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TRANSPORT EQUATIONS IN CELL POPULATION DYNAMICS I

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Abstract. In this article, we study a cell proliferating model, where each cell is characterized by its degree of maturity and its maturation velocity. The boundary conditions in this model generalize the known biological rules. We consider also the degenerate case corresponding to infinite maturation velocity. Then we show that this model is governed by a strongly continuous semigroup and give its explicit expression.

1. Introduction

We consider a cell population in which each cell is distinguished by its degree of maturity μ and its maturation velocity v. At the birth, the degree of maturity of each (daughter) cell is null ($\mu = 0$) and at the division, the degree of maturity of each (mother) cell becomes $\mu = 1$. Between the birth and the division of each cell, its degree of maturity is $0 < \mu < 1$. As each cell may not become less mature, then its maturation velocity v must be positive ($0 \leq a < v < b \leq \infty$). So, if $f = f(t, \mu, v)$ is the cell density with respect to degree of maturity μ and to the maturation velocity v at time $t \geq 0$, then f satisfies the following partial differential equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \mu} = -\sigma f + \int_a^b r(\mu, v, v')f(t, \mu, v')dv' \tag{1.1}$$

where, $r(\mu, v, v')$ is the *transition rate* at which cells change their velocities from v to v' and

$$\sigma(\mu, v) = \int_a^b r(\mu, v', v)dv' \tag{1.2}$$

is the rate of cell mortality or cell loss due to causes other than division.

During each cell division, we suppose there is a kernel correlation $k(v, v')$ between the maturation velocity of a mother cell v' and that of a daughter cell v. This correlation is governed by the *transition biological rule* mathematically described

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by the boundary condition

$$vf(t,0,v) = \beta \int_a^b k(v,v')f(t,1,v')v'dv' \tag{1.3}$$

where, $\beta \geq 0$ is the average number of daughter cells viable per mitotic.

The model (1.1)-(1.3) was introduced in [7] and only a numerical study was made. Since then the model is rarely studied because there are no methods or technics to study such models. For instance, before [2] one did not know whether the model (1.1)-(1.3) was well posed for $\beta > 1$.

When $0 < a < b < \infty$, we have proved, in [2], that the model (1.1)-(1.3) is governed by a strongly continuous semigroup and we have given its explicit expression. Moreover, in spite of the obvious non-compactness of the right hand side of (1.1), we could describe the essential spectrum of that semigroup.

When $0 < a < b = \infty$, then maturation velocities are not bounded and so all announced results in [2] do not hold. This case corresponds to a new serious mathematical difficulty and then the model requires other technics that we will develop in this work. To show the extent of this new difficulty, we have shown, in [3], that the model (1.1)-(1.3) with the perfect memory property (i.e., $k(v,v') = \delta_{v'}(v)$), has a unique solution if and only if $\beta \leq 1$. Namely, there are no solutions for the most interesting and observed case $\beta > 1$ corresponding to an increasing cell density.

In this work, we are concerned with the case $0 < a < b = \infty$ together with the *general biological rule* mathematically described by the following boundary condition

$$f(t,0,v) = [Kf(t,1,\cdot)](v) \tag{1.4}$$

where, K is a linear operator into suitable spaces (see section 2). To study the general model (1.1)-(1.4) in its natural setting $L^1((0,1) \times (a,\infty))$, we organize this work as follows: Introduction, unperturbed model ($r = 0$), construction of the unperturbed semigroup ($r = 0$), and perturbed semigroup (1.1)-(1.4).

In Section 2, we show a generation result of a strongly continuous semigroup for the unperturbed model (1.1)-(1.4) (i.e. $r = 0$). This is obtained by Hille-Yosida's Theorem and Pillips-Lumer's Theorem given as follows.

Lemma 1.1 ([6, Theorem II.3.8]). *Let $(A, D(A))$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$, $M \geq 1$ be constants. Then the following statements are equivalent*

(1) *A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying*
$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad t \geq 0,$$
(2) *$(A, D(A))$ is closed, densely defined and for all $\lambda > \omega$ we have $\lambda \in \rho(A)$ and*
$$\|(\lambda - A)^{-n}\|_{\mathcal{L}(X)} \leq M(\lambda - \omega)^{-n}$$
for all $n \in \mathbb{N}$.

Lemma 1.2 ([6, Theorem II.3.15]). *Let $(A, D(A))$ be a densely defined linear operator on a Banach space X. If A is dissipative and the range $rg(\lambda - A) = X$ for some $\lambda > 0$, then A generates a strongly continuous semigroup of contractions.*

Section 3 deals with the explicit expression of the unperturbed semigroup. This expression will be very useful to describe the asymptotic behavior which is the main goal of [4]. The end of this work concerns the generation theorem for the perturbed model (1.1)-(1.4), where we have applied the following perturbation results

Lemma 1.3 ([6, Theorem III.1.3]). *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let B be a linear bounded operator from X into itself. Then, the operator $C := A + B$ on the domain $D(C) := D(A)$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ given by Trotter's formula*

$$S(t)x = \lim_{n\to\infty} \left[e^{-\frac{t}{n}B} T\left(\tfrac{t}{n}\right)\right]^n x \quad t \geq 0, \tag{1.5}$$

for all $x \in X$.

Lemma 1.4 ([6, Theorem II.2.7]). *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of contractions on a Banach space X and let B be a dissipative operator satisfying $D(A) \subset D(B)$ and*

$$\|Bx\| \leq a\|Ax\| + b\|x\|$$

for all $x \in D(A)$, where, $0 \leq a < 1$ and $b \geq 0$. Then, $A + B$ is the infinitesimal generator of a strongly continuous semigroup of contractions.

Finally, some of these results were announced in [5] and here we explicitly state the detailed conditions and outline all the proofs. For all theoretical results used here, we refer the reader to [6].

2. The unperturbed model $(r = 0)$

In this section, we are going to study the unperturbed model (1.1)-(1.4) (i.e. $r = 0$). So, let us consider the following functional framework $L^1(\Omega)$ whose natural norm is

$$\|\varphi\|_1 = \int_\Omega |\varphi(\mu, v)|\, d\mu\, dv \tag{2.1}$$

where, $\Omega = (0,\ 1) \times (a,\ \infty) := I \times J$. We emphasize that we are only concerned with the following important assumption

$$a > 0 \tag{2.2}$$

until the end of this work. We consider also the partial Sobolev space

$$W^1(\Omega) = \big\{\varphi \in L^1(\Omega),\ v\frac{\partial\varphi}{\partial\mu} \in L^1(\Omega) \text{ and } v\varphi \in L^1(\Omega)\big\}$$

whose norm is

$$\|\varphi\|_{W^1(\Omega)} = \big\|v\frac{\partial\varphi}{\partial\mu}\big\|_1 + \|v\varphi\|_1.$$

Finally, we consider the trace space $Y_1 := L^1(J, vdv)$ endowed with the norm

$$\|\psi\|_{Y_1} = \int_a^\infty |\psi(v)| v dv.$$

Lemma 2.1 ([1]). *The trace mappings $\gamma_0\varphi = \varphi(0,\cdot)$ and $\gamma_1\varphi = \varphi(1,\cdot)$ are linear bounded from $W^1(\Omega)$ into Y_1.*

Thanks to Lemma above, it is easy to check the following useful lemma.

Lemma 2.2. *Let A_0 be the following unbounded operator*

$$\begin{gathered} A_0\varphi = -v\frac{\partial \varphi}{\partial \mu} \text{ on the domain} \\ D(A_0) = \{\varphi \in W^1(\Omega),\ \gamma_0\varphi = 0\}. \end{gathered} \tag{2.3}$$

(1) *The operator A_0 generates, on $L^1(\Omega)$, a positive strongly continuous semigroup $(U_0(t))_{t\geq 0}$ of contractions given by*

$$U_0(t)\varphi(\mu, v) = \chi(\mu, v, t)\varphi(\mu - tv, v) \quad t \geq 0, \tag{2.4}$$

where,

$$\chi(\mu, v, t) = \begin{cases} 1 & \text{if} \quad \mu \geq tv, \\ 0 & \text{if} \quad \mu < tv. \end{cases} \tag{2.5}$$

(2) *Let $\lambda > 0$. Then, for all $g \in L^1(\Omega)$, we have*

$$\|v(\lambda - A_0)^{-1}g\|_1 \leq \|g\|_1. \tag{2.6}$$

(3) *$t \to \gamma_1 U_0(t)\varphi \in Y_1$ is a continuous mapping with respect to $t \geq 0$.*

Now, let us consider a linear boundary operator K from Y_1 into itself until the end of this work. This leads to write the boundary condition (1.4) as

$$\gamma_0\varphi = K\gamma_1\varphi \tag{2.7}$$

and allows us to give a sense, by Lemma 2.1, to the following unbounded operator

$$\begin{gathered} A_K\varphi = -v\frac{\partial \varphi}{\partial \mu} \text{ on the domain} \\ D(A_K) = \{\varphi \in W^1(\Omega),\ \text{satisfying } \gamma_0\varphi = K\gamma_1\varphi\} \end{gathered} \tag{2.8}$$

To study A_K, let us define the operator

$$K_\lambda := \theta_\lambda K, \quad \text{where} \quad \theta_\lambda(\cdot) = e^{-\frac{\lambda}{\cdot}} \tag{2.9}$$

which plays an important role in the sequel.

Lemma 2.3. *Suppose K is bounded satisfying $\|K\|_{\mathcal{L}(Y_1)} < 1$. Then*

(1) *For all $\lambda \geq 0$, K_λ is a linear bounded operator from Y_1 into itself satisfying $\|K_\lambda\|_{\mathcal{L}(Y_1)} < 1$.*

(2) *For all $\lambda > 0$, the resolvent operator of (2.8) is given by*

$$(\lambda - A_K)^{-1}g = \varepsilon_\lambda K(I - K_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}g + (\lambda - A_0)^{-1}g \tag{2.10}$$

for all $g \in L^1(\Omega)$, where $\varepsilon_\lambda(\mu, v) = e^{-\lambda\frac{\mu}{v}}$.

(3) *The operator defined by (2.8) generates, on $L^1(\Omega)$, a strongly continuous semigroup $(U_K(t))_{t\geq 0}$ satisfying*

$$\|U_K(t)\varphi\|_1 \leq \|\varphi\|_1 \quad t \geq 0, \tag{2.11}$$

for all $\varphi \in L^1(\Omega)$.

Proof. (1) This point clearly follows from $\|K_\lambda\|_{\mathcal{L}(Y_1)} \leq \|K\|_{\mathcal{L}(Y_1)} < 1$ for all $\lambda \geq 0$.

(2) Let $\lambda > 0$ and $g \in L^1(\Omega)$. The general solution of

$$\lambda\varphi = -v\frac{\partial \varphi}{\partial \mu} + g \tag{2.12}$$

is given by

$$\varphi = \varepsilon_\lambda \psi + (\lambda - A_0)^{-1}g \tag{2.13}$$

where ψ is any function of the variable v. If $\psi \in Y_1$ then, by (2.6) we clearly have

$$\|v\varphi\|_1 \le \|v\varepsilon_\lambda \psi\|_1 + \|v(\lambda - A_0)^{-1} g\|_1 \le \|\psi\|_{Y_1} + \|g\|_1 < \infty$$

which implies, by (2.12), that

$$\big\|v\frac{\partial\varphi}{\partial\mu}\varphi\big\|_1 \le \lambda\|\varphi\|_1 + \|g\|_1 \le \frac{\lambda}{a}\|v\varphi\|_1 + \|g\|_1 < \infty$$

and therefore $\varphi \in W^1(\Omega)$. Now, $\varphi \in D(A_K)$ if and only if φ satisfies (2.7). This means that ψ is the unique solution of the following system

$$\begin{aligned} \psi &= K\gamma_1\varphi \\ \gamma_1\varphi &= K_\lambda\gamma_1\varphi + \gamma_1(\lambda - A_0)^{-1}g. \end{aligned} \tag{2.14}$$

As $\lambda > 0$, then the first point implies that $(I - K_\lambda)$ is invertible into Y_1 and therefore

$$\psi = K(I - K_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}g$$

which we put in (2.13) to get (2.10).

(3) Let $\varphi \in D(A_K)$. Then, we have

$$\begin{aligned} \big\langle \operatorname{sgn}\varphi,\ A_K\varphi\big\rangle &= -\int_\Omega (\operatorname{sgn}\varphi(\mu,v))\Big(v\frac{\partial\varphi}{\partial\mu}(\mu,v)\Big)\,d\mu\,dv \\ &= -\int_0^1\int_a^\infty v\frac{\partial|\varphi|}{\partial\mu}(\mu,v)\,d\mu\,dv \\ &= \int_a^\infty |\varphi(0,v)|v dv - \int_a^\infty |\varphi(1,v)|v dv \\ &= \|\gamma_0\varphi\|_{Y_1} - \|\gamma_1\varphi\|_{Y_1}. \end{aligned}$$

By (2.7), it follows that

$$\langle \operatorname{sgn}\varphi, A_K\varphi\rangle \le \big(\|K\|_{\mathcal{L}(Y_1)} - 1\big)\,\|\gamma_1\varphi\|_{Y_1} \le 0 \tag{2.15}$$

because of $\|K\|_{\mathcal{L}(Y_1)} < 1$ and hence, A_K is a dissipative operator. Furthermore, A_K is densely defined because of $\mathcal{C}_c(\Omega) \subset D(A_K) \subset L^1(\Omega)$. Now, Lemma 1.2 completes the proof. $\square$

Clearly, Lemma above does not hold for the case $\|K\|_{\mathcal{L}(Y_1)} \ge 1$ because (2.15) can not be satisfied and therefore other ways are needed. So, according to the compactness of the boundary operator K, we have the following result.

Lemma 2.4. *Suppose K is compact satisfying $\|K\|_{\mathcal{L}(Y_1)} \ge 1$. Then*

1. *There exists $\lambda_0 = \lambda_0(K)$, with $\lambda_0 > 0$ if $\|K\|_{\mathcal{L}(Y_1)} > 1$ and $\lambda_0 \ge 0$ if $\|K\|_{\mathcal{L}(Y_1)} = 1$, satisfying $\|K_\lambda\|_{\mathcal{L}(Y_1)} < 1$ for all $\lambda > \lambda_0$ and $\|K_{\lambda_0}\|_{\mathcal{L}(Y_1)} \le 1$.*
2. *For all $\lambda > \lambda_0$, the resolvent operator of (2.8) is given by (2.10).*
3. *The operator defined by (2.8) generates, on $L^1(\Omega)$, a strongly continuous semigroup $(U_K(t))_{t\ge 0}$ satisfying*

$$\|U_K(t)\varphi\|_1 \le e^{\frac{\lambda_0}{a}} e^{t\lambda_0}\|\varphi\|_1 \quad t \ge 0, \tag{2.16}$$

for all $\varphi \in L^1(\Omega)$.

Proof. (1) Let $\lambda, \eta \geq 0$ and let B be the unit ball in Y_1. So we have

$$\begin{aligned}
\|K_\lambda - K_\eta\|_{\mathcal{L}(Y_1)} &= \sup_{\psi \in B} \|K_\lambda \psi - K_\eta \psi\|_{Y_1} \\
&= \sup_{\varphi \in K(B)} \|\theta_\lambda \varphi - \theta_\eta \varphi\|_{Y_1} \\
&\leq \sup_{\varphi \in \overline{K(B)}} \|\theta_\lambda \varphi - \theta_\eta \varphi\|_{Y_1}.
\end{aligned}$$

By the compactness of the set $\overline{K(B)}$, then there exists $\varphi_0 \in \overline{K(B)}$ (so independent of λ and μ) satisfying

$$\|K_\lambda - K_\eta\|_{\mathcal{L}(Y_1)} \leq \|\theta_\lambda \varphi_0 - \theta_\eta \varphi_0\|_{Y_1}$$

which implies

$$\lim_{\lambda \to \eta} \|K_\lambda - K_\eta\|_{\mathcal{L}(Y_1)} \leq \lim_{\lambda \to \eta} \|\theta_\lambda \varphi_0 - \theta_\eta \varphi_0\|_{Y_1} = 0$$

and therefore, the continuity of the mapping

$$\lambda \to \|K_\lambda\|_{\mathcal{L}(Y_1)} \tag{2.17}$$

follows, for $\lambda \geq 0$. Furthermore, proceeding as previously we obtain

$$\|K_\lambda\|_{\mathcal{L}(Y_1)} = \sup_{\psi \in B} \|K_\lambda \psi\|_{Y_1} \leq \sup_{\varphi \in \overline{K(B)}} \|\theta_\lambda \varphi\|_{Y_1} = \|\theta_\lambda \varphi_0\|_{Y_1}$$

and therefore

$$\lim_{\lambda \to \infty} \|K_\lambda\|_{\mathcal{L}(Y_1)} = 0. \tag{2.18}$$

On the other hand, if $\eta > \lambda \geq 0$ then for all $\psi \in Y_1$ we have

$$|K_\eta \psi| = \theta_\eta |K\psi| = \theta_{\eta-\lambda} \theta_\lambda |K\psi| = \theta_{\eta-\lambda} |K_\lambda \psi| < |K_\lambda \psi|$$

which implies

$$\|K_\eta\|_{\mathcal{L}(Y_1)} \leq \|K_\lambda\|_{\mathcal{L}(Y_1)}$$

and therefore the mapping (2.17) is decreasing.

Now, if $\|K\|_{\mathcal{L}(Y_1)} > 1$, then $\|K_0\|_{\mathcal{L}(Y_1)} = \|K\|_{\mathcal{L}(Y_1)} > 1$ together with (2.18) imply that the equation $\|K_\lambda\|_{\mathcal{L}(Y_1)} = 1$ has at least one strictly positive solution ($\lambda_1 > 0$) and therefore the closed set

$$E := \{\lambda \geq 0, \ \|K_\lambda\|_{\mathcal{L}(Y_1)} = 1\} \tag{2.19}$$

is not empty and bounded. Now, it suffices to set

$$\lambda_0 := \max E \geq \lambda_1 > 0$$

Next. If $\|K\|_{\mathcal{L}(Y_1)} = 1$ then $\|K_0\|_{\mathcal{L}(Y_1)} = \|K\|_{\mathcal{L}(Y_1)} = 1$ implies that $\lambda = 0$ is obviously a solution of $\|K_\lambda\|_{\mathcal{L}(Y_1)} = 1$ and therefore the closed set (2.19) is not empty and bounded. Now, it suffices again to set

$$\lambda_0 := \max E \geq 0.$$

Finally, in both cases we have $\|K_\lambda\|_{\mathcal{L}(Y_1)} < 1$ if $\lambda > \lambda_0$ and $\|K_{\lambda_0}\|_{\mathcal{L}(Y_1)} \leq 1$.

(2) Following the proof of the second point of Lemma 2.3, we have only to solve (2.14). This clearly follows from the first point, and therefore $(I - K_\lambda)$ is an invertible operator into Y_1.

(3) First, let us introduce on, $L^1(\Omega)$, the following norm

$$|||\varphi|||_1 = \int_\Omega |\varphi(\mu, v)| h(\mu, v) \, d\mu \, dv \tag{2.20}$$

where $h(\mu,v) = e^{-\lambda_0\frac{(1-\mu)}{v}}$. The norms (2.20) and (2.1) are clearly equivalent because of

$$e^{-\lambda_0/a}\|\varphi\|_1 \le |||\varphi|||_1 \le \|\varphi\|_1 \tag{2.21}$$

for all $\varphi \in L^1(\Omega)$.

Next, let $\lambda > \lambda_0$ and $g \in L^1(\Omega)$. So, the second point means that (2.13) is the unique solution of (2.12) satisfying (2.7). Multiplying (2.12) by $(\operatorname{sgn}\varphi)h$ and integrating it over Ω, we obtain

$$\lambda|||\varphi|||_1 = -\int_\Omega v\frac{\partial|\varphi|}{\partial\mu}h(\mu,v)\,d\mu\,dv + \int_\Omega(\operatorname{sgn}\varphi)hg(\mu,v)\,d\mu\,dv = I + J. \tag{2.22}$$

For the term J, we obviously have

$$J \le |||g|||_1. \tag{2.23}$$

Integrating by parts and using (2.7) and (2.9), the term I becomes

$$\begin{aligned}
I &= \int_a^\infty e^{-\frac{\lambda_0}{v}}|\varphi(0,v)|vdv - \int_a^\infty|\varphi(1,v)|vdv + \lambda_0\int_\Omega|\varphi h|(\mu,v)\,d\mu\,dv\\
&= \int_a^\infty e^{-\frac{\lambda_0}{v}}|K\gamma_1\varphi(v)|vdv - \int_a^\infty|\gamma_1\varphi(v)|vdv + \lambda_0|||\varphi|||_1\\
&= \|K_{\lambda_0}\gamma_1\varphi\|_{Y_1} - \|\gamma_1\varphi\|_{Y_1} + \lambda_0|||\varphi|||_1\\
&\le \left(\|K_{\lambda_0}\|_{\mathcal{L}(Y_1)} - 1\right)\|\gamma_1\varphi\|_{Y_1} + \lambda_0|||\varphi|||_1
\end{aligned}$$

and by the first point, we are led to

$$I \le \lambda_0|||\varphi|||_1. \tag{2.24}$$

Putting (2.24) and (2.23) in (2.22), we obtain

$$|||\varphi|||_1 = |||(\lambda - A_K)^{-1}g|||_1 \le \frac{|||g|||_1}{(\lambda-\lambda_0)}.$$

Moreover, A_K is a closed operator (because of $\rho(A_K) \neq \emptyset$) and densely defined (because of $\mathcal{C}_c(\Omega) \subset D(A_K) \subset L^1(\Omega)$). Therefore, Lemma 1.1 leads to the existence of a strongly continuous semigroup $(U_K(t))_{t\ge0}$ satisfying

$$|||U_K(t)g|||_1 \le e^{t\lambda_0}|||g|||_1, \quad t \ge 0. \tag{2.25}$$

Then (2.21) completes the proof. $\square$

Lemmas 2.3 and 2.4 suggest to set the following definition.

Definition 2.5. K is said to be an *admissible* operator if (K is bounded and $\|K\|_{\mathcal{L}(Y_1)} < 1$) or ($K$ is compact and $\|K\|_{\mathcal{L}(Y_1)} \ge 1$).

In this case, the number

$$\omega_0 = \begin{cases} 0, & \text{if } K \text{ bounded and } \|K\|_{\mathcal{L}(Y_1)} < 1;\\ \lambda_0, & \text{if } K \text{ compact and } \|K\|_{\mathcal{L}(Y_1)} \ge 1. \end{cases} \tag{2.26}$$

is called the *abscissa* of the admissible operator K.

Lemmas 2.3 and 2.4 together with the definition above clearly lead to the main result of this section.

Theorem 2.6. *Let K be an admissible operator whose abscissa is ω_0. Then*

1. $\|K_\lambda\|_{\mathcal{L}(Y_1)} < 1$ *for all* $\lambda > \omega_0$ *and* $\|K_{\omega_0}\|_{\mathcal{L}(Y_1)} \le 1$.
2. *For all* $\lambda > \omega_0$*, the resolvent operator of* (2.8) *is given by* (2.10).

(3) *The operator defined by* (2.8) *generates, on* $L^1(\Omega)$*, a strongly continuous semigroup* $(U_K(t))_{t\geq 0}$ *satisfying*

$$\|U_K(t)\varphi\|_1 \leq e^{\frac{\omega_0}{a}} e^{t\omega_0}\|\varphi\|_1, \quad t \geq 0, \tag{2.27}$$

for all $\varphi \in L^1(\Omega)$.

3. Construction of the unperturbed semigroup ($r = 0$)

In this section, we are going to give the expression of the unperturbed semigroup $(U_K(t))_{t\geq 0}$. This expression is very useful to describe the behavior asymptotic which is the main goal of [4]. So, let us consider the Banach space $Z^1_\omega := L^1((-\infty, 0) \times J, h_\omega)$ ($\omega \geq 0$) whose norm is

$$\|f\|_{Z^1_\omega} = \int_a^\infty \int_{-\infty}^0 |f(x, v)| e^{-\omega\frac{(1-x)}{v}}\, dx\, dv.$$

In this context we have the following result.

Lemma 3.1. *Let* K *be an admissible operator whose abscissa is* ω_0 *and let* $B_K(t)$ *be the operator*

$$B_K(t)\varphi(\mu, v) = \xi(\mu, v, t)(I - H_K)^{-1} V_K\varphi(\mu - tv, v) \quad t \geq 0 \tag{3.1}$$

for almost all $(\mu, v) \in \Omega$*, where, the operators* H_K *and* V_K *are defined as*

$$H_K f(x, v) = \Big(K\big(\xi(1, \cdot, -xv^{-1}) f(1 + xv^{-1}\cdot, \cdot)\big)\Big)(v),$$
$$V_K\varphi(x, v) = \big(K\left(\gamma_1 U_0(-xv^{-1})\varphi\right)\big)(v),$$

with

$$\xi(\mu, v, t) = \begin{cases} 1 & \text{if } \mu < tv; \\ 0 & \text{if } \mu \geq tv. \end{cases} \tag{3.2}$$

Then

(1) H_K *and* V_K *are bounded operators respectively from* Z^1_ω ($\omega \geq 0$) *and* $L^1(\Omega)$ *into* Z^1_ω*. Furthermore, we have*

$$\|H_K\|_{\mathcal{L}(Z^1_\omega)} \leq \|K_\omega\|_{\mathcal{L}(Y_1)}, \tag{3.3}$$
$$\|V_K\|_{\mathcal{L}(L^1(\Omega), Z^1_\omega)} \leq \|K_\omega\|_{\mathcal{L}(Y_1)}. \tag{3.4}$$

(2) *Let* $\omega > \omega_0$*. Then, for all* $t \geq 0$*, the operator* $B_K(t)$ *is linear and bounded from* $L^1(\Omega)$ *into itself.*

(3) *For all* $\varphi \in L^1(\Omega)$*, the mapping* $t \in \mathbb{R}_+ \to B_K(t)\varphi$ *is continuous at :* $t = 0_+$ *and* $B_K(0) = 0$.

(4) *For all* $\varphi \in W^1(\Omega)$ *and for all* $t \geq 0$ *we have*

$$\gamma_0 B_K(t)\varphi - K\gamma_1 B_K(t)\varphi = K\gamma_1 U_0(t)\varphi. \tag{3.5}$$

(5) *Let* K' *be an admissible operator whose abscissa is* ω_0'*. Then, for all* $\omega > \max\{\omega_0,\ \omega_0'\}$*, we have*

$$\|B_K(t) - B_{K'}(t)\|_{\mathcal{L}(L^1(\Omega))} \leq \frac{e^{\omega(\frac{1}{a}+t)}\|K - K'\|_{\mathcal{L}(Y_1)}}{(1 - \|K_\omega\|_{\mathcal{L}(Y_1)})(1 - \|K'_\omega\|_{\mathcal{L}(Y_1)})}. \tag{3.6}$$

Proof. (1) For all $f \in Z^1_\omega$ $(\omega \geq 0)$, we have

$$\begin{aligned}
\|H_K f\|_{\mathcal{L}(Z^1_\omega)} &= \int_a^\infty \int_{-\infty}^0 |\left[K\xi\big(1,\cdot,-\tfrac{x}{v}\big)f\big(1+\tfrac{x}{v}\cdot,\cdot\big)\right](v)|e^{-\omega\frac{(1-x)}{v}}\,dx\,dv \\
&= \int_a^\infty \int_0^\infty |\left[K\xi(1,\cdot,t)f(1-t\cdot,\cdot)\right](v)|e^{-\omega(\frac{1}{v}+t)}v\,dt\,dv
\end{aligned}$$

which leads, by (2.9), to

$$\begin{aligned}
\|H_K f\|_{\mathcal{L}(Z^1_\omega)} &= \int_0^\infty \Big[\int_a^\infty |\left[K_\omega\xi(1,\cdot,t)f(1-t\cdot,\cdot)\right](v)|v dv\Big] e^{-\omega t}dt \\
&\leq \|K_\omega\|_{\mathcal{L}(Y_1)} \int_0^\infty \Big[\int_a^\infty |\xi(1,v,t)f(1-tv,v)|v dv\Big] e^{-\omega t}dt \\
&\leq \|K_\omega\|_{\mathcal{L}(Y_1)} \int_a^\infty \int_{-\infty}^0 |\xi\left(1,v,\tfrac{1-x}{v}\right) f(x,v)|e^{-\omega\frac{(1-x)}{v}}\,dx\,dv \\
&\leq \|K_\omega\|_{\mathcal{L}(Y_1)}\|f\|_{Z^1_\omega}
\end{aligned}$$

and therefore (3.3) holds. A similar calculation implies that

$$\|V_K\varphi\|_{Z^1_\omega} \leq \|K_\omega\|_{\mathcal{L}(Y_1)}\|\varphi\|_1$$

for all $\varphi \in L^1(\Omega)$ and therefore (3.4) holds.

(2) Let $\omega > \omega_0$ and $t \geq 0$. Due to the first point of Theorem 2.6 together with (3.3), we obtain $(I - H_K)$ is an invertible operator into Z^1_ω. So, the operator $B_K(t)$ given by (3.1) is well defined and its linearity follows from those of $(I - H_K)^{-1}$ and V_K. For its boundedness, we have

$$\begin{aligned}
\|B_K(t)\varphi\|_1 &= \int_\Omega \xi(\mu,v,t)|(I-H_K)^{-1}V_K\varphi(\mu-tv,v)|\,d\mu\,dv \\
&\leq \int_a^\infty \int_0^{tv} e^{\omega\frac{\mu}{v}}|(I-H_K)^{-1}V_K\varphi(\mu-tv,v)|\,d\mu\,dv \\
&= \int_a^\infty \int_{-tv}^0 e^{\omega\left(\frac{x}{v}+t\right)}|(I-H_K)^{-1}V_K\varphi(x,v)|\,dx\,dv
\end{aligned}$$

for all $\varphi \in L^1(\Omega)$ and therefore

$$\|B_K(t)\varphi\|_1 \leq e^{\omega\left(\frac{1}{a}+t\right)} \int_a^\infty \int_{-tv}^0 e^{-\omega\frac{(1-x)}{v}}|(I-H_K)^{-1}V_K\varphi(x,v)|\,dx\,dv. \tag{3.7}$$

This implies

$$\|B_K(t)\varphi\|_1 \leq e^{\omega\left(\frac{1}{a}+t\right)}\|(I-H_K)^{-1}V_K\varphi\|_{Z^1_\omega} \tag{3.8}$$

and by (3.3) and (3.4), we clearly have

$$\begin{aligned}
\|B_K(t)\varphi\|_1 &\leq e^{\omega\left(\frac{1}{a}+t\right)}\|(I-H_K)^{-1}V_K\varphi\|_{Z^1_\omega} \\
&= e^{\omega\left(\frac{1}{a}+t\right)}\|\sum_{n\geq 0} H_K^n\|_{Z^1_\omega}\|V_K\varphi\|_{Z^1_\omega} \\
&\leq e^{\omega\left(\frac{1}{a}+t\right)}\frac{1}{1-\|K_\omega\|_{\mathcal{L}(Y_1)}}\|K_\omega\|_{\mathcal{L}(Y_1)}\|\varphi\|_1
\end{aligned}$$

which leads to the boundedness of $B_K(t)$ from $L^1(\Omega)$ into itself.

(2) This point obviously follows from (3.7).

(3) Let $\varphi \in W^1(\Omega)$. Using (3.1), we obtain

$$\begin{aligned}
&\int_a^\infty \int_0^\infty \big|K\gamma_1 U_0(t)\varphi(v) - \gamma_0 B_K(t)\varphi(v) - K\gamma_1 B_K(t)\varphi(v)\big| v\,dt\,dv \\
&= \int_a^\infty \int_0^\infty \Big| V_K\varphi(-tv, v) - (I - H_K)^{-1} V_K(-tv, v) \\
&\quad - K\left[\xi(1,\cdot,t)(I-H_K)^{-1}V_K\varphi(1 - t\cdot,\cdot)\right](v)\Big| v\,dt\,dv.
\end{aligned}$$

The change $x = -tv$ with $dx = -vdt$ infers that

$$\begin{aligned}
&\int_a^\infty \int_0^\infty \big|K\gamma_1 U_0(t)\varphi(v) - \gamma_0 B_K(t)\varphi(v) - K\gamma_1 B_K(t)\varphi(v)\big| v\,dt\,dv \\
&= \int_a^\infty \int_{-\infty}^0 \Big| V_K\varphi(x, v) - (I - H_K)^{-1} V_K(x, v) \\
&\quad - K\left[\xi(1,\cdot,-xv^{-1})(I-H_K)^{-1}V_K\varphi(1 + xv^{-1}\cdot,\cdot)\right](v)\Big|\, dx\,dv \\
&= \int_a^\infty \int_{-\infty}^0 \Big| V_K\varphi(x, v) - (I - H_K)^{-1} V_K(x, v) \\
&\quad - H_K(I-H_K)^{-1}V_K\varphi(x,v)\Big|\, dx\,dv = 0
\end{aligned}$$

and therefore (3.5) holds for almost all $t \in \mathbb{R}_+$. Now, thanks to the third point of Lemma 2.2, we obtain (3.5) holds for all $t \geq 0$.

(4) Let $\omega > \sup\{\omega_0,\ \omega_0'\}$. A simple calculation, like the proof of (3.8), infers that

$$\|B_K(t)\varphi - B_{K'}(t)\varphi\|_1 \leq e^{\omega\left(\frac{1}{a}+t\right)} \|(I-H_K)^{-1}V_K\varphi - (I-H_{K'})^{-1}V_{K'}\varphi\|_{Z^1_\omega}$$

As

$$\begin{aligned}
&(I-H_K)^{-1}V_K - (I-H_{K'})^{-1}V_{K'} \\
&= (I-H_{K'})^{-1}[V_K - V_{K'}] + (I-H_K)^{-1}[H_K - H_{K'}](I-H_{K'})^{-1}V_K,
\end{aligned}$$

we clearly have

$$\begin{aligned}
&\|B_K(t)\varphi - B_{K'}(t)\varphi\|_1 \\
&\leq e^{\omega\left(\frac{1}{a}+t\right)} \|(I-H_{K'})^{-1}\| \|V_K\varphi - V_{K'}\varphi\| \\
&\quad + e^{\omega\left(\frac{1}{a}+t\right)} \|(I-H_K)^{-1}\| \|H_K - H_{K'}\| \|(I-H_{K'})^{-1}\| \|V_K\varphi\|
\end{aligned}$$

and therefore, by (3.3) and (3.4), we are lead to

$$\begin{aligned}
&\|B_K(t)\varphi - B_{K'}(t)\varphi\|_1 \\
&\leq e^{\omega\left(\frac{1}{a}+t\right)} \frac{1}{1 - \|K'_\omega\|_{\mathcal{L}(Y_1)}} \|V_K\varphi - V_{K'}\varphi\|_{Z^1_\omega} \\
&\quad + e^{\omega\left(\frac{1}{a}+t\right)} \frac{1}{(1 - \|K_\omega\|_{\mathcal{L}(Y_1)})} \|H_K - H_{K'}\| \frac{1}{(1 - \|K'_\omega\|_{\mathcal{L}(Y_1)})} \|K_\omega\|_{\mathcal{L}(Y_1)} \|\varphi\|_1.
\end{aligned}$$

Thanks to the obvious linearity of the mappings $K \to H_K$ and $K \to V_K$, (3.3) and (3.4) imply

$$\begin{aligned}\|B_K(t)\varphi - B_{K'}(t)\varphi\|_1 &\le e^{\omega(\frac{1}{a}+t)}\frac{\|K_\omega - K'_\omega\|_{\mathcal{L}(Y_1)}\|\varphi\|_1}{1-\|K'_\omega\|_{\mathcal{L}(Y_1)}} \\ &\quad + e^{\omega(\frac{1}{a}+t)}\frac{\|K_\omega - K'_\omega\|_{\mathcal{L}(Y_1)}\|K_\omega\|_{\mathcal{L}(Y_1)}\|\varphi\|_1}{(1-\|K_\omega\|_{\mathcal{L}(Y_1)})(1-\|K'_\omega\|_{\mathcal{L}(Y_1)})} \\ &\le e^{\omega(\frac{1}{a}+t)}\frac{\|K_\omega - K'_\omega\|_{\mathcal{L}(Y_1)}}{(1-\|K_\omega\|_{\mathcal{L}(Y_1)})(1-\|K'_\omega\|_{\mathcal{L}(Y_1)})}\|\varphi\|_1 \\ &\le e^{\omega(\frac{1}{a}+t)}\frac{\|K - K'\|_{\mathcal{L}(Y_1)}}{(1-\|K_\omega\|_{\mathcal{L}(Y_1)})(1-\|K'_\omega\|_{\mathcal{L}(Y_1)})}\|\varphi\|_1\end{aligned}$$

and therefore (3.6) holds. The proof is complete. $\square$

Theorem 3.2. *Let K be an admissible operator whose abscissa is ω_0. Then, the semigroup $(U_K(t))_{t\ge 0}$ is given by*

$$U_K(t)\varphi = U_0(t)\varphi + B_K(t)\varphi \quad t \ge 0, \tag{3.9}$$

for all $\varphi \in L^1(\Omega)$. Furthermore, the operator $B_K(t)$ satisfies

$$B_K(t)\varphi(\mu,v) = \xi(\mu,v,t)K\gamma_1\big(U_K\left(t-\tfrac{\mu}{v}\right)\varphi\big)(v), \quad t \ge 0, \tag{3.10}$$

for almost all $(\mu,v) \in \Omega$.

Proof. Let $t \ge 0$ and $\varphi \in L^1(\Omega)$ and let $S_K(t)$ be the following operator

$$S_K(t) = U_0(t) + B_K(t), \quad t \ge 0, \tag{3.11}$$

where, $B_K(t)$ is given by (3.1). In the sequel, we are going to prove that $(S_K(t))_{t\ge 0}$ is a strongly continuous semigroup into $L^1(\Omega)$. At the end of this proof, we will show that $U_K(t) = S_K(t)$ for all $t \ge 0$. Then, let us divide the proof in several steps.

Step one. This step deals with a useful expression of the operator $B_K(t)$ like (3.10). So, for all $\varphi \in W^1(\Omega)$ and for almost all $(\mu,v) \in \Omega$, (3.11) implies that

$$S_K\left(t-\tfrac{\mu}{v}\right)\varphi(0,v) = U_0\left(t-\tfrac{\mu}{v}\right)\varphi(0,v) + B_K\left(t-\tfrac{\mu}{v}\right)\varphi(0,v)$$

which leads, by (2.4), to

$$\xi(\mu,v,t)S_K\left(t-\tfrac{\mu}{v}\right)\varphi(0,v) = \xi(\mu,v,t)B_K\left(t-\tfrac{\mu}{v}\right)\varphi(0,v).$$

Using (3.1) we obtain

$$\begin{aligned}&\xi(\mu,v,t)S_K\left(t-\tfrac{\mu}{v}\right)\varphi(0,v) \\ &= \xi(\mu,v,t)\xi\left(0,v,\left(t-\tfrac{\mu}{v}\right)\right)(I-H_K)^{-1}V_K\varphi\left(0-\left(t-\tfrac{\mu}{v}\right)v,v\right) \\ &= \xi(\mu,v,t)(I-H_K)^{-1}V_K\varphi(\mu-tv,v)\end{aligned}$$

and therefore

$$\xi(\mu,v,t)\gamma_0\left(S_K\left(t-\tfrac{\mu}{v}\right)\varphi\right)(v) = B_K(t)\varphi(\mu,v). \tag{3.12}$$

On the other hand, (3.11) implies

$$\begin{aligned}\gamma_0 S_K(t)\varphi - K\gamma_1 S_K(t)\varphi &= \gamma_0 U_0(t)\varphi + \gamma_0 B_K(t)\varphi \\ &\quad - K\gamma_1 U_0(t)\varphi - K\gamma_1 B_K(t)\varphi \\ &= \gamma_0 B_K(t)\varphi - K\gamma_1 U_0(t)\varphi - K\gamma_1 B_K(t)\varphi.\end{aligned}$$

According to (3.5), we obtain

$$\gamma_0 S_K(t)\varphi = K\gamma_1 S_K(t)\varphi$$

and hence

$$\xi(\mu,v,t)\gamma_0\left(S_K\left(t-\tfrac{\mu}{v}\right)\varphi\right)(v) = \xi(\mu,v,t)K\gamma_1\left(S_K\left(t-\tfrac{\mu}{v}\right)\varphi\right)(v). \tag{3.13}$$

Now, combining (3.12) and (3.13), we obtain the following useful relation

$$B_K(t)\varphi(\mu,v) = \xi(\mu,v,t)K\gamma_1\left(S_K\left(t-\tfrac{\mu}{v}\right)\right)\varphi(v). \tag{3.14}$$

Finally, the density of $W^1(\Omega)$ in $L^1(\Omega)$, implies that (3.14) holds again for all $\varphi \in L^1(\Omega)$.

Step two. In this step, we are going to prove that $(S_K(t))_{t\geq 0}$ is a strongly continuous semigroup into $L^1(\Omega)$. So, Lemma 2.2 together with the second and the third points of Lemma 3.1 clearly lead to the linearity and the boundedness of the operator $S_K(t)$ from $L^1(\Omega)$ into itself and $S_K(0) = U_0(0) + B_K(0) = I + 0 = I$ and

$$\lim_{t\to 0_+} \|S_K(t)\varphi - \varphi\|_1 \leq \lim_{t\to 0_+} \|U_0(t)\varphi - \varphi\|_1 + \lim_{t\to 0_+} \|B_K(t)\varphi\|_1 = 0.$$

Now, in order to state that the family operators $(S_K(t))_{t\geq 0}$ is a strongly continuous semigroup into $L^1(\Omega)$, it suffices to prove that

$$G(t,s) := S_K(t)S_K(s) - S_K(t+s) = 0 \tag{3.15}$$

for all $t \geq 0$ and all $s \geq 0$. So, by (3.11) and (3.14), a simple calculation shows that

$$\begin{aligned}
&G(t,s)\varphi(\mu,v)\\
&= \xi(\mu,v,t)K\gamma_1\left(U_K\left(t-\tfrac{\mu}{v}\right)S_K(s)\varphi\right)(v)\\
&\quad + \Big(\chi(\mu,v,t)\xi\left(t+s-\tfrac{\mu}{v}\right) - \xi\left(t+s-\tfrac{\mu}{v}\right)\Big)K\gamma_1\left(S_K\left(t+s-\tfrac{\mu}{v},s\right)\varphi\right)(v)
\end{aligned}$$

for almost all $(\mu,v) \in \Omega$. Furthermore, (2.5) and (3.2) allow to reduce the relation above to

$$G(t,s)\varphi(\mu,v) = \xi(\mu,v,t)K\gamma_1\left(G(t-\tfrac{\mu}{v},s)\varphi\right)(v). \tag{3.16}$$

Next, let $\omega > \omega_0$. On one hand, (3.16) implies

$$\begin{aligned}
\int_0^\infty e^{-\omega t}\|\gamma_1 G(t,s)\varphi\|_{Y_1}\,dt &= \int_0^\infty e^{-\omega t}\int_a^\infty |\gamma_1 G(t,s)\varphi(v)|v\,dv\,dt\\
&= \int_a^\infty\int_0^\infty e^{-\omega t}\xi(1,v,t)|K\gamma_1\left(G(t-\tfrac{1}{v},s)\varphi\right)(v)|v\,dt\,dv\\
&= \int_a^\infty\int_{\frac{1}{v}}^\infty e^{-\omega t}|K\gamma_1\left(G(t-\tfrac{1}{v},s)\varphi\right)(v)|v\,dt\,dv\\
&= \int_a^\infty\int_0^\infty e^{-\omega\left(x+\frac{1}{v}\right)}|K\gamma_1 G(x,s)\varphi(v)|v\,dx\,dv
\end{aligned}$$

which leads, by (2.9), to

$$\begin{aligned}
\int_0^\infty e^{-\omega t}\|\gamma_1 G(t,s)\varphi\|_{Y_1}dt &= \int_0^\infty\int_a^\infty e^{-\omega x}|K_\omega\gamma_1 G(x,s)\varphi(v)|vdvdx\\
&= \int_0^\infty e^{-\omega x}\|K_\omega\gamma_1 G(x,s)\varphi\|_{Y_1}dx
\end{aligned}$$

and therefore,

$$\int_0^\infty e^{-\omega t}\|\gamma_1 G(t,s)\varphi\|_{Y_1}\,dt \le \|K_\omega\|\int_0^\infty e^{-\omega t}\|\gamma_1 G(t,s)\varphi\|_{Y_1}\,dt. \tag{3.17}$$

This obviously means that

$$\int_0^\infty e^{-\omega t}\|\gamma_1 G(t,s)\varphi\|_{Y_1}\,dt = 0 \tag{3.18}$$

because of the first point of Theorem 2.6. On the other hand, (3.16) implies

$$\begin{aligned}
\|G(t,s)\varphi\|_1 &= \int_\Omega \xi(\mu,v,t)|K\gamma_1\left(G\left(t-\tfrac{\mu}{v},s\right)\varphi\right)(v)|\,d\mu\,dv\\
&= \int_a^\infty\int_0^{tv} |K\gamma_1\left(G\left(t-\tfrac{\mu}{v},s\right)\varphi\right)(v)|\,d\mu\,dv\\
&= \int_a^\infty\int_0^{t} |K\gamma_1 G(x,s)\varphi(v)|v\,dx\,dv\\
&= \int_0^t \|K\gamma_1 G(x,s)\varphi\|_{Y_1}\,dx\\
&\le \|K\|_{\mathcal{L}(Y_1)}\int_0^t \|\gamma_1 G(x,s)\varphi\|_{Y_1}\,dx
\end{aligned}$$

and therefore,

$$\|G(t,s)\varphi\|_1 \le \|K\|_{\mathcal{L}(Y_1)}\int_0^\infty \|\gamma_1 G(x,s)\varphi\|_{Y_1}\,dx. \tag{3.19}$$

Now, (3.18) and (3.19) obviously imply $G(t,s)=0$ and therefore (3.15) holds for all $t\ge 0$ and all $s\ge 0$. Hence, $(S_K(t))_{t\ge 0}$ is well a strongly continuous semigroup satisfying

$$S_K(t)\varphi(\mu,v) = U_0(t)\varphi(\mu,v) + \xi(\mu,v,t)K\gamma_1\big(S_K\left(t-\tfrac{\mu}{v}\right)\varphi\big)(v) \tag{3.20}$$

because of (3.11) and (3.14).

Step three. Let $\lambda > \omega_0$ and $\varphi\in L^1(\Omega)$ and let B be the generator of the semigroup $(S_K(t))_{t\ge 0}$. Then, by (3.20) we obtain

$$\begin{aligned}
&(\lambda - B)^{-1}\varphi(\mu,v)\\
&= \int_0^\infty e^{-\lambda t}S_K(t)\varphi(\mu,v)dt\\
&= \int_0^\infty e^{-\lambda t}U_0(t)\varphi(\mu,v)dt + \int_0^\infty e^{-\lambda t}\xi(\mu,v,t)K\gamma_1\left(S_K\left(t-\tfrac{\mu}{v}\right)\varphi\right)(v)dt\\
&= (\lambda - A_0)^{-1}\varphi(\mu,v) + e^{-\lambda\frac{\mu}{v}}\int_0^\infty e^{-\lambda t}K\gamma_1 S_K(t)\varphi(v)dt\\
&= (\lambda - A_0)^{-1}\varphi(\mu,v) + e^{-\lambda\frac{\mu}{v}}K\gamma_1\Big[\int_0^\infty e^{-\lambda t}S_K(t)\varphi dt\Big](v)\\
&= (\lambda - A_0)^{-1}\varphi(\mu,v) + e^{-\lambda\frac{\mu}{v}}K\gamma_1(\lambda - B)^{-1}\varphi(v)
\end{aligned}$$

for almost all $(\mu,v)\in\Omega$, and therefore

$$(\lambda - B)^{-1}\varphi = \varepsilon_\lambda K\gamma_1(\lambda - B)^{-1}\varphi + (\lambda - A_0)^{-1}\varphi \tag{3.21}$$

where, $\varepsilon_\lambda = e^{-\lambda\frac{\mu}{v}}$. Applying γ_1 to (3.21), we infer that

$$\gamma_1(\lambda - B)^{-1}\varphi = K_\lambda \gamma_1(\lambda - B)^{-1}\varphi + \gamma_1(\lambda - A_0)^{-1}\varphi$$

and thanks to the first point of Theorem 2.6, it follows that

$$\gamma_1(\lambda - B)^{-1}\varphi = (I - K_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}\varphi \tag{3.22}$$

because of $\lambda > \omega_0$. Now, putting (3.22) in (3.21), we finally obtain

$$(\lambda - B)^{-1}\varphi = \varepsilon_\lambda K(I - K_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}\varphi + (\lambda - A_0)^{-1}\varphi. \tag{3.23}$$

Step four. By (2.10) and (3.23), we have $(\lambda - A_K)^{-1} = (\lambda - B)^{-1}$ and hence, $U_K(t) = S_K(t)$ for all $t \geq 0$ because of the uniqueness of the generated semigroup. Moreover, (3.10) holds because of (3.14). Now, the proof is complete. □

Corollary 3.3. *Let K be an admissible operator whose abscissa is ω_0. Then we have*

$$\int_a^\infty \int_0^t e^{-\omega x}|\gamma_1\left(U_K(x)\varphi\right)(v)|v\,dx\,dv \leq \frac{\|\varphi\|_1}{1 - \|K_\omega\|_{\mathcal{L}(Y_1)}}, \quad t \geq 0, \tag{3.24}$$

for all $\varphi \in L^1(\Omega)$.

Proof. The corollary is obvious for $t = 0$. So, let $t > 0$ and $\omega > \omega_0$ be a given real and let $\psi \in W^1(\Omega)$. Applying γ_1 to (3.9) and (3.10) we obtain

$$\gamma_1(U_K(x)\psi)(v) = \gamma_1(U_0(x)\psi)(v) + \xi(1, v, x)\left[K\gamma_1\left(U_K\left(x - \tfrac{1}{v}\right)\psi\right)\right](v)$$

for all $x \geq 0$ and for almost all $v \in J$. Multiplying the relation above by $e^{-\omega x}$ and integrating it over $(0, t) \times J$, we infer that

$$\begin{aligned}
&\int_a^\infty \int_0^t e^{-\omega x}|\gamma_1(U_K(x)\psi)(v)|v\,dx\,dv \\
&\leq \int_a^\infty \int_0^t e^{-\omega x}\chi(1, v, x)|\psi(1 - xv, v)|v\,dx\,dv \\
&\quad + \int_a^\infty \int_0^t e^{-\omega x}\xi(1, v, x)|K\gamma_1\left(U_K\left(x - \tfrac{1}{v}\right)\psi\right)(v)|v\,dx\,dv
\end{aligned}$$

where we have used (2.4). A suitable change of variables leads to

$$\begin{aligned}
&\int_a^\infty \int_0^t e^{-\omega x}|\gamma_1(U_K(x)\psi)(v)|v\,dx\,dv \\
&\leq \int_a^\infty \int_{1-tv}^1 e^{-\omega\frac{(1-\mu)}{v}}\chi\left(1, v, \tfrac{1-\mu}{v}\right)|\psi(\mu, v)|\,d\mu\,dv \\
&\quad + \int_a^\infty \int_0^t e^{-\omega y}e^{-\omega\frac{1}{v}}|K\gamma_1\left(U_K(y)\psi\right)(v)|v dy dv \\
&\leq \int_\Omega |\psi(\mu, v)|\,d\mu\,dv + \int_0^t e^{-\omega x}\|K_\omega\gamma_1\left(U_K(x)\psi\right)\|_{Y_1}dx
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_a^\infty \int_0^t e^{-\omega x}|\gamma_1(U_K(x)\psi)(v)|v\,dx\,dv \\
&\leq \|\psi\|_1 + \|K_\omega\|_{\mathcal{L}(Y_1)}\int_a^\infty \int_0^t e^{-\omega x}|\gamma_1(U_K(x)\psi)(v)|v\,dx\,dv.
\end{aligned}$$

Therefore,

$$\int_a^\infty \int_0^t e^{-\omega x}|\gamma_1(U_K(x)\psi)(v)|v\,dx\,dv \le \frac{1}{1-\|K_\omega\|_{\mathcal{L}(Y_1)}}\|\psi\|_1$$

because of the first point of Theorem2.6. Now, the density of $W^1(\Omega)$ in $L^1(\Omega)$ leads to (3.24) for all $\varphi \in L^1(\Omega)$. □

Note that a rank one or a compact boundary operator is admissible and therefore Theorem 2.6 holds. Accordingly, we give three important results very useful for the results in [4]. The first one is as follows.

Theorem 3.4. *Let K and K' be two compact operators. Then, we have*

$$\|U_K(t) - U_{K'}(t)\|_{\mathcal{L}(L^1(\Omega))} \le 4e^{\omega(\frac{1}{a}+t)}\|K-K'\|_{\mathcal{L}(Y_1)}, \quad t \ge 0 \tag{3.25}$$

for all ω big enough.

Proof. Let ω be a positive real and let $t \ge 0$. First, (3.9) and (3.6) clearly lead to

$$\|U_K(t) - U_{K'}(t)\|_{\mathcal{L}(L^1(\Omega))} \le \frac{e^{\omega(\frac{1}{a}+t)}\|K-K'\|_{\mathcal{L}(Y_1)}}{(1-\|K_\omega\|_{\mathcal{L}(Y_1)})(1-\|K'_\omega\|_{\mathcal{L}(Y_1)})} \tag{3.26}$$

Next, let B be the unit ball in Y_1. So we have

$$\|K_\omega\|_{\mathcal{L}(Y_1)} = \sup_{\psi\in B}\|K_\omega\psi\|_{Y_1} = \sup_{\varphi\in K(B)}\|\theta_\omega\varphi\|_{Y_1} \le \sup_{\varphi\in\overline{K(B)}}\|\theta_\omega\varphi\|_{Y_1}.$$

By the compactness of the set $\overline{K(B)}$, then there exists $\varphi_0 \in \overline{K(B)}$ (independent of ω) satisfying

$$\|K_\omega\|_{\mathcal{L}(Y_1)} \le \|\theta_\omega\varphi_0\|_{Y_1}$$

and hence

$$\lim_{\omega\to\infty}\|K_\omega\|_{\mathcal{L}(Y_1)} = \lim_{\omega\to\infty}\|\theta_\omega\varphi_0\|_{Y_1} = 0.$$

Therefore, there exists $\omega_1 > 0$ such that

$$\omega > \omega_1 \Longrightarrow \|K_\omega\|_{\mathcal{L}(Y_1)} < \frac{1}{2}. \tag{3.27}$$

The same calculation above holds for the compact operator K' and therefore there exists $\omega_1' > 0$ such that

$$\omega > \omega_1' \Longrightarrow \|K'_\omega\|_{\mathcal{L}(Y_1)} < \frac{1}{2}. \tag{3.28}$$

Finally, if $\omega > \sup\{\omega_1, \omega_1'\}$ then (3.26) and (3.27) and (3.28) clearly lead to (3.25). The proof is complete. □

Let us end this section by the following result.

Lemma 3.5. *Let K be a rank one operator in Y_1; i.e.,*

$$K\psi = h\int_a^\infty k(v')\psi(v')v'dv', \quad h \in Y_1, \quad k \in L^\infty(J).$$

Then, for all $\varphi \in L^1(\Omega)$, we have

$$U_K(t)\varphi = \sum_{m=0}^\infty U_m(t)\varphi, \quad t \ge 0,$$

where, $U_0(t)$ is given by (2.4) and

$$U_1(t)\varphi(\mu,v)=\xi(\mu,v,t)h(v)\int_a^\infty k(v_1)\chi\left(1,v_1,t-\tfrac{\mu}{v}\right)\varphi\left(1-\left(t-\tfrac{\mu}{v}\right)v_1,v_1\right)v_1dv_1$$

and, for $m\geq 2$, by

$$\begin{aligned}U_m(t)\varphi(\mu,v)&=\xi(\mu,v,t)h(v)\underbrace{\int_a^\infty\cdots\int_a^\infty}_{m\text{ times}}\prod_{j=1}^{m-1}h(v_j)\prod_{j=1}^{m}k(v_j)\\&\quad\times\xi\Big(1,v_{m-1},t-\frac{\mu}{v}-\sum_{i=1}^{(m-2)}\frac{1}{v_i}\Big)\chi\Big(1,v_m,t-\frac{\mu}{v}-\sum_{i=1}^{(m-1)}\frac{1}{v_i}\Big)\\&\quad\times\varphi\Big(1-\Big(t-\frac{\mu}{v}-\sum_{i=1}^{m-1}\frac{1}{v_i}\Big)v_m,v_m\Big)v_1v_2\cdots v_m\,dv_1\cdots dv_m.\end{aligned}$$

Furthermore, for all $t\geq 0$,

$$\lim_{N\to\infty}\big\|U_K(t)-\sum_{m=0}^{N}U_m(t)\big\|_{\mathcal{L}(L^1(\Omega))}=0. \tag{3.29}$$

Proof. Let $\varphi\in L^1(\Omega)$ and let ω be a large real. By Theorem 3.2, it is easy to check, by induction, that for all integer $N\geq 1$ we have

$$U_K(t)=U_0(t)+\sum_{m=1}^{N}U_m(t)+R_N(t)$$

where $R_N(t)$ is given by

$$\begin{aligned}&R_N(t)\varphi(\mu,v)\\&=\xi(\mu,v,t)h(v)\underbrace{\int_a^\infty\cdot\cdot\int_a^\infty}_{(N+1)\text{ times}}\prod_{j=1}^{N}h(v_j)\prod_{j=1}^{N+1}k(v_j)\xi\Big(1,v_N,t-\frac{\mu}{v}-\sum_{i=1}^{(N-1)}\frac{1}{v_i}\Big)\\&\quad\times\gamma_1\Big(U_K\Big(t-\frac{\mu}{v}-\sum_{i=1}^{N}\frac{1}{v_i}\Big)\varphi\Big)(v_{N+1})v_1\cdot\cdot v_{N+1}\,dv_1\cdot\cdot dv_{N+1}.\end{aligned}$$

As $1=e^{\omega\frac{(1-\mu)}{v}}e^{-\omega\frac{(1-\mu)}{v}}\leq e^{\omega/a}e^{-\omega\frac{(1-\mu)}{v}}$ for all $(\mu,v)\in\Omega$, then

$$\begin{aligned}&\|R_N(t)\varphi\|_1\\&\leq e^{\omega/a}\int_\Omega\Big|e^{-\omega\frac{(1-\mu)}{v}}\xi(\mu,v,t)h(v)\underbrace{\int_a^\infty\cdots\int_a^\infty}_{(N+1)\text{ times}}\\&\quad\times\prod_{j=1}^{N}h(v_j)\prod_{j=1}^{N+1}k(v_j)\xi\Big(1,v_N,t-\frac{\mu}{v}-\sum_{i=1}^{(N-1)}\frac{1}{v_i}\Big)\\&\quad\times\gamma_1\Big(U_K\Big(t-\frac{\mu}{v}-\sum_{i=1}^{N}\frac{1}{v_i}\Big)\varphi\Big)(v_{N+1})v_1v_2\cdots v_{N+1}\,dv_1\cdots dv_{N+1}\Big|\,d\mu\,dv.\end{aligned}$$

By the change of variables $x = t - \frac{\mu}{v} - \sum_{i=1}^{N} \frac{1}{v_i}$ with $v dx = -d\mu$, we infer that

$$\begin{aligned}
&\|R_N(t)\varphi\|_1 \\
&\le e^{\omega/a} \int_a^\infty \int_0^t \Big| e^{-\omega\left(x-t+\frac{1}{v}\right)} h(v) \underbrace{\int_a^\infty \cdots \int_a^\infty}_{(N+1) \text{ times}} e^{-\omega\left(\sum_{i=1}^N \frac{1}{v_i}\right)} \prod_{j=1}^{N} h(v_j) \\
&\quad \times \prod_{j=1}^{N+1} k(v_j) \gamma_1\big(U_K(x)\varphi\big)(v_{N+1}) v_1 v_2 \cdots v_N dv_1 \cdots dv_{N+1} \Big| v\, dx\, dv \\
&\le e^{\omega/a} e^{\omega t} \Big[\int_a^\infty e^{-\frac{\omega}{v}} |h(v)| v dv \Big] \Big[\int_a^\infty e^{-\frac{\omega}{v}} |h(v)||k(v)| v dv \Big]^N \\
&\quad \times \int_0^t e^{-\omega x} \int_a^\infty |k(v_{N+1})| |\gamma_1 \left(U_K(x)\varphi\right)(v_{N+1}) v_{N+1}|\, dv_{N+1}\, dx.
\end{aligned}$$

As $k \in L^\infty(J)$, then we obtain

$$\begin{aligned}
\|R_N(t)\varphi\|_1 &\le e^{\omega/a} e^{\omega t} \Big(\|k\|_\infty \int_a^\infty e^{-\frac{\omega}{v}} |h(v_i)| v\, dv \Big)^{N+1} \\
&\quad \times \int_0^t e^{-\omega x} \int_a^\infty |\gamma_1 \left(U_K(x)\varphi\right)(v_{N+1})| v_{N+1}\, dv_{N+1}\, dx \\
&= e^{\omega/a} e^{\omega t} \|K_\omega\|_{\mathcal{L}(Y_1)}^{N+1} \int_a^\infty \int_0^t e^{-\omega x} |\gamma_1 \left(U_K(x)\varphi\right)(v_{N+1})| v_{N+1}\, dx\, dv_{N+1}
\end{aligned}$$

which, by (3.24), implies

$$\|R_N(t)\varphi\|_1 \le e^{\omega/a} e^{\omega t} \frac{\|K_\omega\|_{\mathcal{L}(Y_1)}^{N+1}}{1 - \|K_\omega\|_{\mathcal{L}(Y_1)}} \|\varphi\|_1$$

and therefore

$$\lim_{N\to\infty} \|U_K(t) - \sum_{m=0}^{N} U_m(t)\|_{\mathcal{L}(L^1(\Omega))} = \lim_{N\to\infty} \|R_N(t)\|_{\mathcal{L}(L^1(\Omega))} = 0$$

because of the first point of Theorem 2.6. The proof is now complete. $\square$

4. The perturbed semigroup (1.1)-(1.4)

In this section, we are going to prove that the perturbed model (1.1)-(1.4) is governed by a strongly continuous semigroup like a linear perturbation of the unperturbed semigroup $(U_K(t))_{t\ge 0}$ already studied. So, let us define the following two perturbation operators

$$R\varphi(\mu, v) = \int_a^\infty r(\mu, v, v')\varphi(\mu, v')dv',$$
$$S\varphi(\mu, v) = -\sigma(\mu, v)\varphi(\mu, v),$$

where σ is given by (1.2). Let us impose the following hypothesis

(H1) r is measurable positive, and $\sigma \in L^\infty(\Omega)$.

Denoting

$$\underline{\sigma} := \operatorname{ess\,inf}_{(\mu,v)\in\Omega} \sigma(\mu, v) \quad \text{and} \quad \overline{\sigma} := \operatorname{ess\,sup}_{(\mu,v)\in\Omega} \sigma(\mu, v),$$

we have the following result.

Lemma 4.1. *Suppose that* (H1) *holds. Then, S and R are linear bounded operators from $L^1(\Omega)$ into itself. Furthermore, $S+R$ is a dissipative operator.*

Proof. Let $\varphi \in L^1(\Omega)$. The boundedness of the operators S and R clearly follows from

$$\begin{aligned}
\|R\varphi\|_1 &\le \int_0^1 \int_a^\infty \int_a^\infty r(\mu,v,v')|\varphi(\mu,v')|dv'dvd\mu \\
&= \int_0^1 \int_a^\infty \Big[\int_a^\infty r(\mu,v,v')dv\Big]|\varphi(\mu,v')|dv'd\mu \\
&= \int_0^1 \int_a^\infty \sigma(\mu,v')|\varphi(\mu,v')|dv'd\mu = \|S\varphi\|_1
\end{aligned}$$

and

$$\|S\varphi\|_1 = \int_\Omega \sigma(\mu,v)|\varphi(\mu,v)|\,d\mu\,dv \le \overline{\sigma}\|\varphi\|_1.$$

Furthermore, we have

$$\begin{aligned}
&\langle \operatorname{sgn}\varphi, (S+R)\varphi\rangle \\
&= \int_\Omega \operatorname{sgn}\varphi(\mu,v)\,\big(R\varphi(\mu,v)+S\varphi(\mu,v)\big)\ d\mu\,dv \\
&\le \int_0^1 \int_a^\infty \Big[\int_a^\infty r(\mu,v,v')dv\Big]|\varphi(\mu,v')|dv'd\mu - \int_\Omega \sigma(\mu,v)|\varphi(\mu,v)|\,d\mu\,dv \\
&= \int_\Omega \sigma(\mu,v')|\varphi(\mu,v')|\,d\mu\,dv' - \int_\Omega \sigma(\mu,v)|\varphi(\mu,v)|\,d\mu\,dv
\end{aligned}$$

and therefore

$$\langle \operatorname{sgn}\varphi, (S+R)\varphi\rangle \le 0.$$

The proof is complete. □

Let us define the perturbed operators L_K and T_K as follows

$$\begin{gathered}
L_K := A_K + S, \\
D(L_K) = D(A_K)
\end{gathered} \tag{4.1}$$

and

$$\begin{gathered}
T_K := L_K + R = A_K + S + R, \\
D(T_K) = D(A_K)
\end{gathered} \tag{4.2}$$

for which we have the following generation results.

Lemma 4.2. *Assume* (H1) *and let K be a bounded operator with $\|K\|_{\mathcal{L}(Y_1)} < 1$. Then*

(1) *The operator defined by* (4.1) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(V_K(t))_{t\ge0}$ satisfying*
$$\|V_K(t)\varphi\|_1 \le e^{-t\underline{\sigma}}\|\varphi\|_1, \quad t \ge 0, \tag{4.3}$$
for all $\varphi \in L^1(\Omega)$.

(2) *The operator defined by* (4.2) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(W_K(t))_{t\ge0}$ satisfying*
$$\|W_K(t)\varphi\|_1 \le \|\varphi\|_1, \quad t \ge 0, \tag{4.4}$$
for all $\varphi \in L^1(\Omega)$.

Proof. (1). First, $L_K = A_K + S$ is a bounded linear perturbation of the generator A_K and therefore, Lemma 1.3 implies that L_K is a generator of a strongly continuous semigroup which we denote as $(V_K(t))_{t\geq 0}$. Next, Trotter's formula (1.5) implies
$$V_K(t)\varphi = \lim_{t\to\infty}\Big[e^{-\sigma t/n}U_K\big(\frac{t}{n}\big)\Big]^n\varphi, \quad t\geq 0, \tag{4.5}$$
for all $\varphi \in L^1(\Omega)$. By (2.11), we obtain
$$\|V_K(t)\varphi\|_1 \leq \lim_{t\to\infty}\Big[e^{-\underline{\sigma}t/n}\cdot 1\Big]^n\|\varphi\|_1 \leq e^{-t\underline{\sigma}}\|\varphi\|_1 \quad t\geq 0,$$
and therefore (4.3) holds.

(2) By the third point of Lemma 2.3, the operator A_K generates, on $L^1(\Omega)$, a strongly continuous semigroup of contractions. Furthermore, Lemma 4.1 implies that $S+R$ is a bounded and dissipative operator. As we have, $D(A_K)\subset L^1(\Omega) = D(S+R)$ and
$$\|(S+R)\varphi\|_1 \leq \|S+R\|\|\varphi\|_1 = 0.\|A_K\varphi\|_1 + \|S+R\|\|\varphi\|_1$$
for all $\varphi\in D(A_K)$ then, all conditions of Lemma 1.4 are clearly satisfied. The proof is complete. $\square$

Lemma 4.3. *Assume* (H1) *and let K be a compact operator with $\|K\|_{\mathcal{L}(Y_1)}\geq 1$.*

(1) *The operator defined by* (4.1) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(V_K(t))_{t\geq 0}$ satisfying*
$$\|V_K(t)\varphi\|_1 \leq e^{\frac{\lambda_0}{a}}e^{t(\lambda_0-\underline{\sigma})}\|\varphi\|_1 \quad t\geq 0, \tag{4.6}$$
for all $\varphi\in L^1(\Omega)$.

(2) *The operator defined by* (4.2) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(W_K(t))_{t\geq 0}$.*

Proof. (1) Following the proof of Lemma 4.2, it suffices to show (4.6). So, applying the norm (2.20) to Trotter's formula (4.5), we obtain
$$|||V_K(t)\varphi|||_1 \leq \lim_{t\to\infty}\Big[e^{-\underline{\sigma}t/n}e^{\frac{t}{n}\lambda_0}\Big]^n|||\varphi|||_1 \leq e^{t(\lambda_0-\underline{\sigma})}|||\varphi|||_1, \quad t\geq 0$$
for all $\varphi\in L^1(\Omega)$, where we have used (2.25). Now, (2.21) completes this part of the proof.

(2) Clearly, $T_K = L_K + R$ is a bounded linear perturbation of the generator L_K and therefore, Lemma 1.3 implies that T_K is a generator too. The proof is now complete. $\square$

We can summarize Lemmas 4.2 and 4.3 as follows.

Theorem 4.4. *Suppose that* (H1) *holds and let K be an admissible operator whose abscissa is ω_0.*

(1) *The operator defined by* (4.1) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(V_K(t))_{t\geq 0}$ satisfying*
$$\|V_K(t)\varphi\|_1 \leq e^{\frac{\omega_0}{a}}e^{t(\omega_0-\underline{\sigma})}\|\varphi\|_1 \quad t\geq 0, \tag{4.7}$$
for all $\varphi\in L^1(\Omega)$.

(2) *The operator defined by* (4.2) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(W_K(t))_{t\geq 0}$.*

Let us finish this work with the following Remarks

Remark 4.5. Inequality $a > 0$ has been used in many places in this work. So the open question is: What happens when $a = 0$?

Remark 4.6. According to Theorem 4.4, we can say that the model (1.1)-(1.4) is well-posed. However, the case corresponding to $\|K\|_{\mathcal{L}(Y_1)} < 1$ in Lemma 4.2 is biologically uninteresting because the cell density is decreasing. Indeed, for all t and s with $t > s$ we have

$$\|W_K(t)\varphi\|_1 = \|W_K(t-s)W_K(s)\varphi\|_1 \leq e^{-(t-s)\underline{\sigma}}\|W_K(s)\varphi\|_1 \leq \|W_K(s)\varphi\|_1$$

for all initial data $\varphi \in L^1(\Omega)$. However, the case corresponding to $\|K\|_{\mathcal{L}(Y_1)} > 1$ in Lemma 4.3 means that the cell density is increasing during each mitotic. This corresponds to the most observed and biologically interesting case for which we ask the following natural question: What happens when the cell density is increasing? The answer is given in [4].

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