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REMARK ON WELL-POSEDNESS AND ILL-POSEDNESS FOR THE KDV EQUATION

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Abstract. We consider the Cauchy problem for the KdV equation with low regularity initial data given in the space $H^{s,a}(\mathbb{R})$, which is defined by the norm

$$\|\varphi\|_{H^{s,a}} = \|\langle\xi\rangle^{s-a}|\xi|^a\widehat{\varphi}\|_{L^2_\xi}.$$

We obtain the local well-posedness in $H^{s,a}$ with $s \geq \max\{-3/4, -a-3/2\}$, $-3/2 < a \leq 0$ and $(s,a) \neq (-3/4,-3/4)$. The proof is based on Kishimoto's work [12] which proved the sharp well-posedness in the Sobolev space $H^{-3/4}(\mathbb{R})$. Moreover we prove ill-posedness when $s < \max\{-3/4, -a-3/2\}$, $a \leq -3/2$ or $a > 0$.

1. Introduction

We consider the Cauchy problem of the Korteweg-de Vries equation as follows;

$$\begin{gathered} \partial_t u + \partial_x^3 u - 3\partial_x (u)^2 = 0, \quad (t,x) \in [0,T]\times\mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.1}$$

Here the given data u_0 and an unknown function u are real-valued. We consider (1.1) with initial data given in the space $H^{s,a}(\mathbb{R})$, which is defined by the norm

$$\|\varphi\|_{H^{s,a}} := \|\langle\xi\rangle^{s-a}|\xi|^a\widehat{\varphi}(\xi)\|_{L^2_\xi},$$

where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and $\widehat{u}$ is the Fourier transform of u. The KdV equation was originally derived by Korteweg and de Vries [15] as a model for the propagation of shallow water waves along a canal. This equation is completely integrable in the sense that there are Lax formulations, which have an infinite number of conservation laws as follows;

$$\int u^2 dx, \quad \int (\partial_x u)^2 + 2u^3 dx, \quad \int (\partial_x^2 u)^2 + 5\partial_x(\partial_x u)^2 + \frac{5}{2}u^4 dx, \quad \text{etc.}$$

Our main aim is to prove the local well-posedness (LWP for short) for (1.1) with low regularity initial data given in $H^{s,a}(\mathbb{R})$. The main tool is the Fourier restriction norm method introduced by Bourgain [3].

We recall some known results of LWP for (1.1) with initial data given in the Sobolev space $H^s(\mathbb{R})$. The viscosity method was applied to establish LWP for (1.1)

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with $s > 3/2$, (see [2]). Kenig, Ponce and Vega [9] proved LWP for $s > 3/4$ by the iterative approach exploiting the local smoothing effect for the Airy operator $e^{-t\partial_x^3}$. Bourgain [3] established the Fourier restriction norm method and showed LWP for $s \geq 0$ by this method, which was improved to $s > -3/4$ by Kenig, Ponce and Vega [10]. In [11], they also proved that the data-to-solution map fails to be uniformly continuous as a map from H^s to $C([0,T];H^s)$ for $s < -3/4$, (see also [5]). Kishimoto [12] showed LWP and the global well-posedness for (1.1) at the critical regularity $s = -3/4$, (see also [8]). In [18], Tzvetkov proved the flow map $\dot{H}^s \ni u_0 \mapsto u(t) \in \dot{H}^s$ cannot be C^2 for $s < -3/4$.

Under the following assumptions we obtain the following well-posedness result which is generalization of [12].

$$s \geq \max\{-\frac{3}{4}, -a - \frac{3}{2}\}, \quad -\frac{3}{2} < a \leq 0, \quad (s,a) \neq (-\frac{3}{4}, -\frac{3}{4}) . \tag{1.2}$$

Theorem 1.1. *Let s, a satisfy (1.2). Then (1.1) is locally well-posed in $H^{s,a}$.*

We put $s_a = -a - 3/2$ and $B_r(\mathcal{X}) := \{u \in \mathcal{X}; \|u\|_{\mathcal{X}} \leq r\}$ for a Banach space $\mathcal{X}$. We obtain ill-posedness for (1.1) in the following sense when $s < \max\{-3/4, -a - 3/2\}$, $a \leq -3/2$ or $a > 0$.

Theorem 1.2.
- (i) *Let $r > 1$ and $-3/2 < a < -3/4$. Then, from Proposition 4.1 below, there exist $T > 0$ and the flow map for (1.1) $B_r(H^{s_a,a}) \ni u_0 \mapsto u(t) \in H^{s_a,a}$ for any $t \in (0,T]$. The flow map is discontinuous on $B_r(H^{s_a,a})$ (with $H^{s,a}$ topology) to $H^{s_a,a}$ (with $H^{s,a}$ topology) for any $s < s_a$.*
- (ii) *Let $s < s_a$, $a \leq -3/2$ or $0 < a$. Then there is no $T > 0$ such that the flow map for (1.1), $u_0 \mapsto u(t)$, is C^2 as a map from $B_r(H^{s,a})$ to $H^{s,a}$ for $t \in (0,T]$.*
- (iii) *Let $s < -3/4$ and $a \in \mathbb{R}$. Then there is no $T > 0$ such that the flow map for (1.1), $u_0 \mapsto u(t)$, is C^3 as a map from $B_r(H^{s,a})$ to $H^{s,a}$ for any $t \in (0,T]$.*

We consider (1.1) with initial data given in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$. Noting $\dot{H}^s(\mathbb{R}) = H^{s,a}(\mathbb{R})$ if $s = a$, we immediately obtain the following results.

Corollary 1.3. *Let $-3/4 < s \leq 0$. Then (1.1) is well-posed in $\dot{H}^s$.*

Corollary 1.4.
- (i) *Let $r > 1$, $s_s - s - 3/2$ and $-3/2 < s < -3/4$. Then, from Theorem 1.3, there exists $T > 0$ and the flow map for (1.1) $B_r(H^{s_s,s}) \ni u_0 \mapsto u(t) \in H^{s_s,s}$ for any $t \in (0,T]$. The flow map is discontinuous on $B_r(H^{s_s,s})$ (with $\dot{H}^s$ topology) to $H^{s_s,s}$ (with $\dot{H}^s$ topology).*
- (ii) *Let $s > 0$ or $s \leq -3/2$. Then there is no $T > 0$ such that the flow map for (1.1), $u_0 \mapsto u(t)$, is C^2 as a map from $B_r(\dot{H}^s)$ to $\dot{H}^s$ for $t \in (0,T]$.*

Remark. We do not know whether LWP for (1.1) holds or not in $H^{-3/4,-3/4}$. In the present paper, we only prove LWP when $s \geq \max\{-3/4, -a - 3/2\}$, $-3/2 < a < 0$ and $(s,a) \neq (-3/4, -3/4)$ because the case $a = 0$ is proved in [12].

The main idea is how to define the function space to construct the solution of (1.1). The bilinear estimates of the nonlinear term $\partial_x(u)^2$ play an important role to prove Theorem 1.1. Here the Bourgain space $\hat{X}^{s,a,b}$ is defined by

$$\hat{X}^{s,a,b} := \{f \in \mathcal{Z}'(\mathbb{R}^2); \|f\|_{\hat{X}^{s,a,b}} := \|\langle\xi\rangle^{s-a}|\xi|^a\langle\tau - \xi^3\rangle^b f\|_{L^2_{\tau,\xi}} < \infty\}.$$

Here $\mathcal{Z}'(\mathbb{R}^n)$ denotes the dual space of

$$\mathcal{Z}(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n); D^\alpha \mathcal{F}f(0) = 0 \text{ for every multi-index } \alpha\}.$$

For details on $\mathcal{Z}(\mathbb{R}^n)$, see e.g. [17, pp. 237].

We consider the bilinear estimate in the Bourgain space $\hat{X}^{s,a,b}$ as follows

$$\|\xi f * g\|_{\hat{X}^{s,a,b-1}} \le C\|f\|_{\hat{X}^{s,a,b}}\|g\|_{\hat{X}^{s,a,b}}. \tag{1.3}$$

However, (1.3) fails to hold for any $b \in \mathbb{R}$ when

$$s = -\frac{3}{4}, \quad -\frac{3}{4} < a \le 0, \tag{1.4}$$

$$s = -\frac{3}{4} + \varepsilon_1, \quad a = -\frac{3}{4}, \quad \text{or} \quad s = -a - \frac{3}{2}, \quad -\frac{16}{15} < a < -\frac{3}{4}, \tag{1.5}$$

where ε_1 is a sufficiently small number such that $0 < \varepsilon_1 \le s + 3/4$. Therefore, the standard argument by using the Fourier restriction norm method does not work for (1.4)–(1.5). To overcome this difficulty, we modify the Bourgain space to establish bilinear estimates for (1.4)–(1.5). An idea of a modification of the Bourgain space is used by Bejenaru-Tao [1] to prove LWP at the critical regularity $s = -1$ for the quadratic Schrödinger equation with nonlinear term u^2. We consider counterexamples of (1.3) to find a suitable function space in the case (1.4). Noting Example 5.3 in the appendix, we make a modification to the Besov type space as follows:

$$\|f\|_{\hat{X}^{s,1/2}_{(2,1)}} := \big\|\big\{\|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^{1/2} f\|_{L^2_{\tau,\xi}(A_j\cap B_k)}\big\}_{j\ge0,k\ge0}\big\|_{l^2_j l^1_k},$$

where A_j, B_k are two dyadic decompositions defined by

$$A_j := \{(\tau,\xi) \in \mathbb{R}^2; 2^j \le \langle\xi\rangle < 2^{j+1}\},$$
$$B_k := \{(\tau,\xi) \in \mathbb{R}^2; 2^k \le \langle\tau-\xi^3\rangle < 2^{k+1}\},$$

for $j,k \in \mathbb{N}\cup\{0\}$. For a normed space $\mathcal{X}$ and a set $\Omega \subset \mathbb{R}^n$, $\|\cdot\|_{\mathcal{X}(\Omega)}$ is defined by $\|f\|_{\mathcal{X}(\Omega)} := \|\chi_\Omega f\|_{\mathcal{X}}$ where χ_Ω is the characteristic function of Ω.

From Examples 5.1 and 5.2 in the appendix, we have to take $b = a/3 + 1/2$ on the domain

$$D_0 := \big\{(\tau,\xi) \in \mathbb{R}^2 \ ; \ |\xi| \le 1 \text{ and } |\tau| \sim |\xi|^{-3}\big\}$$

to obtain (1.3) for (1.4). Therefore, we make a modification on the Bourgain norm in the low frequency part $\{|\xi| \le 1\}$ as follows:

$$\|f\|_{\hat{X}^a_L} := \begin{cases} \|f\|_{\hat{X}_L^{a,a/3+1/2}(A_0)} & \text{for } -3/4 < a < 0, \\ \|f\|_{\hat{X}_L^{-3/4,1/4+\varepsilon_1/2}(A_0)} & \text{for } a = -3/4, \\ \|f\|_{\hat{X}_L^{a,1/4+\varepsilon_2/2}(A_0)} & \text{for } -3/2 < a < -3/4. \end{cases}$$

where ε_2 is a sufficiently small number satisfying $0 < \varepsilon_2 \le -(a+3/4)$ and $\hat{X}_L^{a,b}$ is equipped with the norm

$$\|f\|_{\hat{X}_L^{a,b}} := \||\xi|^a\langle\tau-\xi^3\rangle^b f\|_{L^2_{\tau,\xi}(A_0)}.$$

Following the above argument, we define the function space

$$\hat{Z}^{s,a} := \big\{f \in \mathcal{Z}'(\mathbb{R}^2); \|f\|_{\hat{Z}^{s,a}} := \|p_h f\|_{\hat{X}^{s,1/2}_{(2,1)}} + \|p_l f\|_{\hat{X}^a_L} < \infty\big\},$$

where p_h, p_l are the projection operators such that $(p_h f)(\xi) := f(\xi)|_{|\xi|\geq 1}$ and $(p_l f)(\xi) := f(\xi)|_{|\xi|\leq 1}$. Using the function space above, we obtain the following estimates which are the main estimates in this article.

Proposition 1.5. *Let s, a satisfy* (1.2)*. Then*

$$\|\langle \tau - \xi^3 \rangle^{-1} \ \xi \ f * g\|_{\hat{Z}^{s,a}} \leq C\|f\|_{\hat{Z}^{s,a}}\|g\|_{\hat{Z}^{s,a}}, \tag{1.6}$$

$$\|\langle \xi \rangle^{s-a}|\xi|^{a+1}\langle \tau - \xi^3 \rangle^{-1} f * g\|_{L^2_\xi L^1_\tau} \leq C\|f\|_{\hat{Z}^{s,a}}\|g\|_{\hat{Z}^{s,a}}. \tag{1.7}$$

We will use $A \lesssim B$ to denote $A \leq CB$ for some positive constant C and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. The rest of this paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we prove the bilinear estimates. In Section 4, We give the proofs of Theorem 1.1 and 1.2.

2. Preliminaries

In this section, we prepare some lemmas to show the main theorems and the bilinear estimates. When we use the variables (τ, ξ), (τ_1, ξ_1) and (τ_2, ξ_2), we always assume the relation

$$(\tau, \xi) = (\tau_1, \xi_1) + (\tau_2, \xi_2).$$

We state the smoothing estimates for the KdV equation.

Lemma 2.1. *Suppose that f and g are supported on a single A_j for $j \geq 0$. If*

$$K := \inf\{|\xi_1 - \xi_2|; \exists \tau_1, \tau_2 \ s.t. \ (\tau_1, \xi_1) \in \operatorname{supp} f, \ (\tau_2, \xi_2) \in \operatorname{supp} g\} > 0,$$

then we have

$$\||\xi|^{1/2} \ f * g\|_{L^2_{\tau,\xi}} \lesssim K^{-1/2}\|f\|_{\hat{X}^{0,1/2}_{(2,1)}}\|g\|_{\hat{X}^{0,1/2}_{(2,1)}}. \tag{2.1}$$

Lemma 2.2. *Assume that f is supported on A_j and g is an arbitrary test function for $j \geq 0$. If a non-empty set $\Omega \subset \mathbb{R}^2$ satisfies*

$$K := \inf\{|\xi + \xi_1|; \exists \tau, \tau_1 \ s.t. \ (\tau, \xi) \in \Omega, \ (\tau_1, \xi_1) \in \operatorname{supp} f\} > 0,$$

then

$$\|f * g\|_{L^2_{\xi,\tau}(\Omega \cap B_k)} \lesssim 2^{k/2} \ K^{-1/2} \ \|f\|_{\hat{X}^{0,1/2}_{(2,1)}}\||\xi|^{-1/2}g\|_{L^2_{\tau,\xi}}. \tag{2.2}$$

For the proof of these lemmas, refer the reader to [12, Lemmas 3.2 and 3.3]. Here we put $U(t) := \exp(-t\partial_x^3)$ and a smooth cut-off function $\varphi(t)$ satisfying $\varphi(t) = 1$ for $|t| < 1$ and $\varphi(t) = 0$ for $|t| > 2$. For a Banach space $\mathcal{X}$, $\|\cdot\|_{\mathcal{X}}$ denotes $\|u\|_{\mathcal{X}} = \|\widehat{u}\|_{\hat{\mathcal{X}}}$. We mention the linear estimates below.

Proposition 2.3. *Let $s, a \in \mathbb{R}$ and $u(t) = \varphi(t)U(t)u_0$. Then the following estimate holds.*

$$\|u\|_{Z^{s,a}} + \|u\|_{L^\infty_t(\mathbb{R}; H^{s,a}_x)} \lesssim \|u_0\|_{H^{s,a}}.$$

Proposition 2.4. *Let $s, a \in \mathbb{R}$ and*

$$u(t) = \varphi(t)\int_0^t U(t-s)F(s)ds.$$

Then

$$\|u\|_{Z^{s,a}} + \|u\|_{L^\infty_t(\mathbb{R}; H^{s,a}_x)} \lesssim \|\mathcal{F}^{-1}_{\tau,\xi}\langle \tau - \xi^3 \rangle^{-1}\widehat{F}\|_{Z^{s,a}} + \|\langle \xi \rangle^{s-a} \ |\xi|^a \ \langle \tau - \xi^3 \rangle^{-1}\widehat{F}\|_{L^2_\xi L^1_\tau}.$$

The proofs of these two propositions are given in [6].

3. Proof of the bilinear estimates

In this section, we give the proof of the bilinear estimates (1.6) and (1.7). We use the following notation for simplicity,

$$A_{<j_1} := \cup_{j<j_1} A_j, \quad B_{[k_1,k_2)} := \cup_{k_1 \le k < k_2} B_k, \quad \text{etc.}$$

We now prove the key bilinear estimates.

Proposition 3.1. *Let s,a satisfy* (1.2)*. Suppose that f and g are restricted on A_{j_1} and A_{j_2} for $j_1, j_2 \in \mathbb{N} \cup \{0\}$. For $j \ge 0$, we obtain*

$$\|\langle \tau - \xi^3 \rangle^{-1} \xi f * g\|_{\hat{Z}^{s,a}(A_j)} \lesssim C(j,j_1,j_2) \|f\|_{\hat{Z}^{s,a}} \|g\|_{\hat{Z}^{s,a}}, \tag{3.1}$$

$$\big\| \ \langle \xi \rangle^{s-a} \ |\xi|^{a+1} \ \langle \tau - \xi^3 \rangle^{-1} f * g \big\|_{L^2_\xi L^1_\tau (A_j)} \lesssim C(j,j_1,j_2) \|f\|_{\hat{Z}^{s,a}} \|g\|_{\hat{Z}^{s,a}} \tag{3.2}$$

in the following five cases.

- (i) *At least two of j, j_1, j_2 are less than* 20 *and $C(j,j_1,j_2) \sim 1$.*
- (ii) *$j_1, j_2 \ge 20$, $|j_1 - j_2| \le 1$, $0 < j < j_1 - 10$ and $C(j,j_1,j_2) \sim 2^{-\delta j}$ for some $\delta > 0$.*
- (iii) *$j, j_2 \ge 20$, $|j - j_1| \le 10$, $0 < j_2 < j + 11$ and $C(j,j_1,j_2) \sim 2^{-\delta j_2} + 2^{-\delta(j-j_2)}$ for some $\delta > 0$.*
- (iv) *$j_1, j_2 \ge 20$, $j = 0$ and $C(j,j_1,j_2) \sim 1$.*
- (v) *$j, j_1 \ge 20$, $j_2 = 0$ and $C(j,j_1,j_2) \sim 1$.*

We remark that the cases (iii), (v) are also true with j_1 and j_2 exchanged because of symmetry. Using this proposition and $\|f\|^2_{\hat{Z}^{s,a}} \sim \sum_j \|f\|^2_{\hat{Z}^{s,a}(A_j)}$, we obtain (1.6) and (1.7) in the same manner as the proof inc [13, Theorem 2.2].

Proof. We only prove (3.1)–(3.2) in the case $s \ge \max\{-3/4, -a-3/2\}$, $-3/2 < a < 0$ and $(s,a) \ne (-3/4, -3/4)$, because the case $a = 0$ is shown in [12]. In the same manner as [12, Proposition 3.4 (ii) and (iii)], we obtain the desired estimates in the cases (ii) and (iii). Therefore we omit the proof of these cases.

Here we put $2^{k_{\max}} = \max\{2^k,\ 2^{k_1},\ 2^{k_2}\}$. Then we have $2^{k_{\max}} \gtrsim |\xi \xi_1 (\xi - \xi_1)|$. From the definition, we easily obtain

$$\hat{X}^{s,a,1/2+\varepsilon} \hookrightarrow \hat{Z}^{s,a} \hookrightarrow \hat{X}^{s,a,1/4}, \tag{3.3}$$

where $\varepsilon > 0$ is a sufficiently small number. First, we prove (3.1).

(I) Estimate for (i). In this case, we can assume $j, j_1, j_2 \le 30$. From (3.3), the left hand side of (3.1) is bounded by $C\||\xi|^{a+1} \ \langle \tau - \xi^3 \rangle^{-1/2+\varepsilon} f * g\|_{L^2_{\xi,\tau}}$. We use the Hölder inequality and the Young inequality to obtain

$$\begin{aligned} \||\xi|^{a+1} \ \langle \tau - \xi^3 \rangle^{-1/2+\varepsilon} f * g\|_{L^2_{\xi,\tau}} &\lesssim \|f * g\|_{L^\infty_\xi L^4_\tau} \\ &\lesssim \|f\|_{L^2_\xi L^{8/5}_\tau} \|g\|_{L^2_\xi L^{8/5}_\tau} \lesssim \|f\|_{\hat{X}^{s,a,1/4}} \|g\|_{\hat{X}^{s,a,1/4}}. \end{aligned}$$

(II) Estimate for (iv). We prove

$$\|\langle \tau - \xi^3 \rangle^{-1} \ \xi \ f * g\|_{\hat{X}^a_L(A_0)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}. \tag{3.4}$$

(IIa) We consider the estimate (3.4) in the case $|\xi| \le 2^{-2j_1}$. In this case, the left hand side of (3.4) is bounded by $C\||\xi|^{a+1} \langle \tau \rangle^{-1/2+\varepsilon} f * g\|_{L^2_{\tau,\xi}}$ from (3.3). We use

Hölder's inequality and Young's inequality to have

$$\begin{aligned}
\||\xi|^{a+1}\langle\tau\rangle^{-1/2+\varepsilon} f*g\|_{L^2_{\tau,\xi}} &\lesssim 2^{-2sj_1}\||\xi|^{a+1}\|_{L^2_\xi(|\xi|\le 2^{-2j_1})}\|(\langle\xi\rangle^s f)*(\langle\xi\rangle^s g)\|_{L^\infty_\xi L^4_\tau}\\
&\lesssim 2^{-2(s+a+3/2)j_1}\|\langle\xi\rangle^s f\|_{L^2_{\tau,\xi}}\|\langle\xi\rangle^s g\|_{L^2_\xi L^{4/3}_\tau}\\
&\lesssim 2^{-2(s+a+3/2)j_1}\|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}^{s,1/2}_{(2,1)}}.
\end{aligned}$$

We prove only the case $2^{-2j_1}\le|\xi|\le 1$ below.

(IIb) In the case $2^{k_{\max}}=2^{k_2}$, we have $2^{k_2}\gtrsim|\xi|2^{j_1}$. Since $|\xi|^{a+1}\le|\xi|^{-s-1/2}$ and $2^{-k_2/2}\lesssim 2^{-k/4}(|\xi|2^{j_1})^{-1/4}$, we use (2.2) with $K_2\sim 2^{j_1}$ to have

$$\begin{aligned}
\text{(L.H.S.)} &\lesssim 2^{-2sj_1}\sum_{k\ge 0}2^{-k/2}\||\xi|^{a+1}\ (\langle\xi\rangle^s f)*(\langle\xi\rangle^s g)\|_{L^2_{\tau,\xi}(B_k)}\\
&\lesssim 2^{j_1}\sum_{k\ge 0}2^{-k/2}\|\langle|\xi|2^{2j_1}\rangle^{-s-1/2}(\langle\xi\rangle^s f)*(\langle\xi\rangle^s g)\|_{L^2_{\tau,\xi}(B_k)}\\
&\lesssim 2^{j_1}\sum_{k\ge 0}2^{-3k/4}\|\langle|\xi|2^{2j_1}\rangle^{-(s+3/4)}(\langle\xi\rangle^s f)*(\langle\xi\rangle^s\langle\tau-\xi^3\rangle^{1/2}g)\|_{L^2_{\tau,\xi}(B_k)}\\
&\lesssim \sum_{k\ge 0}2^{-k/4}\|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}^{s,1/2}_{(2,1)}}.
\end{aligned}$$

In the same manner as above, we obtain the desired estimate in the case $2^{k_{\max}}=2^{k_1}$.

(IIc) We consider the estimate (3.4) in the case $2^{k_{\max}}=2^k$. If $2^{k_{\max}}\gg|\xi|2^{2j_1}$, then we have $2^{k_{\max}}\sim 2^{k_1}$ or $2^{k_{\max}}\sim 2^{k_2}$. Thus we only consider the case $2^{k_{\max}}\sim|\xi|2^{2j_1}$.

(IIc-1) In the case $-3/4<a<0$, we prove

$$\||\xi|^{a+1}\ \langle\tau-\xi^3\rangle^{a/3-1/2}f*g\|_{L^2_{\tau,\xi}(A_0)}\lesssim\|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}^{s,1/2}_{(2,1)}}. \tag{3.5}$$

(i) We consider (3.5) when $f*g$ is supported on the domain

$$D_1:=\big\{(\tau,\xi)\in\mathbb{R}^2;|\tau|\ge|\xi|^{-3}\text{ and }|\xi|\le 1\big\}.$$

In this case, $2^{-j_1/2}\lesssim|\xi|\le 1$ and $2^{3j_1/2}\lesssim|\tau|\lesssim 2^{4j_1}$. From $|\xi|\sim 2^{k-2j_1}$, we use (2.1) with $K\sim 2^{j_1}$ to obtain

$$\begin{aligned}
\text{(L.H.S.)} &\lesssim \sum_{k\ge 3j_1/2+O(1)}2^{(a/3-1/2)k}\||\xi|^{a+1}f*g\|_{L^2_{\tau,\xi}(B_k)}\\
&\lesssim 2^{(-2s-2a-1)j_1}\sum_{k\ge 3j_1/2+O(1)}2^{4ak/3}\||\xi|^{1/2}\ (\langle\xi\rangle^s f)*(\langle\xi\rangle^s g)\|_{L^2_{\tau,\xi}}\\
&\lesssim 2^{-2(s+3/4)j_1}\|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}^{s,1/2}_{(2,1)}}.
\end{aligned}$$

(ii) We consider (3.5) when $f*g$ is restricted to the domain

$$D_2:=\big\{(\tau,\xi)\in\mathbb{R}^2;|\tau|\le|\xi|^{-3}\text{ and }|\xi|\le 1\big\}.$$

In the present case, we have $2^{-2j_1} \le |\xi| \lesssim 2^{-j_1/2}$ and $1 \lesssim |\tau| \lesssim 2^{3j_1/2}$. We use the Hölder inequality and the Young inequality to have

$$\begin{aligned}
&(\text{L.H.S.})\\
&\lesssim \sum_{k\le 3j_1/2+O(1)} 2^{(a/3-1/2)k} \||\xi|^{a+1} f * g\|_{L^2_{\tau,\xi}(B_k)}\\
&\lesssim 2^{-2sj_1} \sum_{k\le 3j_1/2+O(1)} 2^{(a/3-1/2)k} \||\xi|^{a+1}\|_{L^2_\xi(|\xi|\sim 2^{k-2j_1})} \|(\langle\xi\rangle^s f) * (\langle\xi\rangle^s g)\|_{L^\infty_\xi L^2_\tau}\\
&\lesssim 2^{(-2s-2a-3)j_1} \sum_{k\le 3j_1/2+O(1)} 2^{4(a+3/4)/3} \|\langle\xi\rangle^s f\|_{L^2_\xi L^1_\tau} \|\langle\xi\rangle^s g\|_{L^2_\xi L^2_\tau}\\
&\lesssim 2^{-2(s+3/4)j_1} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}.
\end{aligned}$$

(IIc-2) In the case $a = -3/4$, we prove

$$\||\xi|^{1/4} \langle\tau\rangle^{-3/4+\varepsilon_1/2} f * g\|_{L^2_{\tau,\xi}(A_0)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}. \tag{3.6}$$

From $|\tau| \sim |\xi| 2^{2j_1} \ge 1$, we use Hölder's inequality and Young's inequality to obtain

$$\begin{aligned}
(\text{L.H.S.}) &\lesssim 2^{(-2s-3/2+\varepsilon_1)j_1} \||\xi|^{-1/2+\varepsilon_1/2}\|_{L^2_\xi(|\xi|\le 1)} \|(\langle\xi\rangle^s f) * (\langle\xi\rangle^s g)\|_{L^\infty_\xi L^2_\tau}\\
&\lesssim 2^{(-2s-3/2+\varepsilon_1)j_1} \|\langle\xi\rangle^s f\|_{L^2_\xi L^1_\tau} \|\langle\xi\rangle^s g\|_{L^2_{\tau,\xi}}\\
&\lesssim 2^{(-2s-3/2+\varepsilon_1)j_1} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}.
\end{aligned}$$

Since $-2s - 3/2 + \varepsilon_1 \le -\varepsilon_1$ in present case, we have (3.6).

(IIc-3) In the case $-3/2 < a < -3/4$, we estimate

$$\||\xi|^{a+1}\ \langle\tau\rangle^{-3/4+\varepsilon_2/2} f * g\|_{L^2_{\tau,\xi}(A_0)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}. \tag{3.7}$$

From the assumption, $-s - 3/4 + \varepsilon_2 \le 0$ and $|\xi|^{a+1} \le |\xi|^{-s-1/2}$. Now we use the Hölder inequality and the Young inequality to have

$$\begin{aligned}
(\text{L.H.S.}) &\lesssim 2^{-2sj_1} \||\xi|^{-s-1/2} \langle\tau\rangle^{-3/4+\varepsilon_2/2} (\langle\xi\rangle^s f) * (\langle\xi\rangle^s g)\|_{L^2_{\tau,\xi}}\\
&\lesssim 2^{-\varepsilon_2 j_1} \||\xi|^{-1/2-\varepsilon_2/2}\|_{L^2_\xi(|\xi|\ge 2^{-2j_1})} \|(\langle\xi\rangle^s f) * (\langle\xi\rangle^s g)\|_{L^\infty_\xi L^2_\tau}\\
&\lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}.
\end{aligned}$$

(III) Estimate for (v). We prove

$$2^j \sum_{k\ge 0} 2^{-k/2}\ \|(\langle\xi\rangle^s f) * g\|_{L^2_{\tau,\xi}(B_k)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^a_L(A_0)}. \tag{3.8}$$

In the case $|\xi_2| \le 2^{-2j}$, we use Hölder's inequality and Young's inequality to have

$$\begin{aligned}
(\text{L.H.S.}) &\lesssim 2^j \|(\langle\xi\rangle^s f) * g\|_{L^2_{\tau,\xi}} \lesssim 2^j \|\langle\xi\rangle^s f\|_{L^2_\xi L^1_\tau} \|g\|_{L^1_\xi L^2_\tau(|\xi|\le 2^{-2j})}\\
&\lesssim 2^j \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \||\xi|^{-a}\|_{L^2_\xi(|\xi|\le 2^{-2j})} \||\xi|^a g\|_{L^2_{\tau,\xi}}\\
&\lesssim 2^{2aj} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,0}_L}.
\end{aligned}$$

Therefore we only consider the case $2^{-2j} \le |\xi_2| \le 1$.

(IIIa) We consider the estimate (3.8) in the case $2^{k_{\max}} = 2^k$. From $2^k \ge 2^{k_2}$, we use (2.1) with $K \sim 2^j$ to obtain

$$\begin{aligned}(\text{L.H.S.}) &\lesssim 2^j \|\langle \tau - \xi^3\rangle^{-1/2+\varepsilon}(\langle\xi\rangle^s f) * g\|_{L^2_{\tau,\xi}}\\ &\lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|\langle\tau\rangle^{-1/2+\varepsilon} g\|_{\hat{X}^{0,1/2,1}} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}^{a,1/4}_L}.\end{aligned}$$

(IIIb) We consider the estimate (3.8) in the case $2^{k_{\max}} = 2^{k_1}$. The left hand side of (3.8) is bounded by $C2^j\|\langle\tau - \xi^3\rangle^{-1/2+\varepsilon}(\langle\xi\rangle^s f) * g\|_{L^2_{\tau,\xi}}$. From $|\xi_2| \lesssim 2^{k-2j_1}$, we use the Hölder inequality and the Young inequality to have

$$\begin{aligned}2^j\|\langle\tau-\xi^3\rangle^{-1/2+\varepsilon}(\langle\xi\rangle^s f)*g\|_{L^2_{\tau,\xi}} &\lesssim 2^j\|(\langle\xi\rangle^s f)*g\|_{L^2_\xi L^6_\tau}\\ &\lesssim 2^j\|\langle\xi\rangle^s f\|_{L^2_{\tau,\xi}}\|g\|_{L^1_\xi L^{3/2}_\tau(|\xi|\lesssim 2^{k_1-2j})}\\ &\lesssim 2^j\sum_{k_1}\|\langle\xi\rangle^s f\|_{L^2_{\tau,\xi}(B_{k_1})}2^{k_1/2-j}\|g\|_{L^2_\xi L^{3/2}_\tau},\end{aligned}$$

which is bounded by the right hand side of (3.8).

(IIIc) We consider the estimate (3.8) in the case $2^{k_{\max}} = 2^{k_2}$. If $2^{k_{\max}} = 2^{k_2} \gg |\xi_2|2^{2j}$, then we have $2^{k_{\max}} \sim 2^k$ or $2^{k_{\max}} \sim 2^{k_1}$. Therefore we only consider the case $2^{k_2} \sim |\xi_2|2^{2j}$.

(IIIc-1) In the case $a = -3/4$, we prove

$$2^j\sum_{k\ge 0}2^{-k/2}\|(\langle\xi\rangle^s f)*g\|_{L^2_{\tau,\xi}(B_k)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}_L^{-3/4,1/4+\varepsilon_1/2}}. \tag{3.9}$$

From $2^{(-1/4-\varepsilon_1/2)k_2} \lesssim (|\xi_2|2^{2j})^{-1/4}2^{-\varepsilon_1 k/2}$, we use (2.2) with $K_2 \sim 2^j$ to have

$$\begin{aligned}(\text{L.H.S.}) &\lesssim 2^{j/2}\sum_{k\ge 0}2^{(-1/2-\varepsilon_1/2)k}\|(\langle\xi\rangle^s f)*(|\xi|^{-1/4}\langle\tau\rangle^{1/4+\varepsilon_1/2}g)\|_{L^2_{\tau,\xi}(B_k)}\\ &\lesssim \sum_{k\ge 0}2^{-\varepsilon_1 k/2}\|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}_L^{-3/4,1/4+\varepsilon_1/2}}.\end{aligned}$$

(IIIc-2) In the case $-3/2 < a < -3/4$, we prove

$$2^j\sum_{k\ge 0}2^{-k/2}\|(\langle\xi\rangle^s f)*g\|_{L^2_{\tau,\xi}(B_k)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}_L^{a,1/4+\varepsilon_2/2}}. \tag{3.10}$$

From $|\xi_2|^{-(a+1/2)}\langle\tau_2\rangle^{-1/4-\varepsilon_2/2} \lesssim 2^{-j/2}2^{-\varepsilon_2 k/2}$, we use (2.2) with $K_2 \sim 2^j$ to obtain

$$\begin{aligned}(\text{L.H.S.}) &\lesssim 2^{j/2}\sum_{k\ge 0}2^{(-1/2-\varepsilon_2/2)k}\|(\langle\xi\rangle^s f)*(|\xi|^{a+1/2}\langle\tau\rangle^{1/4+\varepsilon_2/2}g)\|_{L^2_{\tau,\xi}(B_k)}\\ &\lesssim \sum_{k\ge 0}2^{-\varepsilon_2 k/2}\|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}_L^{a,1/4+\varepsilon_2/2}}.\end{aligned}$$

(IIIc-3) In the case $-3/4 < a < 0$, we prove

$$2^j\sum_{k\ge 0}2^{-k/2}\|(\langle\xi\rangle^s f)*g\|_{L^2_{\tau,\xi}(B_k)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}}\|g\|_{\hat{X}_L^{a,a/3+1/2}}. \tag{3.11}$$

(i) We consider (3.11) when g is restricted to D_2. In the present case, $2^{-2j} \le |\xi_2| \lesssim 2^{-j/2}$ and $1 \lesssim |\tau_2| \lesssim 2^{3j/2}$. From $|\xi|^{-a}\langle\tau\rangle^{-a/3-1/2} \sim |\xi|^{-4a/3-1/2}2^{-2aj/3-j}$, we use the Hölder inequality and the Young inequality to obtain

$$\begin{aligned}
\text{(L.H.S.)} &\lesssim 2^j \; \|(\langle\xi\rangle^s f) * g\|_{L^2_\xi L^4_\tau} \lesssim 2^j \|\langle\xi\rangle^s f\|_{L^2_\xi L^{4/3}_\tau} \|g\|_{L^1_\xi L^2_\tau(|\xi|\lesssim 2^{-j/2})} \\
&\lesssim 2^{-2aj/3} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \||\xi_2|^{-4a/3-1/2}\|_{L^2_{\xi_2}(|\xi_2|\lesssim 2^{-j/2})} \|g\|_{\hat{X}^{a,a/3+1/2}_L}.
\end{aligned}$$

Since $\||\xi|^{-4a/3-1/2}\|_{L_{\xi_2}(|\xi_2|\lesssim 2^{-j/2})} \lesssim 2^{2aj/3}$, we have (3.11).

(ii) We consider (3.11) when g is supported on D_1. In this case, we have $2^{-j/2} \lesssim |\xi_2| \le 1$ and $2^{3j/2} \lesssim |\tau| \lesssim 2^{2j}$.

(iia) Firstly, g is restricted to $B_{[3j/2,3j/2+\alpha]}$ with $0 \le \alpha \le j/2$. From $2^{-j/2} \lesssim |\xi_2| \lesssim 2^{-j/2+\alpha}$ and $|\xi_2|^{-a}\langle\tau_2\rangle^{-a/3-1/2} \sim |\xi_2|^{-4a/3-1/2}2^{-2aj/3-j}$, we use Hölder's inequality and Young's inequality to obtain

$$\begin{aligned}
\|\xi\; f * g\|_{\hat{X}^{s,-1/2}_{(2,1)}(B_{\ge 2\alpha})} &\sim 2^j \sum_{k\ge 2\alpha} 2^{-k/2} \|(\langle\xi\rangle^s f) * g\|_{L^2_{\tau,\xi}(B_k)} \\
&\lesssim 2^j \sum_{k\ge 2\alpha} 2^{-k/2} \|\langle\xi\rangle^s f\|_{L^2_\xi L^1_\tau} \|g\|_{L^1_\xi L^2_\tau(|\xi|\lesssim 2^{-j/2+\alpha})} \\
&\lesssim 2^j 2^{-\alpha} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \||\xi|^{-4a/3-1/2}\|_{L^2_\xi(|\xi|\lesssim 2^{-j/2+\alpha})} \|g\|_{\hat{X}^{a,a/3+1/2}_L} \\
&\lesssim 2^{-\frac{4}{3}(a+\frac{3}{4})\alpha} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,a/3+1/2}_L}.
\end{aligned}$$

We put a sufficiently small number ε_3 satisfying $0 < \varepsilon_3 \le 4(a+3/4)/3$. From the above estimate, we have

$$\|\xi f * g\|_{\hat{X}^{s,-1/2}_{(2,1)}(B_{\ge 2\alpha})} \lesssim 2^{-\varepsilon_3\alpha} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,a/3+1/2}_L}. \tag{3.12}$$

(iib) Secondly, g is restricted to $B_{[3j/2+\gamma,2j]}$ with $0 \le \gamma \le 2^{j/2}$. From $2^{-j/2+\gamma} \lesssim |\xi| \le 1$, we use (2.2) with $K_2 \sim 2^j$ to obtain

$$\begin{aligned}
&\|\xi f * g\|_{\hat{X}^{s,-1/2}_{(2,1)}(B_{\le 2\alpha})} \sim 2^j \sum_{k\le 2\alpha} 2^{-k/2} \|(\langle\xi\rangle^s f) * g\|_{L^2_{\tau,\xi}(B_k)} \\
&\lesssim 2^{j/2} \sum_{k\le 2\alpha} 1 \; \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \||\xi|^{-1/2} g\|_{L^2_{\tau,\xi}(2^{-j/2+\gamma}\lesssim|\xi|)} \\
&\lesssim \alpha 2^{-2aj/3-j/2} \||\xi|^{-\frac{4}{3}(a+\frac{3}{4})}\|_{L^2_\xi(2^{-j/2+\gamma}\lesssim|\xi|)} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,a/3+1/2}_L} \\
&\lesssim \alpha 2^{-\frac{4}{3}(a+\frac{3}{4})\gamma} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,a/3+1/2}_L}.
\end{aligned}$$

From the definition of ε_3, we have

$$\|\xi f * g\|_{\hat{X}^{s,-1/2}_{(2,1)}(B_{\le 2\alpha})} \lesssim \alpha 2^{-\gamma\varepsilon_3} \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,a/3+1/2}_L}. \tag{3.13}$$

If g is restricted to $B_{[3j/2+\gamma,3j/2+\alpha]}$ with $\gamma < \alpha$, from (3.12) and (3.13), we have

$$\|\xi f * g\|_{\hat{X}^{s,-1/2}_{(2,1)}} \lesssim \big(2^{-\varepsilon_3\alpha} + \alpha 2^{-\varepsilon_3\gamma}\big) \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,a/3+1/2}_L}. \tag{3.14}$$

Let be the decreasing sequence $\{a_n\}_{n=0}^N$ defined by

$$a_0 = \frac{j}{2}, \quad a_{n+1} = \frac{1}{2}a_n, \quad 0 < a_N \le \frac{1}{2},$$

where N is a minimum integer such that $N \ge \log_2 j$. We first apply with $\alpha = a_0$ and $\gamma = a_1$, next apply with $\alpha = a_1$ and $\gamma = a_2$. Repeating this procedure at the end we apply with $\alpha = a_N$ and $\gamma = 0$. From (3.14), we obtain

$$\|\xi f * g\|_{\hat{X}^{s,-1/2}_{(2,1)}} \lesssim \big(1 + \sum_{n=0}^{N} \frac{1}{a_n}\big) \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{a,a/3+1/2}_{L}(D_1)},$$

which shows the claim since $\sum_{n=0}^{N} \frac{1}{a_n}$ is bounded uniformly in j.

Next, we prove (3.2). From the triangle inequality and the Schwarz inequality, we have

$$\|f\|_{L^1_\tau} \lesssim \sum_{k\geq 0} \|f\|_{L^1_\tau(B_k)} \lesssim \sum_{k\geq 0} 2^{k/2}\ \|f\|_{L^2_\tau(B_k)}. \tag{3.15}$$

From (3.15), we obtain

$$\|\langle\xi\rangle^s \langle\tau - \xi^3\rangle^{-1} \xi f * g\|_{L^2_\xi L^1_\tau(A_j)} \lesssim \|\xi\ f * g\|_{\hat{X}^{s,-1/2}_{(2,1)}(A_j)},$$

for any $j > 0$. Thus we only consider the case (i) and (iv).

(IV) Estimate of (i). In this case, the left hand side of (3.2) is bounded by $C\||\xi|^{a+1}\langle\tau - \xi^3\rangle^{-1/2+\varepsilon} f * g\|_{L^2_{\tau,\xi}}$. In the same manner as (I), we have the desired estimate.

(V) Estimate of (vi). We prove

$$\||\xi|^{a+1}\langle\tau\rangle^{-1} f * g\|_{L^2_\xi L^1_\tau(A_0)} \lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}. \tag{3.16}$$

We easily obtain (3.16) in the case $|\xi| \leq 2^{-2j_1}$. Therefore we only consider the case $2^{-2j_1} \leq |\xi| \leq 1$ below.

(Va) We consider the estimate (3.16) in the case $2^{k_{\max}} = 2^{k_1}$ or 2^{k_2}. Note that the left hand side of (3.16) is bounded by $C\sum_{k\geq 0} 2^{-k/2} \||\xi|^{a+1} f * g\|_{L^2_{\tau,\xi}(B_k)}$ from (3.15). In the same manner as (IIb), we obtain (3.16) in the case $2^{k_{\max}} = 2^{k_1}$ or 2^{k_2}.

(Vb) We consider the estimate (3.16) in the case $2^{k_{\max}} = 2^{k}$. From $|\xi|^{a+1} \leq |\xi|^{-s-1/2}$, we have $|\xi|^{a+1}\langle\tau\rangle^{-1} \lesssim |\xi|^{-3/4} 2^{2sj_1 - j_1/2}$. We use the Hölder inequality and the Young inequality to have

$$\begin{aligned}
\text{(L.H.S.)} &\lesssim 2^{-j_1/2} \||\xi|^{-3/4} (\langle\xi\rangle^s f) * (\langle\xi\rangle^s g)\|_{L^2_\xi L^1_\tau} \\
&\lesssim 2^{-j_1/2} \||\xi|^{-3/4}\|_{L^2_\xi(|\xi|\geq 2^{-2j_1})} \|\langle\xi\rangle^s f\|_{L^2_\xi L^1_\tau} \|\langle\xi\rangle^s g\|_{L^2_\xi L^1_\tau} \\
&\lesssim \|f\|_{\hat{X}^{s,1/2}_{(2,1)}} \|g\|_{\hat{X}^{s,1/2}_{(2,1)}}.
\end{aligned}$$

□

4. Proof of the main results

In this section, we give the proofs of Theorem 1.1 and 1.2. Here $Z^{s,a}_T$ is defined by the norm

$$\|u\|_{Z^{s,a}_T} := \inf\big\{\|v\|_{Z^{s,a}}; u(t) = v(t) \text{ on } t \in [0,T]\big\}.$$

We obtain the following main result.

Proposition 4.1. *Let s, a satisfy* (1.2) *and $r > 1$.*

(Existence) For any $u_0 \in B_r(H^{s,a})$, there exist $T \sim r^{-6/(3+2\min\{s,a\})}$ and $u \in C([0,T]; H^{s,a}) \cap Z^{s,a}_T$ satisfying the following integral form for (1.1)*;*

$$u(t) = U(t)u_0 + 3\int_0^t U(t-s)\partial_x (u(s))^2 ds. \tag{4.1}$$

Moreover the data-to-solution map $B_r(H^{s,a}) \ni u_0 \mapsto u \in C([0,T];H^{s,a}) \cap Z_T^{s,a}$ *is Lipschitz continuous.*

(Uniqueness) Assume that $u, v \in C([0,T];H^{s,a}) \cap Z_T^{s,a}$ *satisfy* (4.1)*. Then,* $u = v$ *on* $t \in [0,T]$*.*

Proof. We first prove the existence of the solution to (4.1). The KdV equation is scale invariant with respect to the transform

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{-2} u(\lambda^{-3}t, \lambda^{-1}x), \quad \lambda \geq 1.$$

A simple calculation shows

$$\|u_\lambda(0,\cdot)\|_{H^{s,a}} \leq \lambda^{-3/2-\min\{s,a\}} \|u_0\|_{H^{s,a}}.$$

Therefore, we can assume that initial data is small enough. From this, we use Propositions 1.5, 2.3 and 2.4 to prove the existence of the solution by Banach's fixed point argument. For the details, see the proof in [14, Proposition 4.1].

We next prove the uniqueness of solutions by the argument in [16]. We define the space $W^{s,a}$ by the norm

$$\|u\|_{W^{s,a}} := \|u\|_{Z^{s,a}} + \|u\|_{L^\infty(\mathbb{R};H^{s,a})}.$$

In the same manner as the proof in [16, Theorem 2.5], we obtain, for $1/2 \leq b < 1$,

$$w \in X^{s,a,b}_{(1,1),T_\lambda}, \quad w(0,x) = 0 \quad \Rightarrow \quad \lim_{\delta \to +0} \|w|_{[0,\delta]}\|_{X^{s,a,b}_{(1,1),\delta}} = 0, \tag{4.2}$$

where $T_\lambda := \lambda^3 T$, $\lambda \geq 1$ and the space $X^{s,a,b}_{(1,1)}$ defined by

$$\|u\|_{X^{s,a,b}_{(1,1)}} := \big\| \big\{ \|\langle\xi\rangle^{s-a} |\xi|^a \langle \tau - \xi^3 \rangle^b \widehat{u}\|_{L^2_{\tau,\xi}(A_j \cap B_k)} \big\}_{j,k \geq 0} \big\|_{l^1_{j,k}}.$$

Let $u \in W^{s,a}$ satisfy $u(0,x) = 0$ and ε is an arbitrary positive number. Since $W^{s,a}$ contains $\mathcal{Z}$ densely, we can choose $v \in \mathcal{Z}$ satisfying $\|v - u\|_{W^{s,a}} < \varepsilon$. From the definition, we have

$$\|v(0)\|_{H^{s,a}} = \|v(0) - u(0)\|_{H^{s,a}} \lesssim \|u - v\|_{W^{s,a}} < \varepsilon.$$

Note that

$$\sup_{t \in \mathbb{R}} \|u\|_{H^{s,a}} \lesssim \|u\|_{W^{s,a}} \lesssim \|u\|_{X^{s,a,b}},$$

for $1/2 < b < 1$. By the above argument, we have

$$\begin{aligned} \|u\|_{W_T^{s,a}} &\lesssim \|u - v\|_{W_T^{s,a}} + \|v - U(t)v(0)\|_{W_T^{s,a}} + \|U(t)v(0)\|_{X_T^{s,a,b}} \\ &\lesssim \varepsilon + \|v - U(t)v(0)\|_{W_T^{s,a}} + \|v(0)\|_{H^{s,a}} \\ &\lesssim \varepsilon + \|v - U(t)v(0)\|_{W_T^{s,a}}. \end{aligned}$$

Since the second term tends to 0 as $T \to 0$ from (4.2), we have

$$\lim_{T \to 0} \|u\|_{W_T^{s,a}} = 0. \tag{4.3}$$

By combining Propositions 1.5, 2.3, 2.4 and (4.3), we have uniqueness. For the details, see [12]. □

Next, we prove Theorem 1.2 (i)–(iii). We first consider Theorem 1.2 (i). In [1], Bejenaru and Tao, for the quadratic Schrödinger equation with nonlinear term u^2, proved the discontinuity of the data-to-solution map for any $s < -1$. We essentially follow their argument to obtain the following proposition.

Proposition 4.2. *Let $s < s_a$, $-3/2 < a < -7/8$ and $0 < \delta \ll 1$. Then there exist $T = T(\delta) > 0$ and a sequence of initial data $\{\phi_{N,\delta}\}_{N=1}^{\infty} \in H^{\infty}$ satisfying the following three conditions for any $t \in (0, T]$,*

(1) $\|\phi_{N,\delta}\|_{H^{s_a,a}} \sim \delta$,
(2) $\|\phi_{N,\delta}\|_{H^{s,a}} \to 0$ *as* $N \to \infty$,
(3) $\|u_{N,\delta}(t)\|_{H^{s,a}} \gtrsim \delta^2$,
where $u_{N,\delta}(t)$ is the solution to (1.1) *obtained in Proposition 4.1 with the initial data $\phi_{N,\delta}$.*

Proof. Let $N \gg 1$. We put the initial data $\phi_{N,\delta}$ as follows;

$$\phi_{N,\delta}(x) = \delta N^{a+5/2} \cos(Nx) \int_{-\gamma}^{\gamma} e^{i\xi x} d\xi,$$

where $\gamma := N^{-2}$. By a simple calculation, we have

$$\widehat{\phi}_{N,\delta}(\xi) \sim \delta N^{a+5/2} \chi_{B^+}(\xi) + \delta N^{a+5/2} \chi_{B^-}(\xi), \tag{4.4}$$

where

$$B^{\pm} := [\pm N - \gamma,\ \pm N + \gamma].$$

Therefore,

$$\|\phi_{N,\delta}\|_{H^{s,a}} \sim \delta N^{s+a+3/2}, \quad \|U(t)\phi_{N,\delta}\|_{H^{s,a}} = \|\phi_{N,\delta}\|_{H^{s,a}} \sim \delta N^{s+a+3/2}. \tag{4.5}$$

Since $\|\phi_{N,\delta}\|_{H^{s_a}} \sim \delta$, we have $T = T(\delta) > 0$ and the solution $u_{N,\delta}$ to (1.1) with the initial data $\phi_{N,\delta}$ by Proposition 4.1. Let $t \in (0, T]$. A quadratic term A_2 of the Taylor expansion is defined by

$$A_2(u_0)(t) := 3 \int_0^t U(t-s) \partial_x (U(s)u_0)^2 ds.$$

A simple calculation shows that

$$\widehat{A}_2(u_0)(t) = \exp(i\xi^3 t) \int \frac{1 - \exp(-iq(\xi,\xi_1)t)}{q(\xi,\xi_1)} \widehat{u}_0(\xi_1) \widehat{u}_0(\xi - \xi_1) d\xi_1, \tag{4.6}$$

where $q(\xi, \xi_1) := 3\xi\xi_1(\xi - \xi_1)$. By similar argument to the proof in [14, Theorem 1.2], we obtain

$$\|A_2(u_0)(t)\|_{H^{s,a}} \gtrsim \delta^2. \tag{4.7}$$

Now we put $v_{N,\delta}(t) := u_{N,\delta}(t) - U(t)\phi_{N,\delta} - A_2(\phi_{N,\delta})(t)$. Since the data-to-solution map is Lipschitz continuous for $s = s_a$, we obtain

$$\|v_{N,\delta}(t)\|_{H^{s_a,a}} \lesssim \delta^3, \tag{4.8}$$

by using Propositions 1.5, 2.3 and 2.4. From (4.5), (4.7) and (4.8), we obtain

$$\|u_{N,\delta}(t)\|_{H^{s,a}} \geq \|A_2(\phi_{N,\delta})(t)\|_{H^{s,a}} - \|v_{N,\delta}(t)\|_{H^{s,a}} - \|U(t)\phi_{N,\delta}\|_{H^{s,a}} \gtrsim \delta^2,$$

for all $N \gg 1$. Since $\|\phi_{N,\delta}\|_{H^{s,a}} \to 0$ as $N \to \infty$, this shows the discontinuity of the flow map.

We next prove Theorem 1.2 (ii). We only prove that the following estimate fails.

$$\|A_2(u_0)(t)\|_{H^{s,a}} \lesssim \|u_0\|_{H^{s,a}}^2, \tag{4.9}$$

for $|t|$ bounded by the general argument. For details, see [7].

Let $N \gg 1$. We put a smooth initial data as follows;

$$\phi_N(x) := N^{-s+1} \cos(Nx) \int_{-\gamma}^{\gamma} e^{i\xi x} d\xi + N^{2a+1} \cos(N^{-2}x) \int_{-\gamma/2}^{\gamma/2} e^{i\xi x} d\xi.$$

A straightforward computation shows that

$$\widehat{\phi}_N(\xi) \sim N^{-s+1}(\chi_{B^+}(\xi) + \chi_{B^-}(\xi)) + N^{2a+1}\chi_C(\xi), \tag{4.10}$$

where $C := [\gamma/2, 3\gamma/2]$. Clearly, $\|\phi_N\|_{H^{s,a}} \sim 1$. Substituting (4.10) into (4.6), we have

$$\begin{aligned} |\widehat{A}_2(\phi_N)(t)| \lesssim\, & N^{-2s}|\xi|\ \chi_{[-\gamma/2,\gamma/2]}(\xi) + N^{-s+a}|\xi|\ \chi_{[\pm N, \pm N+\gamma]}(\xi) \\ & + (\text{remainder terms}). \end{aligned}$$

Therefore,

$$\|A_2(\phi_N)(t)\|_{H^{s,a}} \gtrsim N^{-2s}\Big(\int_{-\gamma/2}^{\gamma/2} |\xi|^{2a+2} d\xi\Big)^{1/2} + N^{-s+2a}\Big(\int_N^{N+\gamma} \langle\xi\rangle^{2s+2}\Big)^{1/2}. \tag{4.11}$$

If $a \le -3/2$, the first term of the right hand side of (4.11) diverges. When we assume $a \ge -3/2$, the right hand side of (4.11) is greater than $C(N^{-2(s+a+3/2)} + N^{2a})$. In the case $0 < a$ or $s < -a - 3/2$, we have $\|A_2(\phi_N)(t)\|_{H^{s,a}} \to \infty$ as $N \to \infty$, which shows the claim since $\|\phi_N\|_{H^{s,a}} \sim 1$.

Finally, we consider Theorem 1.2 (iii). Similar to the proof of Theorem 1.2 (ii), we only prove that the following estimate fails for $|t|$ bounded.

$$\|A_3(u_0)(t)\|_{H^{s,a}} \lesssim \|u_0\|^3_{H^{s,a}}, \tag{4.12}$$

where A_3 is the cubic term of the Taylor expansion. We put the sequence of initial data $\{\psi\}_{N=1}^{\infty} \in H^{\infty}$ as follows;

$$\psi_N(x) = N^{-s+1/4}\ \cos(Nx) \int_{-N^{-1/2}}^{N^{1/2}} e^{i\xi x} d\xi.$$

Similar to this data is used in [4]. In the same manner as the argument in [4], we prove (4.12) fails. □

5. Appendix

We mention the typical counterexamples of (1.3) in the case (1.4).

Example 5.1 (high-high-low interaction)**.** We define the rectangles P_1, P_2 as follows;

$$\begin{gathered} P_1 := \big\{(\tau,\xi) \in \mathbb{R}^2\ ;\ |\xi - N| \le N^{-1/2},\ |\tau - (3N^2\xi - 2N^3)| \le 1/2\big\}, \\ P_2 := \big\{(\tau,\xi) \in \mathbb{R}^2; (-\tau,-\xi) \in A_1\big\}. \end{gathered}$$

Here we put

$$f(\tau,\xi) := \chi_{P_1}(\tau,\xi), \quad g(\tau,\xi) := \chi_{P_2}(\tau,\xi). \tag{5.1}$$

Then

$$f * g(\tau,\xi) \gtrsim N^{-1/2}\ \chi_{R_1}(\tau,\xi), \tag{5.2}$$

where

$$R_1 := \big\{(\tau,\xi) \in \mathbb{R}^2; \xi \in [1/2N^{-1/2}, 3/4N^{-1/2}],\ |\tau - 3N^2\xi| \le 1/2\big\}.$$

Inserting (5.1) and (5.2) into (1.3), the necessary condition for (1.3) is $b \le 4s/3 + a/3 + 3/2$. If (1.3) for $s = -3/4$, $b \le a/3 + 1/2$.

Example 5.2 (high-low-high interaction)**.** We define the rectangle
$$Q := \{(\tau,\xi) \in \mathbb{R}^2; |\xi - 2N^{-1/2}| \le N^{-1/2},\ |\tau - 3N^2\xi| \le 1/2\}.$$
Here we put
$$f(\tau,\xi) = \chi_{P_1}(\tau,\xi), \quad g(\tau,\xi) = \chi_Q(\tau,\xi). \tag{5.3}$$
Then
$$f * g(\tau,\xi) \gtrsim N^{-1/2}\ \chi_{R_2}(\tau,\xi), \tag{5.4}$$
where
$$R_2 := \{(\tau,\xi) \in \mathbb{R}^2; |\xi - N| \le N^{-1/2}/2,\ |\tau - (3N^2\xi - 2N^3)| \le 1/2\}.$$
Substituting (5.3) and (5.4) into (1.3), the necessary condition for (1.3) is $b \ge a/3 + 1/2$.

Example 5.3 (high-high-high interaction)**.** We put
$$f(\tau,\xi) = \chi_{P_1}(\tau,\xi), \quad g(\tau,\xi) = \chi_{P_1}(\tau,\xi). \tag{5.5}$$
Then
$$f * g(\tau,\xi) \gtrsim N^{-1/2}\ \chi_{R_3}(\tau,\xi), \tag{5.6}$$
where
$$R_3 := \{(\tau,\xi) \in \mathbb{R}^2; |\xi - 2N| \le N^{-1/2}/2,\ |\tau - (3N^2\xi - 4N^3)| \le 1/2\}.$$
Inserting (5.5) and (5.6) into (1.3), the necessary condition for (1.3) is $b \le 1/2$ for $s = -3/4$.

On the other hand, we put
$$f(\tau,\xi) = \chi_{R_3}(\tau,\xi), \quad g(\tau,\xi) = \chi_{P_2}(\tau,\xi). \tag{5.7}$$
Then
$$f * g(\tau,\xi) \gtrsim N^{-1/2}\ \chi_{R_4}(\tau,\xi), \tag{5.8}$$
where
$$R_4 := \{(\tau,\xi) \in \mathbb{R}^2; |\xi - N| \le N^{-1/2}/4,\ |\tau - (3N^2\xi - 2N^3)| \le 1/2\}.$$
Substituting (5.7) and (5.8) into (1.3), the necessary condition for (1.3) is $b \ge 1/2$ for $s = -3/4$.

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