)=1
\end{aligned}
$$

(3) we have

$$
M=(1+\frac{|1+\alpha|}{|1+\alpha-\beta|})\int_0^1(1-s)k(s)ds+|\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta k(s)ds.
$$

Then

$$
\begin{aligned}
M &< \big(\frac{|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta}\big) \\
&\quad\times\big(1+|\frac{1+\alpha}{1+\alpha-\beta}|\big)\int_0^1(1-s)ds+|\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^\eta ds \\
&= \frac{1}{2}\big(\tfrac{|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta}\big)(1+|\tfrac{1+\alpha}{1+\alpha-\beta}|)+|\beta\tfrac{1+\alpha}{1+\alpha-\beta}|\eta=1
\end{aligned}
$$

(4) From $\omega=\limsup_{|x|\to\infty}\max_{t\in[0,1]}|\frac{f(t,x)}{x}|$ we deduce that there exists $c>0$ such that for $|x|>c$ we have

$$
|f(t,x)|\le(\omega+\varepsilon)|x|\quad\forall\varepsilon>0.
$$

Set

$$
h(t)=\max\{|f(t,x)|:(t,x)\in[0,1]\times(-c,c)\}.
$$

Then for $(t,x)\in[0,1]\times\mathbb{R}$, with $\varepsilon=\omega$, we obtain

$$
\begin{aligned}
|f(t,x)| &\le 2\omega|x|+h(t) \\
&\le \frac{|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta}|x|+h(t).
\end{aligned}
$$

Setting

$$
k(t)<\frac{|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta},
$$

by applying the above statement we complete the proof. □

Corollary 2.3. *Assume the conditions of Theorem 2.2, and one of the following four conditions:*

(1) *There exists a constant $p>1$ such that*

$$
\int_0^1 k^p(s)ds<\Big[\frac{(1+q)^{1/q}}{1+|\frac{1+\alpha}{1+\alpha-\beta}|+|\beta\frac{1+\alpha}{1+\alpha-\beta}|(1+q)^{1/q}}\Big]^p,\quad(\frac{1}{p}+\frac{1}{q}=1).
$$

(2) *There exists a constant* $\mu > -1$ *such that*

$$k(s) < \frac{(\mu+1)(\mu+2)}{1+|\frac{1+\alpha}{1+\alpha-\beta}|+(|\beta\frac{1+\alpha}{1+\alpha-\beta}|)(\mu+2)}s^{\mu},$$

$$\text{meas}\big\{s\in[0,1]: k(s) < \frac{(\mu+1)(\mu+2)}{1+|\frac{1+\alpha}{1+\alpha-\beta}|+|\beta\frac{1+\alpha}{1+\alpha-\beta}|(\mu+2)}s^{\mu}\big\} > 0$$

(3) *The function* $k(s)$ *satisfies*

$$k(s) < \frac{|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta},$$

$$\text{meas}\Big\{s\in[0,1], k(s) < \frac{|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|\eta}\Big\} > 0$$

(4) *The function* $f(t,x)$ *satisfies*

$$\begin{aligned}\omega &= \limsup_{|x|\to\infty}\max_{t\in[0,1]}|\frac{f(t,x)}{x}|\\ &< 1/2(\frac{|1+\alpha-\beta|}{|(1+\alpha)|+|1+\alpha-\beta|+|\beta(1+\alpha)|})\end{aligned}$$

Then the BVP (1.1)-(1.2) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. Taking into account

$$\begin{aligned}M &= \big(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\big)\int_0^1(1-s)k(s)ds + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^{\eta}k(s)ds\\ &\le \big(1+\frac{|1+\alpha|}{|1+\alpha-\beta|}\big)\int_0^1(1-s)k(s)ds + |\beta\frac{1+\alpha}{1+\alpha-\beta}|\int_0^{1}k(s)ds.\end{aligned}$$

the proof follows. □

3. Examples

To illustrate our results, we give the following examples.

Example 3.1. Consider the three-point BVP

$$\begin{gathered}u'' + \frac{u}{2}(t\sin\sqrt{u} - e^{-t}\cos u) + te^t = 0, \quad 0<t<1\\ u(0) = \frac{1}{2}u'(0), \quad u(1) = \frac{1}{2}u'(\tfrac{1}{2}).\end{gathered}\tag{3.1}$$

Where $\alpha=\beta=\eta=1/2$, $+\alpha-\beta = 1/2 \neq 0$ and

$$f(t,x) = \frac{x}{2}(t\sin\sqrt{x} - e^{-t}\cos x) + te^t,$$

It is easy to see that

$$|f(t,x)| \le \frac{1}{2}(t+e^{-t})|x| + te^t, \quad (t,x)\in[0,1]\times\mathbb{R},$$

set

$$k(t) = \frac{1}{2}(t+e^{-t}) \ge 0, \quad h(t) = te^t \ge 0,$$

we have k, $h \in L^1[0,1]$, $f(t,0) = te^t \neq 0$ and

$$\begin{aligned} M &= \frac{5}{2}\int_0^1 (1-s)k(s)ds + \frac{3}{4}\int_0^{\frac{1}{2}} k(s)ds \\ &= \frac{5}{4}\int_0^1 (1-s)(s+e^{-s})ds + \frac{3}{8}\int_0^{1/2} (s+e^{-s})ds = 0.86261. \end{aligned}$$

Then, by Theorem 2.1, the BVP (3.1) has at least one nontrivial solution u^* in $C[0,1]$.

Example 3.2. Consider the three-point BVP

$$\begin{gathered} u'' + \frac{u^4}{2\sqrt{t^3+1}(1+u^3)} + \cos e^t(1-\sin t) = 0, \quad 0 < t < 1, \\ u(0) = -2u'(0), \quad u(1) = 2u'(1/3) \end{gathered} \tag{3.2}$$

then

$$\begin{aligned} f(t,x) &= \frac{x^4}{2\sqrt{t^3+1}(1+x^3)} + \cos e^t(1-\sin t), \\ |f(t,x)| &\le \frac{1}{4\sqrt{t^3+1}}|x| + \cos e^t(1-\sin t) \\ &= k(t)|x| + h(t), \end{aligned}$$

where

$$k(t) = \frac{1}{4\sqrt{t^3+1}} \ge 0, \quad h(t) = \cos e^t(1-\sin t) \ge 0$$

$$\begin{aligned} M &= \frac{4}{3}\int_0^1 (1-s)k(s)ds + \frac{2}{3}\int_0^{\eta} k(s)ds \\ &= \frac{1}{3}\int_0^1 (1-s)\frac{1}{\sqrt{s^3+1}}ds + \frac{2}{16}\int_0^{1/3} \frac{1}{\sqrt{s^3+1}}ds = 0.201\,41 < 1 \end{aligned}$$

Applying the third statement of Theorem 2.2 we obtain

$$\max_s k(s) = \frac{1}{4} < \frac{|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|\eta}\frac{9}{14} = 0.642\,86\,,$$

$$\begin{aligned} &\operatorname{meas}\{s \in [0,1] : k(s) < \frac{|1+\alpha-\beta|}{|(1+\alpha)| + |1+\alpha-\beta| + |\beta(1+\alpha)|\eta}\} \\ &= \operatorname{meas}\{s \in [0,1] : k(s) < \frac{9}{14} = 0.642\,86\} \\ &= \operatorname{meas}[0,1] > 0\,. \end{aligned}$$

Hence, by Theorem 2.2, BVP (3.2) has at least one nontrivial solution u^* in $C[0,1]$.

Example 3.3. Consider the three-point BVP

$$\begin{gathered} u'' + \frac{u^3}{2(1+e^{-t^2})^4(1+u^2)} - \sqrt{t}\sin t = 0 \\ u(0) = \frac{1}{3}u'(0), \quad u(1) = \frac{-1}{6}u'(\frac{1}{4}). \end{gathered} \tag{3.3}$$

We have

$$f(t,x) = \frac{x^3}{2(1+e^{-t^2})^4(1+x^2)} - \sqrt{t}\sin t,$$

$$|f(t,x)| \le |x|\frac{(1+e^{-t^2})^{-4}}{2} + \sqrt{t}\sin t = k(t)|x| + h(t)$$

$$|\frac{f(t,x)}{x}| \le \frac{(1+e^{-t^2})^{-4}}{2} + \frac{\sqrt{t}\sin t}{|x|} \le 0.142 + \frac{1}{|x|}.$$

Applying the fourth statement in Theorem 2.2 we obtain

$$\begin{aligned}\omega &= \limsup_{|x|\to\infty} \max_{t\in[0,1]} |\frac{f(t,x)}{x}| = \limsup_{|x|\to\infty}(0.142 + \frac{1}{|x|}) \\ &= 0.142 < \frac{1}{2}(\frac{\frac{13}{9}}{\frac{4}{3}+\frac{13}{9}+\frac{1}{27}}) = 0.25658\end{aligned}$$

Hence, BVP (3.3) has at least one nontrivial solution u^* in $C[0,1]$.

Example 3.4. Consider the three-point BVP

$$\begin{gathered} u'' + \frac{\sin u}{2\sqrt{4+t}} - t^2\cos t + t\cos(t^2) = 0, \quad 0<t<1 \\ u(0) = \frac{-1}{4}u'(0), \quad u(1) = \frac{-1}{6}u'(\frac{1}{5}), \end{gathered} \tag{3.4}$$

where

$$f(t,x) = \frac{\sin x}{2\sqrt{4+t}} - t^2\cos t + t\cos(t^2),$$

$$|f(t,x)| \le \frac{|x|}{2\sqrt{4+t}} - t^2\cos t + t\cos(t^2) = k(t)|x| + h(t),$$

$$k(t) = \frac{1}{2\sqrt{4+t}} \ge 0, \quad h(t) = -t^2\cos t + t\cos(t^2) \ge 0$$

We see that $k,\ h \in L^1[0,1]$, $f(t,0) = -t^2\cos t + t\cos(t^2) \ne 0$. The first statement of Corollary 2.3 for $p=2$ yields

$$\begin{aligned}\int_0^1 k^2(s)ds &= \int_0^1 \Big(\frac{1}{2(\sqrt{4+t})}\Big)^2 dt = \frac{1}{4}\int_0^1 \frac{1}{4+t}dt = 0.055786 \\ &< \Big[\frac{(1+q)^{1/q}}{1+|\frac{1+\alpha}{1+\alpha-\beta}| + |\beta\frac{1+\alpha}{1+\alpha-\beta}|(1+q)^{1/q}}\Big]^p = 0.71083.\end{aligned}$$

Hence, BVP (3.4) has at least one nontrivial solution u^* in $C[0,1]$.

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