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SOLUTIONS OF FRACTIONAL DIFFUSION PROBLEMS

RABHA W. IBRAHIM

ABSTRACT. Using the concept of majorant functions, we prove the existence and uniqueness of holomorphic solutions to nonlinear fractional diffusion problems. The analytic continuation of these solutions is studied and the singularity for two cases are posed.

1. Introduction

Fractional calculus and its applications (that is the theory of derivatives and integrals of any arbitrary real or complex order) has importance in several widely diverse areas of mathematical, physical and engineering sciences. It generalized the ideas of integer order differentiation and n-fold integration. Fractional derivatives introduce an excellent instrument for the description of general properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. Also the advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids and rocks, and in many other fields.

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modelling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [4], Erdélyi-Kober operators [6], Weyl-Riesz operators [18], Caputo operators [2] and Grünwald-Letnikov operators [20], have appeared during the past three decades. The existence of positive solution and multi-positive solutions for nonlinear fractional differential equation are established and studied [23]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain are suggested and posed in [7, 8].

The mathematical study of fractional diffusion equations began with the work of Kochubei [11, 12]. Later this study followed by the work of Metzler and Klafter [16] and Zaslavsky [22]. Recently, Mainardi et all obtained the time fractional diffusion equation from the standard diffusion equation [14]. Our aim in this paper is to

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consider the existence and uniqueness of nonlinear diffusion problems of fractional order in the complex domain by employing the concept of the majorant functions. The problems are taken in sense of Riemann-Liouville operators. Also, the analytic continuation of solutions are studied. Finally, the singularity for two cases are posed. In the fractional diffusion problems, we replace the first order time derivative by a fractional derivative. Fractional diffusion problems are useful in physics [5].

2. Preliminaries

One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators (see [9, 10, 17, 19, 20, 21]).

Definition 2.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau.$$

When $a = 0$, we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t)$, where $(*)$ denotes the convolution product (see [19]), $\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$ and $\phi_\alpha(t) = 0$, $t \le 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. The fractional (arbitrary) order derivative of the function f of order $0 \le \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Definition 2.3. The majorant relations described as: if $a(x) = \sum a_i x^i$ and $A(x) = \sum A_i x^i$, $A_i \ge 0$, $\forall\, i$, then we say that $a(x) \ll A(x)$ if and only if $|a_i| \le A_i$ for each i. Likewise, if $g(t,x) = \sum g_{ik}(t-\varepsilon)^i x^k$ and $G(t,x) = \sum G_{ik}(t-\varepsilon)^i x^k$, then we say that $g(t,x) \ll_\varepsilon G(t,x)$ if and only if $|g_{ik}| \le G_{ik}$ for all i and k.

Now we define the following family of majorant functions: for each $i \in \mathbb{N}$, we set

$$\Phi^{(i)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^{i+2}}, \quad (|z| < 1). \tag{2.1}$$

Note that for each $i \in \mathbb{N}$, the family $\Phi^{(i)}$ converges for all $|z| < 1$. Moreover, this family of functions enjoys some interesting majorant relations, as is stated in the following proposition.

Proposition 2.4. The following relations hold.

(i) $\Phi^{(0)}(z)\Phi^{(0)}(z) \ll \Phi^{(0)}(z)$;
(ii) $\Phi^{(0)}(z) \gg \Phi^{(1)}(z) \gg \Phi^{(2)}(z) \gg \dots$;
(iii) $\frac{1}{2^{i+2}}\Phi^{(i-1)}(z) \ll \frac{d}{dz}\Phi^{(i)}(z) \ll \Phi^{(i-1)}(z)$,
$\frac{1}{2^{i+2}}\Phi^{(i-2)}(z) \ll \frac{d^2}{dz^2}\Phi^{(i)}(z) \ll \Phi^{(i-2)}(z), \dots$;
(iv) $\Phi^{(i)}(z)\Phi^{(i)}(z) \ll \Phi^{(i)}(z)$;
(v) $\frac{1}{1-\varepsilon z}\Phi^{(i)}(z) \ll C_{i,\varepsilon}\Phi^{(i)}(z)$, $(0 < \varepsilon < 1,\ C_{i,\varepsilon} > 0)$;
(vi) $\frac{\Phi^{(i-2)}(z)}{(2\mu)^{3(i+2)}} \ll D_a^\alpha \Phi^{(i)}(z)$,

for sufficient large $\mu \ge 1$.

Proof. The first two relations are verified using the definition of $\Phi^{(i)}(z)$. Since

$$\frac{n+1}{(n+2)^{i+2}} < \frac{n+2}{(n+2)^{i+2}} = \frac{1}{(n+2)^{i+1}} \le \frac{1}{(n+1)^{i+1}}$$

then we obtain (iii). Similarly for (iv). To prove (v), by arbitrary choice of ε we consider

$$\varepsilon^n \le \frac{C_{i,\varepsilon}}{(n+1)^{i+2}}$$

which implies

$$\frac{1}{1-\varepsilon z} = \sum_{n=0}^{\infty} \varepsilon^n z^n \ll C_{i,\varepsilon}\Phi^{(i)}(z).$$

But this is equivalent to saying that for all n

$$\frac{1}{1-\varepsilon z}\Phi^{(i)}(z) \ll C_{i,\varepsilon}\Phi^{(i)}(z).$$

Finally, by using the approximation

$$D_a^\alpha \Phi^{(i)}(t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)(n+1)^{i+2}} t^{n-\alpha}, \quad (0 < t < 1)$$

we obtain the desired relation (vi) for sufficient large $\mu \ge 1$. $\square$

Similarly we can verify the following property.

Proposition 2.5. *If $f(z)$ is holomorphic in a neighborhood of $|z| \le r_0$, then $f(z)$ is majorized by*

$$f(z) \ll \frac{M}{1-\frac{z}{r_0}} \ll \frac{M}{1-\frac{\varepsilon z}{r}} \times \Phi^{(i)}(\frac{z}{r}) \ll MC_{i,\varepsilon}\Phi^{(i)}(\frac{z}{r}),$$

for any $0 < r < \varepsilon r_0$.

3. Fractional Diffusion Problems

Let $F(t,z,u,v)$, $t \in J = [a,A]$ be a function which is holomorphic in a neighborhood of the point $(a,b,c,d) \in J \times \mathbb{C}^3$, and let $\varphi(z)$ be a function which is holomorphic in a neighborhood of $z = b$ and satisfies $\varphi(b) = c$ and $\frac{\partial^2\varphi}{\partial z^2}(b) = d$. Consider the initial value problem:

$$\begin{gathered} \frac{\partial^\alpha u(t,z)}{\partial t^\alpha} = F(t,z,u,\frac{\partial^2 u}{\partial z^2}), \\ u(a,z) = \varphi(z), \quad \text{in a neighborhood of } z = b. \end{gathered} \tag{3.1}$$

For $0 < \alpha < 1$, problem (3.1) is called sub-diffusion problem [15] and for $1 < \alpha < 2$, it is called intermediate processes (see [3, 13]).

Then we have the following unique solvability result.

Theorem 3.1. *Initial value problem* (3.1) *has one and only one solution $u(t,z)$ which is holomorphic in a neighborhood of $(a,b) \in J \times \mathbb{C}$.*

Proof. Translating the setting from the point (a,b) into the origin $(0,0)$. Also we perform a change of variable by setting $w(t,z) = u(t,z) - \varphi(z)$, where $w(t,z)$ is the new unknown function. Then the initial value problem (3.1) is equivalent to the problem

$$\begin{gathered}\frac{\partial^\alpha w(t,z)}{\partial t^\alpha}=G(t,z,w,\frac{\partial^2 w}{\partial z^2}),\\ w(0,0)=0,\quad \text{in a neighborhood of } z=0.\end{gathered}\tag{3.2}$$

Here, the function $G(t,z,w,\frac{\partial^2 w}{\partial z^2})$ is holomorphic in a neighborhood of the origin $\in I\times\mathbb{C}^3$, $t\in I=[0,1]$ and $G(0,z,0,0)\equiv 0$ near $z=0$. Thus it is sufficient to consider the reduced initial value problem (3.2). Equation 3.2 has a unique solution takes the form

$$w(t,z)=\sum_{k=0}^{\infty}w_k(z)t^k,\quad (t\in I).$$

The uniqueness comes from the idea given in [1]; Solutions to fractional Cauchy problems are obtained by subordinating the solution to the original Cauchy problem.

We proceed to prove that $w(t,z)$ converges. Let $r_0>0$ and $\rho>0$ be small enough and suppose that the function $G(t,z,w,v)$ is holomorphic in a neighborhood of the set $\{(t,z,w,v)\in I\times\mathbb{C}^3; t<\tau\le 1, |z|\le r_0, |w|\le\rho \text{ and } |v|\le\rho\}$. Suppose further that G is bounded by M in this domain. Since G is holomorphic, we may expand it into

$$G(t,z,w,v)=\sum_{p,q,s}a_{p,q,s}(z)t^p w^q v^s,\quad \Big(t\in I,\ (w,v)\in(\mathbb{C}\times\mathbb{C})\Big).$$

By Cauchy's inequality and the fact that the coefficient $a_{p,q,s}(z)$ is holomorphic in a neighborhood of $\{z\in\mathbb{C}; 0<|z|\le r_0\}$, we have

$$a_{p,q,s}(z)\ll\frac{M}{\tau^p\rho^{q+s}}\frac{1}{1-(\frac{z}{r_0})^2}.\tag{3.3}$$

Now the problem returns to find a function $g(t,z)$ satisfying the majorant relations

$$\begin{gathered}\frac{\partial^\alpha g(t,z)}{\partial t^\alpha}\gg\sum_{p,q,s}\frac{M}{\tau^p\rho^{q+s}}\frac{1}{1-(\frac{z}{r_0})^2}t^p g^q(\frac{\partial^2 g}{\partial z^2})^s,\quad t\in I\\ g(0,0)\gg 0,\end{gathered}\tag{3.4}$$

then the function $g(t,z)$ majorizes the formal solution $w(t,z)$. Assume $0<r<r_0$ and define

$$g(t,z)=L\Phi^{(2)}\Big(t+(\frac{z}{r})^2\Big),\quad (L>0).\tag{3.5}$$

By taking the fractional derivative for both sides of (3.5) with respect to t we obtain

$$\frac{\partial^\alpha g(t,z)}{\partial t^\alpha}=L\frac{\partial^\alpha\Phi^{(2)}\big(t+(\frac{z}{r})^2\big)}{\partial t},\quad (L>0).\tag{3.6}$$

Then by Proposition 2.4 (vi) we have

$$\frac{\partial^\alpha g(t,z)}{\partial t^\alpha}\gg\frac{L}{C_\mu}\Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big),\tag{3.7}$$

where $C_\mu := (2\mu)^{12}$. For a constant $K_0 > 0$ again in view of Proposition 2.4 (ii) and (iii) we realize that

$$
\begin{aligned}
&\sum_{p,q,s} \frac{M}{\rho^{q+s}} \frac{1}{1-(\frac{z}{r_0})^2} \frac{1}{1-\frac{t}{\tau}} g^q (\frac{\partial^2 g}{\partial z^2})^s \\
&\ll \sum_{p,q,s} \frac{M}{\rho^{q+s}} \frac{1}{1-(\frac{z}{r_0})^2 - \frac{t}{\tau}} \{L\Phi^{(2)}\Big(t+(\frac{z}{r})^2\Big)\}^q \{\frac{2L}{r^2}\Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big)\}^s \\
&\ll \sum_{p,q,s} \frac{M}{\rho^{q+s}} \frac{1}{1-(\frac{z}{r_0})^2 - \frac{t}{\tau}} \{L\Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big)\}^q \{\frac{2L}{r^2}\Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big)\}^s \\
&\ll \frac{MK_0}{1-L/\rho-2L/\rho r^2} \Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big)
\end{aligned} \tag{3.8}
$$

if $\frac{L}{\rho} + \frac{2L}{\rho r^2} < 1$. Comparing (3.7) and (3.8), if the relation

$$
\frac{L}{C_\mu} \geq \frac{MK_0}{1-L/\rho-2L/\rho r^2} \tag{3.9}
$$

holds then the majorant relations in (3.4) will be satisfied by $g(t,z)$ defined in (3.5). Note that relation (3.9) holds by choosing a sufficiently small L, the condition $\frac{L}{\rho} + \frac{2L}{\rho r^2} < 1$ is satisfied. Hence $g(t,z)$ in (3.5) majorizes the formal solution $w(t,z)$. This now implies that $w(t,z)$ converges in a domain containing $\{(t,z) \in I \times \mathbb{C}; |t + (\frac{z}{r})^2| < 1\}$. $\square$

4. Analytic Continuation of solution

Let Ω be a neighborhood of the origin $(0,0)$. Let $F(t,z,u,v)$, $t \in I$, be a holomorphic function in $\Omega \times \mathbb{C}_u \times \mathbb{C}_v$ and consider the nonlinear fractional partial differential equation

$$
\frac{\partial^\alpha u}{\partial t} = F(t,z,u,\frac{\partial^2 u}{\partial z^2}). \tag{4.1}
$$

Then we may expand it into the convergent series

$$
F(t,z,u,v) = \sum_{j,p} a_{j,p}(t,z) u^j v^p. \tag{4.2}
$$

Let $S_0 = \{(j,p) \in \mathbb{N}^2; a_{j,p}(t,z) \neq 0\}$ and $S = \{(j,p) \in S_0; j+p \geq 2\}$. Note that F is linear if and only if $S = \emptyset$; it is nonlinear otherwise. Assume henceforth that F is nonlinear, that is S is nonempty. In the following, we will write the coefficients as

$$
a_{j,p}(t,z) = t^{k_{j,p}} b_{j,p}(t,z), \tag{4.3}
$$

where $k_{j,p}$ is a nonnegative integer and $b_{j,p}(0,z) = 0$. Using (4.1) may now be written as

$$
\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{j,p} t^{k_{j,p}} b_{j,p}(t,z) u^j (\frac{\partial^2 u}{\partial z^2})^p. \tag{4.4}
$$

For $\kappa \in \mathbb{R}$ we define the quantity

$$
\delta(\kappa) := \inf_{j,p \in S} \Big(k_{(j,p)} + 1 + \kappa(j+p-1)\Big). \tag{4.5}
$$

Note that when $\kappa = 0$, then $\delta(\kappa) \geq 1$. Moreover, if

$$\kappa > \sup_{(j,p)\in S} \frac{-(k_{(j,p)}+1)}{j+p-1},$$

then $\delta(\kappa)$ is positive.

In this section, our aim is to show that any solution $u(t,z) = O(t^\kappa)$ of the problem (4.4) which is holomorphic in Ω, is analytically continued up to some neighborhood of the origin.

Theorem 4.1. *Suppose $u(t,z)$ is a solution of (4.4) which is holomorphic in Ω. If for some $\kappa \in \mathbb{R}$ satisfying $\delta(\kappa) > 0$, we have*

$$\sup_{z\in\mathbb{C}} |u(t,z)| = O(t^\kappa), \quad (t \to 0),$$

then the solution $u(t,z)$ can be extended analytically as a holomorphic solution of (4.4) up to a neighborhood of the origin.

Proof. Assume that $u(t,z)$ is a solution for the problem (4.4) which is holomorphic in Ω. Furthermore, we suppose that expansion (4.2) is valid in the domain D where

$$D := \{(t,z,u,v) : t \leq 2\tau,\ |z| \leq 2r,\ |u| \leq \rho, |v| \leq \rho\},$$

such that $\tau < 1$, $2r < 1$ and ρ is positive number. Let M be a bound of F in D.

Now we consider the following initial value problem in $w(t,z) := \sum_{k=0}^\infty w_k(z)(t-\varepsilon)^k$

$$\begin{gathered} \frac{\partial^\alpha w(t,z)}{\partial t^\alpha} = \sum_{j,p} t^{k_{j,p}} b_{j,p}(t,z) w^j (\frac{\partial^2 w}{\partial z^2})^p, \quad t \in I \\ w(\varepsilon,z) = u(\varepsilon,z). \end{gathered} \tag{4.6}$$

Our aim is to show that the formal solution $w(t,z)$ converges in some domain containing the origin. This then poses that $u(t,z)$ is analytically continued by $w(t,z)$ up to some neighborhood of the origin. First, since $u(t,z) = O(t^\kappa)$ as $t \to 0$, there exists a constant N such that $|u(\varepsilon,z)| \leq N\varepsilon^\kappa$ uniformly in z. Hence in view of Proposition 2.5, for some constant C_1, we have

$$u(\varepsilon,z) \ll N\varepsilon^\kappa C_1 \Phi^{(2)}(\frac{t-\varepsilon}{c\tau} + (\frac{z}{r})^2). \tag{4.7}$$

Assume that $\kappa < 1$ (without lose generality). To construct an inequality to be satisfied by the majorant function, we will first majorize the expression $t^{k_{j,p}} b_{j,p}(t,z)$ by using $\Phi^{(0)}(z)$. Let

$$Z := \frac{t-\varepsilon}{c\tau} + (\frac{z}{r})^2$$

then t is majorized by

$$t = \varepsilon + (t-\varepsilon) \ll_\varepsilon \Big(\varepsilon + 4c\tau\Big)\Big(1 + \frac{t-\varepsilon}{4c\tau}\Big) \ll_\varepsilon \Big(\varepsilon + 4c\tau\Big)\Phi^{(0)}(Z). \tag{4.8}$$

Now, we may expand the function $b_{j,p}(t,z)$ as follows

$$b_{j,p}(t,z) = \sum_{m=0}^\infty b_{j,p}^{(m)}(z) t^m,$$

where each $b_{j,p}^{(m)}$ is holomorphic in a neighborhood of $\{|z| \leq 2r\}$ and satisfies

$$|b_{j,p}^{(m)}(z)| \leq \frac{M}{\rho^{j+p}(2\tau)^{m+k_{j,p}}}.$$

This estimate implies

$$b_{j,p}^{(m)}(z) \ll \frac{MC_1\Phi^{(0)}(Z)}{\rho^{j+p}(2\tau)^{m+k_{j,p}}} \tag{4.9}$$

where C_1 is the same constant as in (4.7). Combining relations (4.8) and (4.9) and by using Proposition 2.4 (i), we obtain

$$\begin{aligned} t^{k_{j,p}} b_{j,p}(t,z) &\ll_\varepsilon \sum_{m=0}^{\infty} \Big[(\varepsilon + 4c\tau)\Phi^{(0)}(Z)\Big]^{m+k_{j,p}} \Big[\frac{MC_1\Phi^{(0)}(Z)}{\rho^{j+p}(2\tau)^{m+k_{j,p}}}\Big] \\ &\ll_\varepsilon \frac{C_1 M}{\rho^{j+p}} \Phi^{(0)}(Z) \sum_{m=0}^{\infty} (4c)^{m+k_{j,p}}, \end{aligned} \tag{4.10}$$

by choosing $\varepsilon = \frac{c\tau}{2}$ and $0 < c \le 1$ and fixing r so that $4cr < 1$, $0 < c \le 1$, we finally have the relation

$$t^{k_{j,p}} b_{j,p}(t,z) \ll_\varepsilon \frac{2C_1 M}{\rho^{j+p}} \Phi^{(0)}(Z).$$

Thus, any function $W(t,z)$ found to satisfy the majorant relations

$$\begin{gathered} \frac{\partial^\alpha W}{\partial t^\alpha} \gg_\varepsilon \sum_{j,p} \frac{2C_1 M}{\rho^{j+p}} \Phi^{(0)}(Z) W^j (\frac{\partial^2 W}{\partial Z^2})^p, \quad t \in I \\ W(\varepsilon, z) \gg_\varepsilon N\varepsilon^\kappa C_1 \Phi^{(2)}(Z) \end{gathered} \tag{4.11}$$

is one majorant function for the formal solution $w(t,z)$. In the similar manner of the proof of Theorem 3.1 and by choosing suitable values for $\rho > 0$ and $c > 0$, and setting $\varepsilon = \frac{c\tau}{2}$, the function

$$W(t,z) = \varepsilon^\kappa N C_1 \Phi^{(2)}(Z)$$

satisfies the majorant relations given in (4.11). Hence $W(t,z)$ is holomorphic in a domain containing the origin; consequently must be true for $w(t,z)$. □

5. Singularity

Again, we state our assumptions in studying the equation

$$t\frac{\partial^\alpha w(t,z)}{\partial t^\alpha} = G(t,z,w,\frac{\partial^2 w}{\partial z^2}). \tag{5.1}$$

Let G be a function holomorphic in some neighborhood of the origin in $I \times \mathbb{C}^3$ and suppose $G(0,z,0,0)$ is identically zero near $z = 0$. In this section, our aim is to seek a holomorphic solution $w(t,z)$ of (5.1) for special case, which satisfies $w(0,z) \equiv 0$. First we write G as

$$G(t,z,w,\frac{\partial^2 w}{\partial z^2}) = a(z)t + b(z)w + c(z)\frac{\partial^2 w}{\partial z^2} + R(t,z,w,\frac{\partial^2 w}{\partial z^2}), \tag{5.2}$$

where R is the remainder of the Taylor expansion of G. Then we have the following result.

Theorem 5.1. *Assume that the coefficient $c(z) \equiv 0$ in (5.2). If $b(z)$ does not take values in $\mathbb{N} \cup \{0\}$ at the origin, then (5.1) has a unique holomorphic solution satisfying $w(0,z) \equiv 0$.*

Proof. Consider a formal solution of the form $w(t,z)=\sum_{k=0}^{\infty} w_k(z)t^k$. Then we expand $G(t,z,w,\frac{\partial^2 w}{\partial z^2})$ as follows

$$G(t,z,w,\frac{\partial^2 w}{\partial z^2}) = a(z)t + b(z)w + \sum_{p+q+j\geq 2} a_{p,q,j}(z)t^p w^q (\frac{\partial^2 w}{\partial z^2})^j.$$

Suppose that this expansion is convergent in a neighborhood of the set

$$\mathcal{S} := \{(t,z,w,\frac{\partial^2 w}{\partial z^2}) : t \leq \tau, |z| \leq r_0, |w| \leq \rho, |\frac{\partial^2 w}{\partial z^2}| \leq \rho\}$$

and that G is bounded in $\mathcal{S}$ by M. Since $b(z)$ does not take values in $\mathbb{N}\cup\{0\}$ at the origin then there exists a constant B such that

$$|\frac{k}{k-b(z)}| \leq B, \quad \textit{for all } k \in \mathbb{N}\cup\{0\} \textit{ and } |z| \leq r_0.$$

Then any function $\omega(t,z)$ satisfying the following relations is a majorant of the formal solution:

$$\begin{gathered} t\frac{\partial^\alpha \omega(t,z)}{\partial t^\alpha} \gg \frac{BM}{1-\frac{z}{r_0}}\frac{t}{\tau} + \sum_{p+q+j\geq 2} \frac{BM}{\tau^p \rho^{q+j}} \frac{t^p \omega^q}{1-\frac{z}{r_0}} (\frac{\partial^2 \omega}{\partial z^2})^j \\ \omega|_{t=0} = 0. \end{gathered} \tag{5.3}$$

Let $0 < r < r_0$. Assume that

$$\omega(t,z) = Lt\Phi^{(2)}\Big(t + (\frac{z}{r})^2\Big), \quad (L>0), \tag{5.4}$$

yields

$$\frac{\partial^\alpha \omega(t,z)}{\partial t^\alpha} = L\frac{\partial^\alpha t\Phi^{(2)}\Big(t+(\frac{z}{r})^2\Big)}{\partial t^\alpha}, \quad (L>0).$$

Applying the Leibniz rule for fractional differentiation [20, Eq. 2.202] and using the relation (vi), the left hand side of (5.3) becomes

$$t\frac{\partial^\alpha \omega(t,z)}{\partial t^\alpha} \gg \binom{\alpha}{0} Lt^2 \frac{\Phi^{(0)}(t)}{C_\mu} + \binom{\alpha}{1} Lt \frac{\Phi^{(0)}(t)}{C_\mu} \gg C_{\alpha,\mu} L \Phi^{(0)}(t)(t^2+t). \tag{5.5}$$

Meanwhile, the right hand side becomes

$$\begin{aligned} &\frac{BM}{1-(\frac{z}{r_0})^2}\frac{t}{\tau} + \sum_{p+q+j\geq 2} \frac{BM}{\tau^p\rho^{q+j}} \frac{t^p\omega^q}{1-(\frac{z}{r_0})^2}(\frac{\partial^2\omega}{\partial z^2})^j \\ &\ll \frac{BM}{1-(\frac{z}{r_0})^2}\frac{t}{\tau} \\ &\quad + \sum_{p+q+j\geq 2} \frac{BM}{1-(\frac{z}{r_0})^2}(\frac{t}{\tau})^p \Big[\frac{Lt\Phi^{(2)}\big(t+(\frac{z}{r})^2\big)}{\rho}\Big]^q \Big[\frac{2Lt}{r^2\rho}\frac{d^2\Phi^{(2)}\big(t+(\frac{z}{r})^2\big)}{dz^2}\Big]^j \\ &\ll \sum_{p=1}^{\infty} \frac{BM}{1-(\frac{z}{r_0})^2}(\frac{t}{\tau})^p + \sum_{(p+q+j\geq 2, q+j\geq 1)} \frac{BM}{1-(\frac{z}{r_0})^2}(\frac{t}{\tau})^p \Big[\frac{Lt}{\rho}\Big]^q \Big[\frac{2Lt}{r^2\rho}\Big]^j \Phi^{(0)}\big(t+(\frac{z}{r})^2\big). \end{aligned} \tag{5.6}$$

But

$$\sum_{p=1}^{\infty} \frac{BM}{1-(\frac{z}{r_0})^2}(\frac{t}{\tau})^p \ll \frac{BM}{1-(\frac{z}{r_0})^2-\frac{t}{\tau}}(\frac{t}{\tau})\Phi^{(1)}(t) \ll \frac{BMC_1}{\tau} t\Phi^{(0)}(t) \tag{5.7}$$

and

$$\begin{aligned}
&\sum_{(p+q+j\geq 2, q+j\geq 1)} \frac{BM}{1-(\frac{z}{r_0})^2}(\frac{t}{\tau})^p\Big[\frac{Lt}{\rho}\Big]^q\Big[\frac{2Lt}{r^2\rho}\Big]^j\Phi^{(0)}\big(t+(\frac{z}{r})^2\big)\\
&\ll \Big(\frac{t}{\tau}+\frac{Lt}{\rho}+\frac{2Lt}{r^2\rho}\Big)^2\frac{BM}{1-(\frac{z}{r_0})^2-\frac{t}{\tau}-\frac{Lt}{\rho}-\frac{2Lt}{r^2\rho}}\Phi^{(0)}(t)\\
&\ll \Big(\frac{t}{\tau}+\frac{Lt}{\rho}+\frac{2Lt}{r^2\rho}\Big)^2 C_2BM\Phi^{(0)}(t),
\end{aligned} \tag{5.8}$$

where

$$\frac{1}{\tau}+\frac{L}{\rho}+\frac{2L}{r^2\rho}\leq\frac{1}{\tau}.$$

Comparing the majorant relations (5.7) and (5.8) to the one in relation (5.6) that $\omega(t,z)$ satisfies (5.3) if we could force

$$L\geq\frac{BMC_1}{\tau C_{\alpha,\mu}}$$

and

$$L\geq\frac{\big(\frac{1}{\tau}+\frac{L}{\rho}+\frac{2L}{r^2\rho}\big)^2C_2BM}{C_{\alpha,\mu}}.$$

The last two conditions in are satisfied by choosing L large enough, fixing it, and then choosing a sufficiently small value for C_1, C_2 and $C_{\alpha,\mu}$. We thus have shown that the function $\omega(t,z)$ defined in (5.4) majorizes the formal solution $w(t,z)$. This implies that the formal solution converges in some neighborhood of the origin. $\square$

If $c(z)$ is not identically zero in (5.2), then we write

$$c(z)=z^p\widetilde{c}(z), \tag{5.9}$$

where p is a nonnegative integer and $\widetilde{c}(z)\neq 0$. We now state the following result.

Theorem 5.2. *Suppose $p=1$ in (5.9). If a positive constant ν exists such that*

$$|k-b(0)-\widetilde{c}(z)\ell|\geq\nu(k+\nu+1),\quad (k,\ell)\in\mathbb{N}\cup\{0\}\times\mathbb{N}, \tag{5.10}$$

then (5.1) has one and only one holomorphic solution satisfying $w(0,z)\equiv 0$.

Proof. Equation (5.1) may be written as

$$\begin{aligned}
&t\frac{\partial^\alpha w}{\partial t^\alpha}-b(0)w-\widetilde{c}(0)(z\frac{\partial^2 w}{\partial z^2})\\
&=z\beta(z)w+z\gamma(z)(z\frac{\partial^2 w}{\partial z^2})+a(z)t+\sum_{p+q+j\geq 2}a_{p,q,j}(z)t^pw^q(\frac{\partial^2 w}{\partial z^2})^j,
\end{aligned} \tag{5.11}$$

where

$$b(z)=b(0)+z\beta(z)\quad\text{and}\quad\widetilde{c}(z)=\widetilde{c}(0)+z\gamma(z).$$

Suppose that this expansion is convergent in a neighborhood of the set

$$\mathcal{S}:=\{(t,z,w,\frac{\partial^2 w}{\partial z^2}):t\leq\tau, 0<r\leq|z|\leq r_0, |w|\leq\rho, |\frac{\partial^2 w}{\partial z^2}|\leq\rho\}$$

and that G is bounded in $\mathcal{S}$ by M. Moreover, assume that $a(z),\beta(z)$ and $\gamma(z)$ are bounded by A, B and C respectively. Consider a formal solution of the form

$w(t,z)=\sum_{k=0}^{\infty} w_k(z)t^k$. Now, it can be shown that this formal solution is majorized by any function $W(t,z)$ satisfying these relations:

$$\begin{gathered}\nu\Big(t\frac{\partial^\alpha}{\partial t^\alpha}+z\frac{\partial^2}{\partial z^2}+1\Big)W(t,z)\\ \gg \frac{BzW+At}{1-(\frac{z}{r_0})^2}+\frac{Cz}{1-(\frac{z}{r_0})^2}(z\frac{\partial^2 W}{\partial z^2})+\sum_{p+q+j\ge 2}\frac{M}{\tau^p\rho^{q+j}}\frac{t^pW^q}{1-(\frac{z}{r_0})^2}(\frac{\partial^2 W}{\partial z^2})^j \quad (5.12)\\ W|_{t=0}=0\end{gathered}$$

such that W can be found in the form

$$W(t,z)=Lt\Phi^{(2)}\Big(t+(\frac{z}{r})^2\Big),\quad (L>0), \quad (5.13)$$

The summation in (5.12) is estimated as in Theorem 5.1. Our aim is to estimate the terms

$$\frac{BzW+At}{1-(\frac{z}{r_0})^2}+\frac{Cz}{1-(\frac{z}{r_0})^2}(z\frac{\partial^2 W}{\partial z^2})\quad\text{and}\quad \nu z(\frac{\partial^2 W}{\partial z^2})$$

in (5.12). We thus have, for some constant $K>0$,

$$\begin{aligned}&\frac{BzW+At}{1-(\frac{z}{r_0})^2}+\frac{Cz}{1-(\frac{z}{r_0})^2}(z\frac{\partial^2 W}{\partial z^2})\\ &\ll BLKzt\Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big)+CLKzt\frac{2z}{r^2}\Phi^{(0)}_{\mu}\Big(t+(\frac{z}{r})^2\Big) \quad (5.14)\\ &\ll (BLK+2CLK)zt\Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big).\end{aligned}$$

On the other hand, the left-hand side is estimated by using Proposition 2.4, (iii) as follows:

$$\nu z(\frac{\partial^2 W}{\partial z^2})\gg\frac{2\nu Ltz}{r}\frac{d^2\Phi^{(2)}\Big(t+(\frac{z}{r})^2\Big)}{dz^2}\gg\frac{\nu Ltz}{8r}\Phi^{(0)}\Big(t+(\frac{z}{r})^2\Big). \quad (5.15)$$

Therefore, in order for $W(t,z)$ to satisfy the majorant relations in (5.12) we must impose the condition

$$K(B+2C)\le\frac{\nu}{8r}.$$

This completes the proof. $\square$

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References

[1] B. Baeumer and M.M. Meerschaert; Stochastic solutions for fractional Cauchy problems. Frac. Calc. Appl. Anal. 4, (2001) 481-500.

[2] B. Bonilla , M. Rivero, J. J. Trujillo; On systems of linear fractional differential equations with constant coefficients, App. Math. Comp. 187 (2007) 68-78.

[3] A. V. Chechkin, J. Klafter and I.M. Sokolov; Fractional Fokker-Plank equation for ultraslow kinetics. Europhys. Lett. 63(3), (2003) 326-332.

[4] K. Diethelm, N. Ford; Analysis of fractional differential equations, J. Math. Anal. Appl., 265 (2002) 229-248.

[5] R. Hilfer; Fractional diffusion based on Riemann-Liouville fractional derivatives, J. Phys. Chem. Bio. 104(2000) 3914-3917.

[6] R. W. Ibrahim , S. Momani; On the existence and uniqueness of solutions of a class of fractional differential equations, J. Math. Anal. Appl. 334 (2007) 1-10.

[7] R. W. Ibrahim, M. Darus; Subordination and superordination for analytic functions involving fractional integral operator, Complex Variables and Elliptic Equations, 53 (2008) 1021-1031.
[8] R. W. Ibrahim, M. Darus; Subordination and superordination for univalent solutions for fractional differential equations, J. Math. Anal. Appl. 345 (2008) 871-879.
[9] A. A. Kilbas, H. M. Srivastava and J.J. Trujillo; Theory and applications of fractional differential equations. North-Holland, Mathematics Studies, Elsevier, 2006.
[10] V. Kiryakova; Generalized Fractional Calculus and Applications, Pitman Res. Notes Math. Ser., Vol. 301, Longman/Wiley, New York, 1994.
[11] A. N. Kochubei; The Cauchy problem for evolution equations of fractional order. Diff. Eq. 25, (1989) 967-974.
[12] A. N. Kochubei; Diffusion of fractional order, Differential Equations 26 (1990) 485-492.
[13] A. N. Kochubei; Distributed order calculus and equations of ultraslow diffusion. J. Math. Anal. Appl. 340 (2008) 252-281.
[14] F. Mainardi, G. Pagnini and R. Gorenflo; Some aspects of fractional diffusion equations of single and distributed order, App. Math. Compu., Vol. 187, No 1, (2007) 295-305.
[15] F. Mainardi, A. Mura, G. Pagnini and R. Gorenflo; Sub-diffusion equations of fractional order and their fundamental solutions, Invited lecture by F. Mainardi at the 373. WE-Heraeus-Seminar on Anomalous Transport: Experimental Results and Theoretical Challenges, Physikzentrum Bad-Honnef (Germany), 12-16 July 2006.
[16] R. Metzler and J. Klafter; The random walk's guide to anomalous diffusion: A fractional dynamics approach. Phys. Rep. 339, (2000) 1-77.
[17] K. S. Miller and B. Ross; An Introduction to The Fractional Calculus and Fractional Differential Equations, John-Wily and Sons, Inc., 1993.
[18] S. M. Momani, R. W. Ibrahim; On a fractional integral equation of periodic functions involving Weyl-Riesz operator in Banach algebras, J. Math. Anal. Appl. 339 (2008) 1210-1219.
[19] K. B. Oldham and J. Spanier; The Fractional Calculus, Math. in Science and Engineering, Acad. Press, New York/London, 1974.
[20] I. Podlubny; Fractional Differential Equations, Acad. Press, London, 1999.
[21] S. G. Samko, A. A. Kilbas, O. I. Marichev; Fractional Integrals and Derivatives (Theory and Applications), Gorden and Breach, New York, 1993.
[22] G. Zaslavsky; Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. Phys. D 76, (1994) 110-122.
[23] S. Zhang; The existence of a positive solution for a nonlinear fractional differential equation, J. Math. Anal. Appl. 252 (2000) 804-812.

Rabha W. Ibrahim
School of Mathematical Sciences, Faculty of Sciences and Technology, UKM, Malaysia
E-mail address: rabhaibrahim@yahoo.com