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CONCENTRATION-COMPACTNESS PRINCIPLE FOR VARIABLE EXPONENT SPACES AND APPLICATIONS

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Abstract. In this article, we extend the well-known concentration - compactness principle by Lions to the variable exponent case. We also give some applications to the existence problem for the $p(x)$-Laplacian with critical growth.

1. Introduction

When dealing with nonlinear elliptic equations with critical growth (in the sense of the Sobolev embeddings) the concentration - compactness principle by Lions, see [12], have been proved to be a fundamental tool for proving existence of solutions. Just to cite a few references, we have [1, 2, 3, 7, 4, 11] but there is an impressive list of references on this topic.

Recently in the analysis of some new models, that are called electrorheological fluids, the following equation has been studied

$$-\Delta_{p(x)} u = f(x, u) \quad \text{in } \Omega. \tag{1.1}$$

The operator $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian. When $p(x) \equiv p$ is the well-known p-Laplacian.

In recent years a vast amount of literature that deal with the existence problem for (1.1) with different boundary conditions (Dirichlet, Neumann, nonlinear, etc) have appeared. See, for instance [5, 6, 8, 13, 14] and references therein.

However, up to our knowledge, no results are available for (1.1) when the source term f is allowed to have critical growth at infinity (see the remark after the introduction for more on this). That is,

$$|f(x,t)| \le C(1 + |t|^{q(x)})$$

with $q(x) \le p^*(x) := Np(x)/(N - p(x))$ (if $p(x) < N$) and $\{q(x) = p^*(x)\} \neq \emptyset$. This article attempts to begin filling this gap. So, the objective is to extend the concentration - compactness principle by Lions to the variable exponent setting.

The method of the proof follows the lines of the ones in the original work of P.L. Lions and the main novelty in our result is the fact that we do not require the exponent $q(x)$ to be critical everywhere. Moreover, we show that the delta masses are concentrated in the set where $q(x)$ is critical.

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Finally, as an application of our result, we prove the existence of solutions to the problem

$$\begin{gathered} -\Delta_{p(x)} u = |u|^{q(x)-2}u + \lambda(x)|u|^{r(x)-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $r(x) < p^*(x) - \delta$, $q(x) \le p^*(x)$ with $\{q(x) = p^*(x)\} \neq \emptyset$.

1.1. **Statement of the results.** As we already mentioned, the main result of the paper is the extension of Lions concentration - compactness method to the variable exponent case. More precisely, we prove the following result.,

Theorem 1.1. *Let $q(x)$ and $p(x)$ be two continuous functions such that*

$$1 < \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) < n \quad \text{and} \quad 1 \le q(x) \le p^*(x) \quad \text{in } \Omega.$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a weakly convergent sequence in $W_0^{1,p(x)}(\Omega)$ with weak limit u, and such that:

- *$|\nabla u_j|^{p(x)} \rightharpoonup \mu$ weakly-* in the sense of measures.*
- *$|u_j|^{q(x)} \longrightarrow \nu$ weakly-* in the sense of measures.*

Also assume that $\mathcal{A} = \{x \in \Omega \colon q(x) = p^*(x)\}$ is nonempty. Then, for some countable index set I, we have:

$$\nu = |u|^{q(x)} + \sum_{i\in I} \nu_i \delta_{x_i} \quad \nu_i > 0 \tag{1.3}$$

$$\mu \ge |\nabla u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i} \quad \mu_i > 0 \tag{1.4}$$

$$S\nu_i^{1/p^*(x_i)} \le \mu_i^{1/p(x_i)} \quad \forall i \in I. \tag{1.5}$$

where $\{x_i\}_{i\in I} \subset \mathcal{A}$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = S_q(\Omega) := \inf_{\phi\in C_0^\infty(\Omega)} \frac{\||\nabla\phi|\|_{L^{p(x)}(\Omega)}}{\|\phi\|_{L^{q(x)}(\Omega)}}.$$

We remark that in Theorem 1.1 is not required the exponent $q(x)$ to be critical *everywhere* and that the point masses are located in the *criticality set* $\mathcal{A} = \{x \in \Omega \colon q(x) = p^*(x)\}$.

Now, as an application of Theorem 1.1, following the techniques in [11], we prove the existence of solutions to

$$\begin{gathered} -\Delta_{p(x)} u = |u|^{q(x)-2}u + \lambda(x)|u|^{r(x)-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.6}$$

In the spirit of [11], we have two types of results, depending on $r(x)$ being smaller or bigger that $p(x)$. More precisely, we prove the following two theorems.

Theorem 1.2. *Let $p(x)$ and $q(x)$ be as in Theorem 1.1 and let $r(x)$ be continuous. Moreover, assume that $\max_{\overline{\Omega}} p < \min_{\overline{\Omega}} q$ and $\max_{\overline{\Omega}} r < \min_{\overline{\Omega}} p$. Then, there exists a constant $\lambda_1 > 0$ depending only on p, q, r, N and Ω such that if $\lambda(x)$ verifies $0 < \inf_{x\in\Omega} \lambda(x) \le \|\lambda\|_{L^\infty(\Omega)} < \lambda_1$, then there exists infinitely many solutions to (1.6) in $W_0^{1,p(x)}(\Omega)$.*

Theorem 1.3. *Let $p(x)$ and $q(x)$ be as in Theorem 1.1 and let $r(x)$ be continuous. Moreover, assume that $\max_{\overline{\Omega}} p < \min_{\overline{\Omega}} r$ and that there exists $\eta > 0$ such that $r(x) \le p^*(x) - \eta$ in Ω.*

Then, there exists $\lambda_0 > 0$ depending only on p, q, r, N and Ω, such that if

$$\inf_{x\in \mathcal{A}_\delta} \lambda(x) > \lambda_0 \quad \text{for some } \delta > 0,$$

problem (1.6) *has at least one nontrivial solution in $W_0^{1,p(x)}(\Omega)$. Here, $\mathcal{A}_\delta$ is the δ-tubular neighborhood of $\mathcal{A}$, namely*

$$\mathcal{A}_\delta := \cup_{x\in\mathcal{A}}(B_\delta(x)\cap\Omega).$$

Organization of this article. After finishing this introduction, in Section 2 we give a very short overview of some properties of variable exponent Sobolev spaces that will be used throughout the paper. In Section 3 we deal with the main result of the paper. Namely the proof of the concentration - compactness principle (Theorem 1.1). In Section 4, we begin analyzing problem (1.6) and prove Theorem 1.3. Finally, in Section 5, we prove Theorem 1.2.

Comment on a related result. After this paper was written, we found out that a similar result was obtained independently by Yongqiang Fu [10]. Even the techniques in Fu's work are similar to the ones in this paper (and both are related to the original work by Lions), we want to remark that our results are slightly more general than those in [10]. For instance, we do not require $q(x)$ to be critical everywhere (as is required in [10]) and we obtain that the delta functions are located in the criticality set $\mathcal{A}$ (see Theorem 1.1).

Also, in our application, again as we do not required the source term to be critical everywhere, so the result in [10] is not applicable directly. Moreover, in Theorem 1.3 our approach allows us to consider $\lambda(x)$ not necessarily a constant and the restriction that λ is large is only needed in an L^∞-norm in the criticality set.

We believe that these improvements are significant and made our result more flexible that those in [10].

2. Results on variable exponent Sobolev spaces

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as

$$L^{p(x)}(\Omega) = \{u \in L^1_{\rm loc}(\Omega) \colon \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

This space is endowed with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \{u \in W^{1,1}_{\rm loc}(\Omega) \colon u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

The corresponding norm for this space is

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}$$

Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the $W^{1,p(x)}(\Omega)$ norm. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces when $1 < \inf_\Omega p \le \sup_\Omega p < \infty$.

As usual, we denote $p'(x) = p(x)/(p(x)-1)$ the conjugate exponent of $p(x)$. Define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \ge N\,. \end{cases}$$

The following results are proved in [9].

Proposition 2.1 (Hölder-type inequality)**.** *Let $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$. Then the following inequality holds*

$$\int_\Omega |f(x)g(x)|\,dx \le C_p \|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^{p'(x)}(\Omega)}\,.$$

Proposition 2.2 (Sobolev embedding)**.** *Let $p, q \in C(\overline{\Omega})$ be such that $1 \le q(x) \le p^*(x)$ for all $x \in \overline{\Omega}$. Assume moreover that the functions p and q are log-Hölder continuous. Then there is a continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Moreover, if $\inf_\Omega(p^* - q) > 0$ then, the embedding is compact.

Proposition 2.3 (Poincaré inequality)**.** *There is a constant $C > 0$, such that*

$$\|u\|_{L^{p(x)}(\Omega)} \le C \||\nabla u|\|_{L^{p(x)}(\Omega)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Remark 2.4. By Proposition 2.3, we know that $\||\nabla u|\|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

In this article, the following notation will be used: Given $q\colon \Omega \to \mathbb{R}$ bounded, we denote

$$q^+ := \sup_\Omega q(x), \quad q^- := \inf_\Omega q(x).$$

The following proposition is also proved in [9] and it will be very useful here.

Proposition 2.5. *Set $\rho(u) := \int_\Omega |u(x)|^{p(x)}\,dx$. For $u, \in L^{p(x)}(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^{p(x)}(\Omega)$, we have*

$$u \ne 0 \Rightarrow \Big(\|u\|_{L^{p(x)}(\Omega)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1\Big). \tag{2.1}$$

$$\|u\|_{L^{p(x)}(\Omega)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1). \tag{2.2}$$

$$\|u\|_{L^{p(x)}(\Omega)} > 1 \Rightarrow \|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)}. \tag{2.3}$$

$$\|u\|_{L^{p(x)}(\Omega)} < 1 \Rightarrow \|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}. \tag{2.4}$$

$$\lim_{k\to\infty} \|u_k\|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0. \tag{2.5}$$

$$\lim_{k\to\infty} \|u_k\|_{L^{p(x)}(\Omega)} = \infty \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = \infty. \tag{2.6}$$

3. Concentration compactness principle

Let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $W_0^{1,p(x)}(\Omega)$ and let $q \in C(\overline{\Omega})$ be such that $q \le p^*$ with $\{x \in \Omega\colon q(x) = p^*(x)\} \neq \emptyset$. Then there exists a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, such that

- $u_j \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$,
- $u_j \to u$ strongly in $L^{r(x)}(\Omega)$ $\forall 1 \le r(x) < p^*(x)$,
- $|u_j|^{q(x)} \rightharpoonup \nu$ weakly * in the sense of measures,
- $|\nabla u_j|^{p(x)} \rightharpoonup \mu$ weakly * in the sense of measures.

Consider $\phi \in C^\infty(\overline{\Omega})$, from the Poincaré inequality for variable exponents, we obtain

$$\|\phi u_j\|_{L^{q(x)}(\Omega)} S \le \|\nabla(\phi u_j)\|_{L^{p(x)}(\Omega)}. \tag{3.1}$$

On the other hand,

$$|\|\nabla(\phi u_j)\|_{L^{p(x)}(\Omega)} - \|\phi \nabla u_j\|_{L^{p(x)}(\Omega)}| \le \|u_j \nabla\phi\|_{L^{p(x)}(\Omega)}.$$

We first assume that $u = 0$. Then, we observe that the right side of the inequality converges to 0. In fact, if, for instance $\||u|^{p(x)}\|_{L^1(\Omega)} \ge 1$,

$$\begin{aligned}\|u_j \nabla\phi\|_{L^{p(x)}(\Omega)} &\le (\|\nabla\phi\|_{L^\infty(\Omega)} + 1)^{p^+} \|u_j\|_{L^{p(x)}(\Omega)} \\ &\le (\|\nabla\phi\|_{L^\infty(\Omega)} + 1)^{p^+} \||u|^{p(x)}\|_{L^1(\Omega)}^{1/p_-} \to 0\end{aligned}$$

Now we want to take the limit in (3.1). To do this, we need the following Lemma.

Lemma 3.1. *Let $\{\nu_j\}_{j\in\mathbb{N}}, \nu$ be nonnegative, finite Radon measures in Ω such that $\nu_j \rightharpoonup \nu$ weakly* in the sense of measures. Then*

$$\|\phi\|_{L^{q(x)}_{\nu_j}(\Omega)} \to \|\phi\|_{L^{q(x)}_{\nu}(\Omega)} \quad \text{as } j \to \infty,$$

for all $\phi \in C^\infty(\overline{\Omega})$.

Proof. First, observe that for $\phi \in C^\infty(\overline{\Omega})$ fixed and for any nonnegative, finite Radon measure μ, the function

$$h_\mu(\lambda) := \int_\Omega \Big|\frac{\phi(x)}{\lambda}\Big|^{q(x)} d\mu$$

is continuous, decreasing with $h_\mu(0) = +\infty$ and $h_\mu(+\infty) = 0$. Hence, if $\lambda_\mu = \|\phi\|_{L^{p(x)}_\mu(\Omega)}$ we have that

$$\int_\Omega \Big|\frac{\phi(x)}{\lambda_\mu}\Big|^{q(x)} d\mu = 1.$$

Now, let $\lambda = \|\phi\|_{L^{q(x)}_\nu(\Omega)} + \varepsilon$. Hence

$$\int_\Omega \Big|\frac{\phi(x)}{\lambda}\Big|^{q(x)} d\nu < 1.$$

Now, as $\nu_j \to \nu$ weakly* in the sense of measures,

$$\int_\Omega \Big|\frac{\phi(x)}{\lambda}\Big|^{q(x)} d\nu_j \to \int_\Omega \Big|\frac{\phi(x)}{\lambda}\Big|^{q(x)} d\nu < 1.$$

Therefore, for j large,

$$\|\phi\|_{L^{q(x)}_{\nu_j}(\Omega)} < \lambda = \|\phi\|_{L^{q(x)}_\nu(\Omega)} + \varepsilon,$$

and so

$$\limsup_{j\to\infty} \|\phi\|_{L^{q(x)}_{\nu_j}(\Omega)} \le \|\phi\|_{L^{q(x)}_{\nu}(\Omega)}.$$

Let $\lambda_0 := \liminf_{j\to\infty} \|\phi\|_{L^{q(x)}_{\nu_j}(\Omega)}$ and assume that $\lambda_0 < \|\phi\|_{L^{q(x)}_{\nu}(\Omega)}$.

We can assume that $\lambda_0 := \lim_{j\to\infty} \|\phi\|_{L^{q(x)}_{\nu_j}(\Omega)}$. It is easy to see that

$$f_j(x) := \Big|\frac{\phi(x)}{\lambda_j}\Big|^{q(x)} \to f_0(x) := \Big|\frac{\phi(x)}{\lambda_0}\Big|^{q(x)} \quad \text{as } j\to\infty$$

uniformly in $\overline{\Omega}$ and so, as $j\to\infty$,

$$1 = \int_\Omega \Big|\frac{\phi(x)}{\lambda_j}\Big|^{q(x)}\, d\nu_j \to \int_\Omega \Big|\frac{\phi(x)}{\lambda_0}\Big|^{q(x)}\, d\nu < 1,$$

a contradiction. The proof is completed. $\square$

Finally, if we take the limit for $j\to\infty$ in (3.1), by Lemma 3.1, we have

$$\|\phi\|_{L^{q(x)}_{\nu}(\Omega)} S \le \|\phi\|_{L^{p(x)}_{\mu}(\Omega)} \tag{3.2}$$

Now we need a lemma that is the key role in the proof of Theorem 1.1.

Lemma 3.2. *Let μ,ν be two non-negative and bounded measures on $\overline{\Omega}$, such that for $1\le p(x) < r(x) < \infty$ there exists some constant $C>0$ such that*

$$\|\phi\|_{L^{r(x)}_{\nu}(\Omega)} \le C\|\phi\|_{L^{p(x)}_{\mu}(\Omega)}$$

Then, there exist $\{x_j\}_{j\in J}\subset\overline{\Omega}$ and $\{\nu_j\}_{j\in J}\subset(0,\infty)$, such that

$$\nu = \Sigma\nu_i\delta_{x_i}$$

For the proof of the lemma above, we need a couple of preliminary results.

Lemma 3.3. *Let ν be a non-negative bounded measure. Assume that there exists $\delta>0$ such that for all A Borelian, $\nu(A)=0$ or $\nu(A)\ge\delta$. Then, there exist $\{x_i\}$ and $\nu_i>0$ such that*

$$\nu = \sum \nu_i\delta_{x_i}$$

The proof of the above lemma is elementary and is omitted.

Lemma 3.4. *Let ν be non-negative and bounded measures, such that*

$$\|\psi\|_{L^{r(x)}_{\nu}(\Omega)} \le C\|\psi\|_{L^{p(x)}_{\nu}(\Omega)}$$

Then there exist $\delta>0$ such that for all A Borelian, $\nu(A)=0$ or $\nu(A)\ge\delta$.

Proof. First, observe that if $\nu(A)\ge 1$,

$$\int_\Omega \Big(\frac{\chi_A(x)}{\nu(A)^{\frac{1}{p-}}}\Big)^{p(x)}\, d\nu \le \int_\Omega \Big(\frac{\chi_A(x)}{\nu(A)^{\frac{1}{p(x)}}}\Big)^{p(x)}\, d\nu = 1.$$

Then $\nu(A)^{\frac{1}{p-}} \ge \|\chi_A\|_{L^{p(x)}_{\nu}}$. On the other hand,

$$\int_\Omega \Big(\frac{\chi_A(x)}{\nu(A)^{\frac{1}{r+}}}\Big)^{r(x)}\, d\nu \ge \int_\Omega \frac{\chi_A(x)}{\nu(A)}\, d\nu = 1.$$

Then $\nu(A)^{\frac{1}{r+}} \le \|\chi_A\|_{L^{r(x)}_{\nu}}$. So we conclude that

$$\nu(A)^{\frac{1}{r+}} \le C\nu(A)^{\frac{1}{p-}}.$$

Now, if $\nu(A) < 1$, we obtain
$$\nu(A)^{\frac{1}{r-}} \leq C\nu(A)^{\frac{1}{p+}}.$$
Combining all these facts, we arrive at
$$\min\{\nu(A)^{\frac{1}{r-}}, \nu(A)^{\frac{1}{r+}}\} \leq C\max\{\nu(A)^{\frac{1}{p-}}, \nu(A)^{\frac{1}{p+}}\}.$$
Now, if $\nu(A) \leq 1$, we have
$$\nu(A)^{\frac{1}{r-}} \leq C\nu(A)^{\frac{1}{p+}}.$$
Then, $\nu(A) = 0$ or
$$\nu(A) \geq (\frac{1}{C})^{\frac{p^+r^-}{r^- - p^+}}.$$
Finally,
$$\nu(A) \geq \min\{(\frac{1}{C})^{\frac{p^+r^-}{r^- - p^+}}, 1\}$$
This completes the proof. $\square$

In the rest of the proofs we will use the following notation: Given a Radon measure μ in Ω and a funcion $f \in L^1_\mu(\Omega)$ we denote the restriction of μ to f by
$$\mu\lfloor f(E) := \int_E f\, d\mu.$$

Proof of Lemma 3.2. By reverse Hölder inequality (3.2), the measure ν is absolutely continuous with respect to μ. As consequence there exists $f \in L^1_\mu(\Omega)$, $f \geq 0$, such that $\nu = \mu\lfloor f$. Also by (3.2), we have
$$\min\left\{\nu(A)^{\frac{1}{r-}}, \nu(A)^{\frac{1}{r+}}\right\} \leq C\max\left\{\mu(A)^{\frac{1}{p-}}, \mu(A)^{\frac{1}{p+}}\right\}$$
for any Borel set $A \subset \Omega$. In particular, $f \in L^\infty_\mu(\Omega)$. On the other hand the Lebesgue decomposition of μ with respect to ν gives us
$$\mu = \nu\lfloor g + \sigma, \text{ where } g \in L^1_\nu(\Omega), g \geq 0$$
and σ is a bounded positive measure, singular with respect to ν.

Now consider (3.2) applying the test function
$$\phi = g^{\frac{1}{r(x)-p(x)}}\chi_{\{g\leq n\}}\psi.$$
We obtain
$$\begin{aligned}
&\|g^{\frac{1}{r(x)-p(x)}}\chi_{\{g\leq n\}}\psi\|_{L^{r(x)}_\nu}\\
&\leq C\|g^{\frac{1}{r(x)-p(x)}}\chi_{\{g\leq n\}}\psi\|_{L^{p(x)}_\mu}\\
&= C\|g^{\frac{1}{r(x)-p(x)}}\chi_{\{g\leq n\}}\psi\|_{L^{p(x)}_{g d\nu + d\sigma}}\\
&\leq C\|g^{\frac{r(x)}{p(x)(r(x)-p(x))}}\chi_{\{g\leq n\}}\psi\|_{L^{p(x)}_\nu} + C\|g^{\frac{1}{r(x)-p(x)}}\chi_{\{g\leq n\}}\psi\|_{L^{p(x)}_\sigma}
\end{aligned}$$
Since $\sigma \perp \nu$, we have
$$\|g^{\frac{1}{r(x)-p(x)}}\chi_{\{g\leq n\}}\psi\|_{L^{r(x)}_\nu} \leq C\|g^{\frac{r(x)}{p(x)(r(x)-p(x))}}\chi_{\{g\leq n\}}\psi\|_{L^{p(x)}_\nu}$$
Hence calling $d\nu_n = g^{\frac{r(x)}{(r(x)-p(x))}}\chi_{g\leq n}d\nu$ the following reverse Hölder inequality holds
$$\|\psi\|_{L^{r(x)}_{\nu_n}} \leq C\|\psi\|_{L^{p(x)}_{\nu_n}}.$$

By Lemma 3.3 and Lemma 3.4, there exists x_i^n and $K_i^n > 0$ such that $\nu_n = \sum_{i\in I} K_i^n \delta_{x_i^n}$. On the other hand, $\nu_n \nearrow g^{\frac{r(x)}{r(x)-p(x)}}\nu$. Then, the points x_i^n are in fact independent of n, and there will denoted by x_i, and the numbers K_i^n are monotone in n. Then, we have

$$g^{\frac{r(x)}{r(x)-p(x)}}\nu = \sum_{i\in I} K_i \delta_{x_i}$$

where $K_i = g^{\frac{r(x_i)}{r(x_i)-p(x_i)}}(x_i)\nu(x_i)$. This finishes the proof. $\square$

The following Lemma follows exactly as in the constant exponent case and the proof is omitted.

Lemma 3.5. *Let $f_n \to f$ a.e and $f_n \rightharpoonup f$ in $L^{p(x)}(\Omega)$ then*

$$\lim_{n\to\infty}\Big(\int_\Omega |f_n|^{p(x)}dx - \int_\Omega |f-f_n|^{p(x)}dx\Big) = \int_\Omega |f|^{p(x)}dx$$

Now we are in position to prove the main results.

Proof of Theorem 1.1. Given any $\phi \in C^\infty(\Omega)$, we write $v_j = u_j - u$ and by Lemma 3.5, we have

$$\lim_{j\to\infty}\Big(\int_\Omega |\phi|^{q(x)}|u_j|^{q(x)} - \int_\Omega |\phi|^{q(x)}|v_j|^{q(x)}dx\Big) = \int_\Omega |\phi|^{q(x)}|u|^{q(x)}dx.$$

On the other hand, by reverse Hölder inequality (3.2) and Lemma 3.2, taking limits we obtain the representation

$$\nu = |u|^{q(x)} + \sum_{j\in I}\nu_j\delta_{x_j}$$

Let us now show that the points x_j actually belong to the *critical set* $\mathcal{A}$. In fact, assume by contradiction that $x_1 \in \Omega\setminus\mathcal{A}$. Let $B = B(x_1, r) \subset\subset \Omega - \mathcal{A}$. Then $q(x) < p^*(x) - \delta$ for some $\delta > 0$ in $\overline{B}$ and, by Proposition 2.2, The embedding $W^{1,p(x)}(B) \hookrightarrow L^{q(x)}(B)$ is compact. Therefore, $u_j \to u$ strongly in $L^{q(x)}(B)$ and so $|u_j|^{q(x)} \to |u|^{q(x)}$ strongly in $L^1(B)$. This is a contradiction to our assumption that $x_1 \in B$.

Now we proceed with the proof. Applying (3.1) to ϕu_j and taking into account that $u_j \to u$ in $L^{p(x)}(\Omega)$, we have

$$S\|\phi\|_{L_\nu^{q(x)}(\Omega)} \le \|\phi\|_{L_\mu^{p(x)}(\Omega)} + \|(\nabla\phi)u\|_{L^{p(x)}(\Omega)}.$$

Consider $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \phi \le 1$, $\phi(0) = 1$ and supported in the unit ball of $\mathbb{R}^n$. Fixed $j \in I$, we consider $\varepsilon > 0$ be arbitrary.

We denote by $\phi_{\varepsilon,j}(x) := \varepsilon^{-n}\phi((x-x_j)/\varepsilon)$. By decomposition of ν, we have:

$$\begin{aligned}
\rho_\nu(\phi_{i_0,\varepsilon}) &:= \int_\Omega |\phi_{i_0,\varepsilon}|^{q(x)}\,d\nu \\
&= \int_\Omega |\phi_{i_0,\varepsilon}|^{q(x)}|u|^{q(x)}\,dx + \sum_{i\in I}\nu_i\phi_{i_0,\varepsilon}(x_i)^{q(x_i)} \ge \nu_{i_0}.
\end{aligned}$$

For the rest of this article, we will denote

$$q_{i,\varepsilon}^+ := \sup_{B_\varepsilon(x_i)} q(x), \quad q_{i,\varepsilon}^- := \inf_{B_\varepsilon(x_i)} q(x),$$
$$p_{i,\varepsilon}^+ := \sup_{B_\varepsilon(x_i)} p(x), \quad p_{i,\varepsilon}^- := \inf_{B_\varepsilon(x_i)} p(x).$$

If $\rho_\nu(\phi_{i_0,\varepsilon}) < 1$ then

$$\|\phi_{i_0,\varepsilon}\|_{L^{q(x)}_\nu(\Omega)} = \|\phi_{i_0,\varepsilon}\|_{L^{q(x)}_\nu(B_\varepsilon(x_{i_0}))} \geq \rho_\nu(\phi_{i_0,\varepsilon})^{1/q^-_{i,\varepsilon}} \geq \nu_{i_0}^{1/q^-_{i,\varepsilon}}.$$

Analogously, if $\rho_\nu(\phi_{i_0,\varepsilon}) > 1$, then

$$\|\phi_{i_0,\varepsilon}\|_{L^{q(x)}_\nu(\Omega)} \geq \nu_{i_0}^{1/q^+_{i,\varepsilon}}.$$

Then

$$\min\{\nu_i^{\frac{1}{q^+_{i,\varepsilon}}}, \nu_i^{\frac{1}{q^-_{i,\varepsilon}}}\} S \leq \|\phi_{i,\varepsilon}\|_{L^{p(x)}_\mu(\Omega)} + \|(\nabla\phi_{i,\varepsilon})u\|_{L^{p(x)}(\Omega)}.$$

By Proposition 2.5,

$$\|(\nabla\phi_{i,\varepsilon})u\|_{L^{p(x)}(\Omega)} \leq \max\{\rho((\nabla\phi_{i,\varepsilon})u)^{1/p^-}; \rho((\nabla\phi_{i,\varepsilon})u)^{1/p^+}\}.$$

Then, by Hölder inequality, we have

$$\begin{aligned}
\rho((\nabla\phi_{i,\varepsilon})u) &= \int_\Omega |\nabla\phi_{i,\varepsilon}|^{p(x)}|u|^{p(x)}\,dx \\
&\leq \||u|^{p(x)}\|_{L^{\alpha(x)}(B_\varepsilon(x_i))}\||\nabla\phi_{i,\varepsilon}|^{p(x)}\|_{L^{\alpha'(x)}(B_\varepsilon(x_i))},
\end{aligned}$$

where $\alpha(x) = n/(n-p(x))$ and $\alpha'(x) = n/p(x)$.

Moreover, using that $\nabla\phi_{i,\varepsilon} = \nabla\phi\left(\frac{x-x_i}{\varepsilon}\right)\frac{1}{\varepsilon}$, we obtain

$$\||\nabla\phi_{i,\varepsilon}|^{p(x)}\|_{L^{\alpha'(x)}(B_\varepsilon(x_i))} \leq \max\{\rho(|\nabla\phi_{i,\varepsilon}|^{p(x)})^{p^+/n}; \rho(|\nabla\phi_{i,\varepsilon}|^{p(x)})^{p^-/n}\},$$

and

$$\begin{aligned}
\rho(|\nabla\phi_{i,\varepsilon}|^{p(x)}) &= \int_{B_\varepsilon(x_i)} |\nabla\phi_{i,\varepsilon}|^n\,dx \\
&= \int_{B_\varepsilon(x_i)} |\nabla\phi(\frac{x-x_i}{\varepsilon})|^n\frac{1}{\varepsilon^n}\,dx \\
&= \int_{B_1(0)} |\nabla\phi(y)|^n\,dy.
\end{aligned}$$

Then $\nabla\phi_{i,\varepsilon}u \to 0$ strongly in $L^{p(x)}(\Omega)$. On the other hand,

$$\int_\Omega |\phi_{i,\varepsilon}|^{p(x)}\,d\mu \leq \mu(B_\varepsilon(x_i)).$$

Therefore,

$$\begin{aligned}
\|\phi_{i,\varepsilon}\|_{L^{p(x)}(\Omega)} &= \|\phi_{i,\varepsilon}\|_{L^{p(x)}(B_\varepsilon(x_i))} \\
&\leq \max\{\rho_\mu(\phi_{i,\varepsilon})^{1/p^+_{i,\varepsilon}}, \rho_\mu(\phi_{i,\varepsilon})^{1/p^-_{i,\varepsilon}}\} \\
&\leq \max\{\mu(B_\varepsilon(x_i))^{1/p^+_{i,\varepsilon}}, \mu(B_\varepsilon(x_i))^{1/p^-_{i,\varepsilon}}\},
\end{aligned}$$

so we obtain,

$$S\min\{\nu_i^{\frac{1}{q^+_{i,\varepsilon}}}, \nu_i^{\frac{1}{q^-_{i,\varepsilon}}}\} \leq \max\{\mu(B_\varepsilon(x_i))^{1/p^+_{i,\varepsilon}}, \mu(B_\varepsilon(x_i))^{\frac{1}{p^-_{i,\varepsilon}}}\}.$$

As p and q are continuous functions and as $q(x_i) = p^*(x_i)$, letting $\varepsilon \to 0$, we get

$$S\nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)},$$

where $\mu_i := \lim_{\varepsilon\to 0}\mu(B_\varepsilon(x_i))$.

Finally, we show that $\mu \geq |\nabla u|^{p(x)} + \Sigma\mu_i\delta_{x_i}$. In fact, we have that $\mu \geq \sum\mu_i\delta_{x_i}$. On the other hand $u_j \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$ then $\nabla u_j \rightharpoonup \nabla u$ weakly in

$L^{p(x)}(U)$ for all $U \subset \Omega$. By weakly lower semi continuity of norm we obtain that $d\mu \geq |\nabla u|^{p(x)}\,dx$ and, as $|\nabla u|^{p(x)}$ is orthogonal to μ_1, we conclude the desired result. This completes the proof. □

4. Applications

In this section, we apply Theorem 1.1 to study the existence of nontrivial solutions of the problem

$$\begin{gathered} -\Delta_{p(x)}u = |u|^{q(x)-2}u + \lambda(x)|u|^{r(x)-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

where $r(x) < p^*(x) - \varepsilon$, $q(x) \leq p^*(x)$ and $\mathcal{A} = \{x \in \Omega \colon q(x) = p^*(x)\} \neq \emptyset$. We define $A_\delta := \bigcup_{x\in\mathcal{A}}(B_\delta(x) \cap \Omega) = \{x \in \Omega \colon \operatorname{dist}(x, \mathcal{A}) < \delta\}$. The ideas for this application follow those in [11].

For (weak) solutions of (4.1) we understand critical points of the functional

$$\mathcal{F}(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} - \frac{|u|^{q(x)}}{q(x)} - \lambda(x)\frac{|u|^{r(x)}}{r(x)}\,dx$$

4.1. **Proof of Theorem 1.3.** We begin by proving the Palais-Smale condition for the functional $\mathcal{F}$, below certain level of energy.

Lemma 4.1. *Assume that $r \leq q$. Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ a Palais-Smale sequence then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.*

Proof. By definition $\mathcal{F}(u_j) \to c$ and $\mathcal{F}'(u_j) \to 0$. Now, we have

$$c + 1 \geq \mathcal{F}(u_j) = \mathcal{F}(u_j) - \frac{1}{r-}\langle \mathcal{F}'(u_j), u_j\rangle + \frac{1}{r-}\langle \mathcal{F}'(u_j), u_j\rangle,$$

where

$$\langle \mathcal{F}'(u_j), u_j\rangle = \int_\Omega \big(|\nabla u_j|^{p(x)} - |u_j|^{q(x)} - \lambda(x)|u_j|^{r(x)}\big)\,dx.$$

Then, if $r(x) \leq q(x)$, we conclude that

$$c + 1 \geq \big(\frac{1}{p+} - \frac{1}{r-}\big)\int_\Omega |\nabla u_j|^{p(x)}\,dx - \frac{1}{r-}|\langle \mathcal{F}'(u_j), u_j\rangle|.$$

We can assume that $\|u_j\|_{W_0^{1,p(x)}(\Omega)} \geq 1$. As $\|\mathcal{F}'(u_j)\|$ is bounded we have that

$$c + 1 \geq \big(\frac{1}{p+} - \frac{1}{r-}\big)\|u_j\|^{p^-}_{W_0^{1,p(x)}(\Omega)} - \frac{C}{r-}\|u_j\|_{W_0^{1,p(x)}(\Omega)}.$$

We deduce that u_j is bounded. This completes the proof. □

From the fact that $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence it follows, by Lemma 4.1, that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Hence, by Theorem 1.1, we have

$$|u_j|^{q(x)} \rightharpoonup \nu = |u|^{q(x)} + \sum_{i\in I}\nu_i\delta_{x_i} \quad \nu_i > 0, \tag{4.2}$$

$$|\nabla u_j|^{p(x)} \rightharpoonup \mu \geq |\nabla u|^{p(x)} + \sum_{i\in I}\mu_i\delta_{x_i} \quad \mu_i > 0, \tag{4.3}$$

$$S\nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)}. \tag{4.4}$$

Note that if $I = \emptyset$ then $u_j \to u$ strongly in $L^{q(x)}(\Omega)$. We know that $\{x_i\}_{i\in I} \subset \mathcal{A}$.

Let us show that if $c < \big(\frac{1}{p^+} - \frac{1}{q_{\mathcal{A}}^-}\big)S^n$ and $\{u_j\}_{j\in\mathbb{N}}$ is a Palais-Smale sequence, with energy level c, then $I = \emptyset$. In fact, suppose that $I \neq \emptyset$. Then let $\phi \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball of $\mathbb{R}^n$. Consider, as in the previous section, the rescaled functions $\phi_{i,\varepsilon}(x) = \phi(\frac{x-x_i}{\varepsilon})$.

As $\mathcal{F}'(u_j) \to 0$ in $(W_0^{1,p(x)}(\Omega))'$, we obtain that

$$\lim_{j\to\infty} \langle \mathcal{F}'(u_j), \phi_{i,\varepsilon} u_j\rangle = 0.$$

On the other hand,

$$\langle \mathcal{F}'(u_j), \phi_{i,\varepsilon} u_j\rangle = \int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla(\phi_{i,\varepsilon} u_j) - \lambda(x)|u_j|^{r(x)}\phi_{i,\varepsilon} - |u_j|^{q(x)}\phi_{i,\varepsilon}\, dx$$

Then, passing to the limit as $j \to \infty$, we obtain

$$\begin{aligned} 0 = \lim_{j\to\infty} \Big(\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla(\phi_{i,\varepsilon}) u_j\, dx\Big) \\ + \int_\Omega \phi_{i,\varepsilon}\, d\mu - \int_\Omega \phi_{i,\varepsilon}\, d\nu - \int_\Omega \lambda(x)|u|^{r(x)}\phi_{i,\varepsilon}\, dx. \end{aligned}$$

By Hölder inequality, it is easy to check that

$$\lim_{j\to\infty} \int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla(\phi_{i,\varepsilon}) u_j\, dx = 0.$$

On the other hand,

$$\lim_{\varepsilon\to 0} \int_\Omega \phi_{i,\varepsilon}\, d\mu = \mu_i\phi(0), \quad \lim_{\varepsilon\to 0} \int_\Omega \phi_{i,\varepsilon}\, d\nu = \nu_i\phi(0), \lim_{\varepsilon\to 0} \int_\Omega \lambda(x)|u|^{r(x)}\phi_{i,\varepsilon}\, dx = 0.$$

So, we conclude that $(\mu_i - \nu_i)\phi(0) = 0$; i.e., $\mu_i = \nu_i$. Then

$$S\nu_i^{1/p^*(x_i)} \le \nu_i^{1/p(x_i)};$$

so it is clear that $\nu_i = 0$ or $S^n \le \nu_i$.

On the other hand, as $r^- > p^+$,

$$\begin{aligned} c &= \lim_{j\to\infty} \mathcal{F}(u_j) = \lim_{j\to\infty} \mathcal{F}(u_j) - \frac{1}{p+}\langle \mathcal{F}'(u_j), u_j\rangle \\ &= \lim_{j\to\infty} \int_\Omega \big(\frac{1}{p(x)} - \frac{1}{p+}\big)|\nabla u_j|^{p(x)}\, dx + \int_\Omega \big(\frac{1}{p+} - \frac{1}{q(x)}\big)|u_j|^{q(x)}\, dx \\ &\quad + \lambda \int_\Omega \big(\frac{1}{p+} - \frac{1}{r(x)}\big)|u_j|^{r(x)}\, dx \\ &\ge \lim_{j\to\infty} \int_\Omega \big(\frac{1}{p+} - \frac{1}{q(x)}\big)|u_j|^{q(x)}\, dx \\ &\ge \lim_{j\to\infty} \int_{\mathcal{A}_\delta} \big(\frac{1}{p+} - \frac{1}{q(x)}\big)|u_j|^{q(x)}\, dx \\ &\ge \lim_{j\to\infty} \int_{\mathcal{A}_\delta} \big(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}_\delta}^-}\big)|u_j|^{q(x)}\, dx\,. \end{aligned}$$

However,

$$\begin{aligned} \lim_{j\to\infty} \int_{\mathcal{A}_\delta} \big(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}_\delta}^-}\big)|u_j|^{q(x)}\, dx &= \big(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}_\delta}^-}\big)\Big(\int_{\mathcal{A}_\delta} |u|^{q(x)}\, dx + \sum_{j\in I}\nu_j\Big) \\ &\ge \big(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}_\delta}^-}\big)\nu_i \end{aligned}$$

$$\geq (\frac{1}{p+} - \frac{1}{q_{\mathcal{A}_\delta}^-})S^n.$$

As δ is positive and arbitrary, and q is continuous, we have

$$c \geq (\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-})S^n.$$

Therefore, if

$$c < (\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-})S^n,$$

the index set I is empty.

Now we are ready to prove the Palais-Smale condition below level c.

Theorem 4.2. *Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence, with energy level c. If $c < \big(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-}\big)S^n$, then there exist $u \in W_0^{1,p(x)}(\Omega)$ and $\{u_{j_k}\}_{k\in\mathbb{N}} \subset \{u_j\}_{j\in\mathbb{N}}$ a subsequence such that $u_{j_k} \to u$ strongly in $W_0^{1,p(x)}(\Omega)$.*

Proof. We have that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. Then, for a subsequence that we still denote $\{u_j\}_{j\in\mathbb{N}}$, $u_j \to u$ strongly in $L^{q(x)}(\Omega)$. We define $\mathcal{F}'(u_j) := \phi_j$. By the Palais-Smale condition, with energy level c, we have $\phi_j \to 0$ in $(W_0^{1,p(x)}(\Omega))'$.

By definition $\langle \mathcal{F}'(u_j), z\rangle = \langle \phi_j, z\rangle$ for all $z \in W_0^{1,p(x)}(\Omega)$; i.e.,

$$\int_\Omega |\nabla u_j|^{p(x)-2}\nabla u_j \nabla z\,dx - \int_\Omega |u_j|^{q(x)-2}u_j z\,dx - \int_\Omega \lambda(x)|u_j|^{r(x)-2}u_j z\,dx = \langle \phi_j, z\rangle.$$

Then, u_j is a weak solution of the following equation.

$$\begin{gathered} -\Delta_{p(x)}u_j = |u_j|^{q(x)-2}u_j + \lambda(x)|u_j|^{r(x)-2}u_j + \phi_j =: f_j \quad \text{in } \Omega, \\ u_j = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.5}$$

We define $T\colon (W_0^{1,p(x)}(\Omega))' \to W_0^{1,p(x)}(\Omega)$, $T(f) := u$ where u is the weak solution of the equation

$$\begin{gathered} -\Delta_{p(x)}u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.6}$$

Then T is a continuous invertible operator.

It is sufficient to show that f_j converges in $(W_0^{1,p(x)}(\Omega))'$. We only need to prove that $|u_j|^{q(x)-2}u_j \to |u|^{q(x)-2}u$ strongly in $(W_0^{1,p(x)}(\Omega))'$. In fact,

$$\begin{aligned} \langle |u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u, \psi\rangle &= \int_\Omega (|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\psi\,dx \\ &\leq \|\psi\|_{L^{q(x)}(\Omega)} \|(|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\|_{L^{q'(x)}(\Omega)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|(|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\|_{(W_0^{1,p(x)}(\Omega))'} \\ &= \sup_{\substack{\psi\in W_0^{1,p(x)}(\Omega) \\ \|\psi\|_{W_0^{1,p(x)}(\Omega)}=1}} \int_\Omega (|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\psi\,dx \\ &\leq \|(|u_j|^{q(x)-2}u_j - |u|^{q(x)-2}u)\|_{L^{q'(x)}(\Omega)} \end{aligned}$$

and now, by the Dominated Convergence Theorem this last term approaches zero as $j \to \infty$. The proof is complete. □

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. In view of the previous result, we seek for critical values below level c. For that purpose, we want to use the Mountain Pass Theorem. Hence we have to check the following condition:

(1) There exist constants $R, r > 0$ such that when $\|u\|_{W^{1,p(x)}(\Omega)} = R$, then $\mathcal{F}(u) > r$.
(2) There exist $v_0 \in W^{1,p(x)}(\Omega)$ such that $\mathcal{F}(v_0) < r$.

Let us first check (1). We suppose that $\||\nabla u|\|_{L^{p(x)}(\Omega)} \le 1$ and $\|u\|_{L^{p(x)}(\Omega)} \le 1$. The other cases can be treated similarly.

By Poincaré inequality (Proposition 3.1), we have

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} - \frac{|u|^{q(x)}}{q(x)} - \lambda(x)\frac{|u|^{r(x)}}{r(x)}\,dx \\
&\ge \frac{1}{p+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{1}{q-}\int_\Omega |u|^{q(x)}\,dx - \frac{\|\lambda\|_\infty}{r-}\int_\Omega |u|^{r(x)}\,dx \\
&\ge \frac{1}{p+}\||\nabla u|\|^{p+} - \frac{1}{q-}\|u\|^{q-}_{L^{q(x)}(\Omega)} - \frac{\|\lambda\|_\infty}{r-}\|u\|^{r-}_{L^{r(x)}(\Omega)} \\
&\ge \frac{1}{p+}\||\nabla u|\|^{p+} - \frac{C}{q-}\||\nabla u|\|^{q-}_{L^{p(x)}(\Omega)} - \frac{C\|\lambda\|_\infty}{r-}\||\nabla u|\|^{r-}_{L^{p(x)}(\Omega)}.
\end{aligned}
$$

Let $g(t) = \frac{1}{p+}t^{p+} - \frac{C}{q-}t^{q-} - \frac{C\|\lambda\|_\infty}{r-}t^{r-}$, then it is easy to check that $g(R) > r$ for some $R, r > 0$. This proves (1).

Now (2) is immediate as for a fixed $w \in W_0^{1,p(x)}(\Omega)$ we have

$$
\lim_{t\to\infty} \mathcal{F}(tw) = -\infty.
$$

Now the candidate for critical value according to the Mountain Pass Theorem is

$$
c = \inf_{g\in\mathcal{C}} \sup_{t\in[0,1]} \mathcal{F}(g(t)),
$$

where $\mathcal{C} = \{g : [0,1] \to W_0^{1,p(x)}(\Omega) \colon g \text{ continuous and } g(0) = 0, g(1) = v_0\}$.

We will show that, if $\inf_{x\in\mathcal{A}_\delta} \lambda(x)$ is big enough for some $\delta > 0$ then $c < \big(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-}\big)S^n$ and so the local Palais-Smale condition (Theorem 4.2) can be applied. We fix $w \in W_0^{1,p(x)}(\Omega)$. Then, if $t < 1$, we have

$$
\begin{aligned}
\mathcal{F}(tw) &\le \int_\Omega t^{p(x)}\frac{|\nabla w|^{p(x)}}{p-} - t^{q(x)}\frac{|w|^{q(x)}}{q+} - \lambda(x)t^{r(x)}\frac{|w|^{r(x)}}{r+}\,dx \\
&\le \frac{t^{p-}}{p-}\int_\Omega |\nabla w|^{p(x)}\,dx - \frac{t^{r+}}{r+}\int_\Omega \lambda(x)|w|^{r(x)}\,dx \\
&\le \frac{t^{p-}}{p-}\int_\Omega |\nabla w|^{p(x)}\,dx - \frac{t^{r+}}{r+}\int_{\mathcal{A}_\delta} \lambda(x)|w|^{r(x)}\,dx \\
&\le \frac{t^{p-}}{p-}\int_\Omega |\nabla w|^{p(x)}\,dx - \frac{t^{r+}}{r+}\int_{\mathcal{A}_\delta} (\inf_{x\in\mathcal{A}_\delta} \lambda(x))|w|^{r(x)}\,dx
\end{aligned}
$$

We define $g(t) := \frac{t^{p-}}{p-}a_1 - (\inf_{x\in\mathcal{A}_\delta} \lambda(x))\frac{t^{r+}}{r+}a_3$, where a_1 and a_2 are given by $a_1 = \||\nabla w|^{p(x)}\|_{L^1(\Omega)}$ and $a_3 = \||w|^{r(x)}\|_{L^1(\mathcal{A}_\delta)}$.

The maximum of g is attained at $t_\lambda = \big(\frac{a_1}{(\inf_{x\in\mathcal{A}_\delta}\lambda(x))a_3}\big)^{\frac{1}{r+-p-}}$. So, we conclude that there exists $\lambda_0 > 0$ such that if $(\inf_{x\in\mathcal{A}_\delta}\lambda(x)) \geq \lambda_0$ then

$$\mathcal{F}(tw) < \big(\frac{1}{p+} - \frac{1}{q_{\mathcal{A}}^-}\big)S^n$$

This completes the proof. □

Remark 4.3. Observe that if $\lambda(x)$ is continuous it suffices to assume that $\lambda(x)$ is large in the *criticality set* $\mathcal{A}$.

4.2. **Proof of Theorem 1.2.** Now it remains to prove Theorem 1.2. So we begin by checking the Palais-Smale condition for this case.

Lemma 4.4. *Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence for $\mathcal{F}$ then $\{u_j\}_{j\in\mathbb{N}}$ is bounded.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence; that is, $\mathcal{F}(u_j) \to c$ and $\mathcal{F}'(u_j) \to 0$. Therefore there exists a sequence $\varepsilon_j \to 0$ such that

$$|\mathcal{F}'(u_j)w| \leq \varepsilon_j \|w\|_{W_0^{1,p(x)}(\Omega)} \quad \text{for all } w \in W_0^{1,p(x)}(\Omega).$$

Now we have

$$\begin{aligned} c+1 &\geq \mathcal{F}(u_j) - \frac{1}{q^-}\mathcal{F}'(u_j)u_j + \frac{1}{q^-}\mathcal{F}'(u_j)u_j \\ &\geq \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\int_\Omega |\nabla u_j|^{p(x)}\,dx + \int_\Omega \big(\frac{\lambda(x)}{q^-} - \frac{\lambda(x)}{r^-}\big)|u_j|^{r(x)}\,dx + \frac{1}{q^-}\mathcal{F}'(u_j)u_j \end{aligned}$$

We can assume that $\||\nabla u_j|\|_{L^{p(x)}(\Omega)} > 1$. Then we have, by Proposition 2.5 and by Poincaré inequality,

$$\begin{aligned} c+1 &\geq \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\||\nabla u_j|\|_{L^{p(x)}(\Omega)}^{p^-} + \|\lambda\|_\infty\big(\frac{1}{q^-} - \frac{1}{r^-}\big)\|u_j\|_{L^{r(x)}(\Omega)}^{r^+} \\ &\quad - \frac{1}{q^-}\|u_j\|_{W_0^{1,p(x)}(\Omega)}\varepsilon_j \\ &\geq \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\||\nabla u_j|\|_{L^{p(x)}(\Omega)}^{p^-} + \|\lambda\|_\infty\big(\frac{1}{q^-} - \frac{1}{r^-}\big)C\||\nabla u_j|\|_{L^{p(x)}(\Omega)}^{r^+} \\ &\quad - \frac{1}{q^-}\|u_j\|_{W_0^{1,p(x)}(\Omega)} \end{aligned}$$

from where it follows that $\|u_j\|_{W_0^{1,p(x)}(\Omega)}$ is bounded (recall that $p^+ \leq q^-$ and $r^+ < p^-$). □

Let $\{u_j\}_{j\in\mathbb{N}}$ be a Palais-Smale sequence for $\mathcal{F}$. Therefore, by the previous Lemma, it follows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Then, by Theorem 1.1 we can assume that there exist two measures μ, ν and a function $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_j \rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega), \tag{4.7}$$

$$|\nabla u_j|^{p(x)} \rightharpoonup \mu \quad \text{weakly in the sense of measures}, \tag{4.8}$$

$$|u_j|^{q(x)} \rightharpoonup \nu \quad \text{weakly in the sense of measures}, \tag{4.9}$$

$$\nu = |u|^{q(x)} + \sum_{i\in I}\nu_i\delta_{x_i}, \tag{4.10}$$

$$\mu \geq |\nabla u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i}, \tag{4.11}$$

$$S\nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)}. \tag{4.12}$$

As before, assume that $I \neq \emptyset$. Now the proof follows exactly as in the previous case, until we get to

$$c \geq \big(\frac{1}{p^+} - \frac{1}{q^-}\big) \int_\Omega |u|^{q(x)}\,dx + \big(\frac{1}{p^+} - \frac{1}{q^-}\big) S^n + \|\lambda\|_{L^\infty(\Omega)} \big(\frac{1}{p^+} - \frac{1}{r^-}\big) \int_\Omega |u|^{r(x)}\,dx.$$

Applying now Hölder inequality, we find

$$\begin{aligned} c \geq &\big(\frac{1}{p^+} - \frac{1}{q^-}\big) \int_\Omega |u|^{q(x)}\,dx + \big(\frac{1}{p^+} - \frac{1}{q^-}\big) S^n \\ &+ \|\lambda\|_{L^\infty(\Omega)} \big(\frac{1}{p^+} - \frac{1}{r^-}\big) \||u|^{r(x)}\|_{L^{q(x)/r(x)}(\Omega)} |\Omega|^{\frac{q^+}{q^- - r^+}}. \end{aligned}$$

If $\||u|^{r(x)}\|_{L^{q(x)/r(x)}(\Omega)} \geq 1$, we have

$$c \geq c_1 \||u|^{r(x)}\|^{(q/r)^-}_{L^{q(x)/r(x)}(\Omega)} + c_3 - \|\lambda\|_{L^\infty(\Omega)} c_2 \||u|^{r(x)}\|_{L^{q(x)/r(x)}(\Omega)},$$

so, if $f_1(x) := c_1 x^{(q/r)^-} - \|\lambda\|_{L^\infty(\Omega)} c_2 x$, this function reaches its absolute minimum at $x_0 = \big(\frac{\|\lambda\|_{L^\infty(\Omega)} c_2}{c_1 (q/r)^-}\big)^{\frac{1}{(q/r)^- - 1}}$.

On the other hand, if $\||u|^{r(x)}\|_{L^{q(x)/r(x)}(\Omega)} < 1$, then

$$c \geq c_1 \||u|^{r(x)}\|^{(q/r)^+}_{L^{q(x)/r(x)}(\Omega)} + c_3 - \|\lambda\|_{L^\infty(\Omega)} c_2 \|u\|_{L^{q(x)/r(x)}(\Omega)},$$

so, if $f_2(x) = c_1 x^{(q/r)^+} - \|\lambda\|_{L^\infty(\Omega)} c_2 x$, this function reaches its absolute minimum at $x_0 = \big(\frac{\|\lambda\|_{L^\infty(\Omega)} c_2}{c_1 (q/r)^+}\big)^{\frac{1}{(q/r)^+ - 1}}$. Then

$$c \geq \big(\frac{1}{p+} - \frac{1}{q^-}\big) S^n + K \min\{\|\lambda\|^{\frac{(q/r)^-}{(q/r)^- - 1}}_{L^\infty(\Omega)}, \|\lambda\|^{\frac{(q/r)^+}{(q/r)^+ - 1}}_{L^\infty(\Omega)}\},$$

which contradicts our hypothesis. Therefore $I = \emptyset$ and so $u_j \to u$ strongly in $L^{q(x)}(\Omega)$.

With these preliminaries the Palais-Smale condition can now be easily checked.

Lemma 4.5. *Let $(u_j) \subset W_0^{1,p(x)}(\Omega)$ be a Palais-Smale sequence for $\mathcal{F}$, with energy level c. There exists a constant K depending only on p, q, r and Ω such that, if $c < \big(\frac{1}{p+} - \frac{1}{q^-}\big) S^n + K \min\{\|\lambda\|^{\frac{(q/r)^-}{(q/r)^- - 1}}_{L^\infty(\Omega)}, \|\lambda\|^{\frac{(q/r)^+}{(q/r)^+ - 1}}_{L^\infty(\Omega)}\}$, then there exists a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}} \subset \{u_j\}_{j\in\mathbb{N}}$ that converges strongly in $W_0^{1,p(x)}(\Omega)$.*

The proof of the above lemma follows by the continuity of the solution operator as in Theorem 4.2.

Assume now that $\||\nabla u|\|_{L^{p(x)}(\Omega)} \leq 1$. Then, applying Poincaré inequality, we have

$$\begin{aligned} \mathcal{F}(u) &\geq \frac{1}{p^+} \||\nabla u|\|^{p^+}_{L^{p(x)}(\Omega)} - \frac{1}{q^-} \|u\|^{q^-}_{L^{q(x)}(\Omega)} - \frac{\|\lambda\|_{L^\infty(\Omega)}}{r^-} \|u\|^{r^-}_{L^{r(x)}(\Omega)} \\ &\geq \frac{1}{p^+} \||\nabla u|\|^{p^+}_{L^{p(x)}(\Omega)} - \frac{C}{q^-} \||\nabla u|\|^{q^-}_{L^{p(x)}(\Omega)} - \frac{\|\lambda\|_{L^\infty(\Omega)} C}{r^-} \||\nabla u|\|^{r^-}_{L^{p(x)}(\Omega)} \\ &=: J_1(\||\nabla u|\|_{L^{p(x)}(\Omega)}), \end{aligned}$$

where $J_1(x) = \frac{1}{p^+}x^{p^+} - \frac{C}{q^-}x^{q^-} - \frac{\|\lambda\|_{L^\infty(\Omega)}C}{r^-}x^{r^-}$. We recall that $p^+ \le q^-$ and $r^- < r^+ < p^- < p^+$.

As J_1 attains a local, but not a global, minimum (J_1 is not bounded below), we have to perform some sort of truncation. To this end let x_0, x_1 be such that $m < x_0 < M < x_1$ where m is the local minimum and M is the local maximum of J_1 and $J_1(x_1) > J_1(m)$. For these values x_0 and x_1 we can choose a smooth function $\tau_1(x)$ such that $\tau_1(x) = 1$ if $x \le x_0$, $\tau_1(x) = 0$ if $x \ge x_1$ and $0 \le \tau_1(x) \le 1$.

If $\||\nabla u|\|_{L^{p(x)}(\Omega)} > 1$, we argue similarly and obtain

$$\begin{aligned}\mathcal{F}(u) &\ge \frac{1}{p^+}\||\nabla u|\|^{p^-}_{L^{p(x)}(\Omega)} - \frac{C}{q^-}\||\nabla u|\|^{q^+}_{L^{p(x)}(\Omega)} - \frac{\|\lambda\|_{L^\infty(\Omega)}C}{r^-}\||\nabla u|\|^{r^+}_{L^{p(x)}(\Omega)}\\ &=: J_2(\||\nabla u|\|_{L^{p(x)}(\Omega)})\end{aligned}$$

where

$$J_2(x) = \frac{1}{p^+}x^{p^-} - \frac{C}{q^-}x^{q^+} - \frac{\|\lambda\|_{L^\infty(\Omega)}C}{r^-}x^{r^+}.$$

As in the previous case, J_2 attains a local but not a global minimum. So let x_0, x_1 be such that $m < x_0 < M < x_1$ where m is the local minimum of j and M is the local maximum of J_2 and $J_2(x_1) > J_2(m)$. For these values x_0 and x_1 we can choose a smooth function $\tau_2(x)$ with the same properties as τ_1. Finally, we define

$$\tau(x) = \begin{cases}\tau_1(x) & \text{if } x \le 1\\ \tau_2(x) & \text{if } x > 1.\end{cases}$$

Next, let $\varphi(u) = \tau(\||\nabla u|\|_{L^{p(x)}(\Omega)})$ and define the truncated functional as follows,

$$\tilde{\mathcal{F}}(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{|u|^{q(x)}}{q(x)}\varphi(u)\,dx - \int_\Omega \frac{\lambda(x)}{r(x)}|u|^{r(x)}\,dx$$

Next we state a Lemma that contains the main properties of $\tilde{\mathcal{F}}$.

Lemma 4.6. *$\tilde{\mathcal{F}}$ is C^1, if $\tilde{\mathcal{F}}(u) \le 0$ then $\|u\|_{W_0^{1,p(x)}(\Omega)} < x_0$ and $\mathcal{F}(v) = \tilde{\mathcal{F}}(v)$ for every v close enough to u. Moreover there exists $\lambda_1 > 0$ such that if $0 < \|\lambda\|_{L^\infty(\Omega)} < \lambda_1$ then $\tilde{\mathcal{F}}$ satisfies a local Palais-Smale condition for $c \le 0$.*

Proof. We have to check only the local Palais-Smale condition. Observe that every Palais-Smale sequence for $\tilde{\mathcal{F}}$ with energy level $c \le 0$ must be bounded, therefore by Lemma 4.5 if λ verifies

$$0 < (\frac{1}{p+} - \frac{1}{q^-})S^n + K\min\{\|\lambda\|^{\frac{(q/r)^-}{(q/r)^- - 1}}_{L^\infty(\Omega)}, \|\lambda\|^{\frac{(q/r)^+}{(q/r)^+ - 1}}_{L^\infty(\Omega)}\},$$

then there exists a convergent subsequence. □

The following Lemma gives the final ingredients needed in the proof.

Lemma 4.7. *For every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that*

$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \ge n$$

where $\tilde{\mathcal{F}}^{-\varepsilon} = \{u \in W_0^{1,p(x)}(\Omega) \colon \tilde{\mathcal{F}}(u) \le -\varepsilon\}$ and γ is the Krasnoselskii genus.

Proof. Let $E_n \subset W_0^{1,p(x)}(\Omega)$ be a n-dimensional subspace. Hence we have, for $u \in E_n$ such that $\|u\|_{W_0^{1,p(x)}(\Omega)} = 1$,

$$\begin{aligned}\tilde{\mathcal{F}}(tu) &= \int_\Omega \frac{|\nabla(tu)|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{|tu|^{q(x)}}{q(x)}\varphi(tu)\,dx - \int_\Omega \frac{\lambda(x)}{r(x)}|tu|^{r(x)}\,dx\\ &\le \int_\Omega \frac{|\nabla(tu)|^{p(x)}}{p^-}\,dx - \int_\Omega \frac{|tu|^{q(x)}}{q^+}\varphi(tu)\,dx - \int_\Omega \frac{\lambda(x)}{r^+}|tu|^{r(x)}\,dx.\end{aligned}$$

If $t < 1$, then

$$\begin{aligned}\tilde{\mathcal{F}}(tu) &\le \int_\Omega \frac{t^{p^-}|\nabla u|^{p(x)}}{p^-}\,dx - \int_\Omega \frac{t^{q^+}|u|^{q(x)}}{q^+}\,dx - \int_\Omega \frac{\inf_{x\in\Omega}\lambda(x)}{r^+}t^{r^+}|u|^{r(x)}\,dx\\ &\le \frac{t^{p^-}}{p^-} - \frac{t^{q^+}}{q^+}a_n - \inf_{x\in\Omega}\lambda(x)\frac{t^{r^+}}{r^+}b_n,\end{aligned}$$

where

$$\begin{aligned}a_n &= \inf\Big\{\int_\Omega |u|^{q(x)}\,dx\colon u \in E_n, \|u\|_{W_0^{1,p(x)}(\Omega)} = 1\Big\},\\ b_n &= \inf\Big\{\int_\Omega |u|^{r(x)}\,dx\colon u \in E_n, \|u\|_{W_0^{1,p(x)}(\Omega)} = 1\Big\}.\end{aligned}$$

Then

$$\tilde{\mathcal{F}}(tu) \le \frac{t^{p^-}}{p^-} - \frac{t^{q^+}}{q^+}a_n - \inf_{x\in\Omega}\lambda(x)\frac{t^{r^+}}{r^+}b_n \le \frac{t^{p^-}}{p^-} - \inf_{x\in\Omega}\lambda(x)\frac{t^{r^+}}{r^+}b_n\,.$$

Observe that $a_n > 0$ and $b_n > 0$ because E_n is finite dimensional. As $r^+ < p^-$ and $t < 1$ we obtain that there exists positive constants ρ and ε such that

$$\tilde{\mathcal{F}}(\rho u) < -\varepsilon \quad \text{for } u \in E_n, \|u\|_{W_0^{1,p(x)}(\Omega)} = 1.$$

Therefore, if we set $S_{\rho,n} = \{u \in E_n : \|u\| = \rho\}$, we have that $S_{p,n} \subset \tilde{\mathcal{F}}^{-\varepsilon}$. Hence by monotonicity of the genus

$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \ge \gamma(S_{\rho,n}) = n$$

as we wanted to show. □

Theorem 4.8. *Let*

$$\Sigma = \{A \subset W_0^{1,p(x)}(\Omega) - 0\colon A \text{ is closed, } A = -A\}, \quad \Sigma_k = \{A \subset \Sigma\colon \gamma(A) \ge k\},$$

where γ stands for the Krasnoselskii genus. Then

$$c_k = \inf_{A\in\Sigma_k}\sup_{u\in A}\mathcal{F}(u)$$

is a negative critical value of $\mathcal{F}$ and moreover, if $c = c_k = \dots = c_{k+r}$, then $\gamma(K_c) \ge r + 1$, where $K_c = \{u \in W^{1,p(x)}(\Omega)\colon \mathcal{F}(u) = c, \mathcal{F}'(u) = 0\}$.

The proof follows exactly the steps in in [11], using Lemma 4.7.

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References

[1] Claudianor O. Alves; Existence of positive solutions for a problem with lack of compactness involving the p-Laplacian. *Nonlinear Anal.*, 51(7):1187–1206, 2002.

[2] Claudianor O. Alves and Yanheng Ding; Existence, multiplicity and concentration of positive solutions for a class of quasilinear problems. *Topol. Methods Nonlinear Anal.*, 29(2):265–278, 2007.

[3] Abbas Bahri and Pierre-Louis Lions; On the existence of a positive solution of semilinear elliptic equations in unbounded domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(3):365–413, 1997.

[4] Julián Fernández Bonder, Sandra Martínez, and Julio D. Rossi; Existence results for gradient elliptic systems with nonlinear boundary conditions. *NoDEA Nonlinear Differential Equations Appl.*, 14(1-2):153–179, 2007.

[5] Alberto Cabada and Rodrigo L. Pouso; Existence theory for functional p-Laplacian equations with variable exponents. *Nonlinear Anal.*, 52(2):557–572, 2003.

[6] Teodora-Liliana Dinu; Nonlinear eigenvalue problems in Sobolev spaces with variable exponent. *J. Funct. Spaces Appl.*, 4(3):225–242, 2006.

[7] Pavel Drábek and Yin Xi Huang; Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbf{R}^N$ with critical Sobolev exponent. *J. Differential Equations*, 140(1):106–132, 1997.

[8] Xian-Ling Fan and Qi-Hu Zhang; Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal.*, 52(8):1843–1852, 2003.

[9] Xianling Fan and Dun Zhao; On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, 263(2):424–446, 2001.

[10] Yongqiang Fu; The principle of concentration compactness in $L^{p(x)}(\Omega)$ spaces and its application. *Nonlinear Anal.*, 71(5-6):1876–1892, 2009.

[11] J. García Azorero and I. Peral Alonso; Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Amer. Math. Soc.*, 323(2):877–895, 1991.

[12] P.-L. Lions; The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[13] Mihai Mihăilescu; Elliptic problems in variable exponent spaces. *Bull. Austral. Math. Soc.*, 74(2):197–206, 2006.

[14] Mihai Mihăilescu and Vicenţiu Rădulescu; On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent. *Proc. Amer. Math. Soc.*, 135(9):2929–2937 (electronic), 2007.

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