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MONOTONE ITERATIVE METHOD FOR SEMILINEAR IMPULSIVE EVOLUTION EQUATIONS OF MIXED TYPE IN BANACH SPACES

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Abstract. We use a monotone iterative method in the presence of lower and upper solutions to discuss the existence and uniqueness of mild solutions for the initial value problem

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = x_0, \end{gathered}$$

where $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a strongly continuous semigroup $T(t)(t \geq 0)$ in E. Under wide monotonicity conditions and the non-compactness measure condition of the nonlinearity f, we obtain the existence of extremal mild solutions and a unique mild solution between lower and upper solutions requiring only that $-A$ generate a strongly continuous semigroup.

1. Introduction

The theory of impulsive differential equations is a new and important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background; hence it has emerged as an important area of research in the previous decades, see for example [6]. Consequently, some basic results on impulsive differential equations have been obtained and applications to different areas have been considered by many authors; see [1, 4, 7, 10] and their references.

In this article, we use a monotone iterative method in the presence of lower and upper solutions to discuss the existence of mild solutions to the initial value problem (IVP) of first order semilinear impulsive integro-differential evolution equations of Volterra type in an ordered Banach space E

$$\begin{gathered} u'(t) + Au(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = x_0, \end{gathered} \tag{1.1}$$

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where $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a strongly continuous semigroup (C_0-semigroup, in short) $T(t)(t \geq 0)$ in E; $f \in C(J \times E \times E, E)$, $J = [0, a]$, $a > 0$ is a constant, $0 < t_1 < t_2 < \dots < t_m < a$; $I_k \in C(E, E)$ is an impulsive function, $k = 1, 2, \dots, m$; $x_0 \in E$; and

$$Tu(t) = \int_0^t K(t,s)u(s)ds \tag{1.2}$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \{(t,s) : 0 \leq s \leq t \leq a\}$; $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$; i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively. Let $PC(J, E) = \{u : J \to E, u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \dots, m\}$. Evidently, $PC(J, E)$ is a Banach space with the norm $\|u\|_{PC} = \sup_{t \in J} \|u(t)\|$. Denote E_1 by the norm $\|\cdot\|_1 = \|\cdot\| + \|A \cdot\|$. Let $J' = J \backslash \{t_1, t_2, \dots, t_m\}$. An abstract function $u \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ is called a solution of IVP (1.1) if $u(t)$ satisfies all the equalities in (1.1).

The monotone iterative technique in the presence of lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces. Recently, Du and Lakshmikantham [3], Sun and Zhao [12] investigated the existence of minimal and maximal solutions to initial value problem of ordinary differential equation without impulse by using the method of upper and lower solutions and the monotone iterative technique. Guo and Liu [4] developed the monotone iterative method for impulsive integro-differential equations, and built a monotone iterative method for impulsive ordinary integro-differential equation for the IVP in E

$$\begin{gathered} u'(t) = f(t, u(t), Tu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = x_0. \end{gathered} \tag{1.3}$$

They proved that if (1.3) has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and the nonlinear term f satisfies the monotonicity condition

$$\begin{gathered} f(t, x_2, y_2) - f(t, x_1, y_1) \geq -M(x_2 - x_1) - M_1(y_2 - y_1), \\ v_0(t) \leq x_1 \leq x_2 \leq w_0(t), \quad Tv_0(t) \leq y_1 \leq y_2 \leq Tw_0(t), \quad \forall t \in J. \end{gathered} \tag{1.4}$$

They also required that the nonlinear term f and the impulsive function I_k satisfy the noncompactness measure condition

$$\alpha(f(t, U, V)) \leq L_1 \alpha(U) + L_2 \alpha(V), \tag{1.5}$$

$$\alpha(I_k(D)) \leq M_k \alpha(D), \quad k = 1, 2, \dots, m, \tag{1.6}$$

where $U, V, D \subset E$ are arbitrarily bounded sets, L_1, L_2 and M_k are positive constants and satisfy

$$2a(M + L_1 + aK_0 L_2) + \sum_{k=1}^m M_k < 1, \tag{1.7}$$

where $K_0 = \max_{(t,s) \in \Delta} K(t,s)$, $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness in E. Then IVP(1.3) has minimal and maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively. Latter, Li and Liu [7] expanded the results in [4], they obtained the existence of the extremal solutions to the initial value problem for impulsive ordinary integro-differential equation (1.3), but did not require the noncompactness measure condition (1.6) for impulsive function I_k and the restriction condition (1.7).

On the other hand, some authors consider the initial value problem of evolution equations, see [1, 8, 9, 10, 13, 14] and the reference therein. But they all require the semigroup $T(t)(t \geq 0)$ generated by $-A$ be equicontinuous semigroup, this is a very strong assumption. In this paper, we will study the initial value problem of impulsive integro-differential evolution equation (1.1) not requiring the equicontinuity of the semigroup $T(t)(t \geq 0)$ generated by $-A$. We obtain the existence of extremal mild solutions and a unique mild solution between lower and upper solutions only requiring the semigroup $T(t)(t \geq 0)$ generated by $-A$ is a C_0-semigroup in E.

2. Preliminaries

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{x \in E : x \geq 0\}$ is normal with normal constant N. Let $C(J, E)$ denote the Banach space of all continuous E-value functions on interval J with the norm $\|u\|_C = \max_{t\in J} \|u(t)\|$. Evidently, $C(J, E)$ is also an ordered Banach space reduced by the convex cone $P' = \{u \in Y | u(t) \geq 0, t \in J\}$, and P' is also a normal cone. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [2]. For any $B \subset C(J, E)$ and $t \in J$, set $B(t) = \{u(t) : u \in B\} \subset E$. If B is bounded in $C(J, E)$, then $B(t)$ is bounded in E, and $\alpha(B(t)) \leq \alpha(B)$.

We first give the following lemmas to be used in proving our main results.

Lemma 2.1 ([5]). *Let $B = \{u_n\} \subset PC(J, E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\Big(\Big\{\int_J u_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int_J \alpha(B(t))dt.$$

Let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ in E. Then there exist constants $C > 0$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq Ce^{\delta t}, \quad t \geq 0.$$

Let $I = [t_0, T](t_0 \geq 0)$, $T > t_0$ be a constant. It is well-known [11, Chapter 4, Theorem 2.9] that for any $x_0 \in D(A)$ and $h \in C^1(I, E)$, the initial value problem of the linear evolution equation

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t \in I, \\ u(t_0) = x_0, \end{gathered} \tag{2.1}$$

has a unique classical solution $u \in C^1(I, E) \cap C(I, E_1)$ given by

$$u(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)h(s)ds, \quad t \in I. \tag{2.2}$$

If $x_0 \in E$ and $h \in C(I, E)$, the function u given by (2.2) belongs to $C(I, E)$, we call it a mild solution [11] of IVP(2.1).

To prove our main results, for any $h \in PC(J,E)$, we consider the initial value problem (IVP) of linear impulsive evolution equation in E

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1,2,\dots,m, \\ u(0) = x_0, \end{gathered} \tag{2.3}$$

where $y_k \in E$, $k = 1,2,\dots,m$, $x_0 \in E$.

Lemma 2.2. *Let $T(t)(t \geq 0)$ be a C_0-semigroup in E generated by $-A$, for any $h \in PC(J,E)$, $x_0 \in E$ and $y_k \in E$, $k = 1,2,\dots,m$, then the linear IVP (2.3) has a unique mild solution $u \in PC(J,E)$ given by*

$$u(t) = T(t)x_0 + \int_0^t T(t-s)h(s)ds + \sum_{0<t_k<t} T(t-t_k)y_k, \quad t \in J. \tag{2.4}$$

Proof. Let $y_0 = \theta$, $J_k = [t_{k-1}, t_k]$, $k = 1,2,\dots,m+1$, where $t_0 = 0$ and $t_{m+1} = a$. If $u \in PC(J,E)$ is a mild solution of IVP(2.3), then the restriction of u on J_k satisfies the initial value problem of linear evolution equation without impulse

$$\begin{gathered} u'(t) + Au(t) = h(t), \quad t_{k-1} < t \leq t_k, \\ u(t_{k-1}^+) = u(t_{k-1}) + y_{k-1}. \end{gathered}$$

Hence, on $(t_{k-1}, t_k], u(t)$ can be expressed by

$$u(t) = T(t - t_{k-1})(u(t_{k-1}) + y_{k-1}) + \int_{t_{k-1}}^t T(t-s)h(s)ds.$$

Iterating successively in the above equality with $u(t_j)$, $j = k-1, k-2, \dots, 1$, we see that u satisfies (2.4).

Inversely, we can verify directly that the function $u \in PC(J,E)$ defined by (2.4) is a mild solution of IVP(2.3). Hence IVP(2.3) has a unique mild solution $u \in PC(J,E)$ given by (2.4). □

Definition 2.3. If a function $v_0 \in PC(J,E) \cap C^1(J',E) \cap C(J',E_1)$ satisfies

$$\begin{gathered} v_0'(t) + Av_0(t) \leq f(t, v_0(t), Tv_0(t)) \quad t \in J', \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k)), \quad k = 1,2,\dots,m, \\ v_0(0) \leq x_0, \end{gathered} \tag{2.5}$$

we call it a lower solution of IVP(1.1); if all the inequalities in (2.5) are reversed, we call it an upper solution of IVP(1.1).

Definition 2.4. A C_0-semigroup $T(t)(t \geq 0)$ in E is called to be positive, if order inequality $T(t)x \geq \theta$ holds for each $x \geq \theta, x \in E$ and $t \geq 0$.

It is easy to see that for any $M \geq 0$, $-(A + MI)$ also generates a C_0-semigroup $S(t) = e^{-Mt}T(t)(t \geq 0)$ in E. And $S(t)(t \geq 0)$ is a positive C_0-semigroup if $T(t)(t \geq 0)$ is a positive C_0-semigroup (about the positive C_0-semigroup, see [8]).

Evidently, $PC(J,E)$ is also an ordered Banach space with the partial order $\leq$ induced by the positive cone $K_{PC} = \{u \in PC(J,E) : u(t) \geq 0, t \in J\}$. K_{PC} is also normal with the same normal constant N. For $v, w \in PC(J,E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in PC(J,E) : v \leq u \leq w\}$ in $PC(J,E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E : v(t) \leq u(t) \leq w(t), t \in J\}$ in E.

3. Main results

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator, the positive C_0-semigroup $T(t)(t \geq 0)$ generated by $-A$ is compact in $E, f \in C(J \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \dots, m$. Assume that IVP(1.1) has a lower solution $v_0 \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ with $v_0 \leq w_0$. Suppose also that the following conditions are satisfied:*

- (H1) *There exist a constant $M > 0$ such that*
$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -M(u_2 - u_1),$$
for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Tv_0(t) \leq v_1 \leq v_2 \leq Tw_0(t)$.
- (H2) *$I_k(u)$ is increasing on order interval $[v_0(t), w_0(t)]$ for $t \in J, k = 1, 2, \dots, m$.*

Then the IVP(1.1) has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0.

Proof. Let $\overline{M} = \sup_{t\in J} \|S(t)\|$, we define the mapping $Q : [v_0, w_0] \to PC(J, E)$ by
$$\begin{aligned} Qu(t) = S(t)x_0 &+ \int_0^t S(t-s)(f(s, u(s), Tu(s)) + Mu(s))ds \\ &+ \sum_{0<t_k<t} S(t-t_k)I_k(u(t_k)). \end{aligned} \tag{3.1}$$
Clearly, $Q : [v_0, w_0] \to PC(J, E)$ is continuous. By Lemma 2.2, the mild solution of IVP(1.1) is equivalent to the fixed point of the operator Q. Since $S(t)(t \geq 0)$ is a positive C_0-semigroup, combine this with the assumptions (H1) and (H2), Q is increasing in $[v_0, w_0]$.

We first show $v_0 \leq Qv_0$, $Qw_0 \leq w_0$. Let $h(t) = v_0'(t) + Av_0(t) + Mv_0(t)$, by (2.5), $h \in PC(J, E)$ and $h(t) \leq f(t, v_0(t), Tv_0(t)) + Mv_0(t), t \in J'$. By Lemma 2.2, we have
$$\begin{aligned} v_0(t) &= S(t)v_0(0) + \int_0^t S(t-s)h(s)ds + \sum_{0<t_k<t} S(t-t_k)\Delta v_0|_{t=t_k} \\ &\leq S(t)x_0 + \int_0^t S(t-s)(f(s, v_0(s), Tv_0(s)) + Mv_0(s))ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(v_0(t_k)) \\ &= Qv_0(t), \quad t \in J, \end{aligned}$$
namely, $v_0 \leq Qv_0$. Similarly, it can be show that $Qw_0 \leq w_0$. So $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.

Next, we show that $Q : [v_0, w_0] \to [v_0, w_0]$ is completely continuous. Let
$$\begin{aligned} (Wu)(t) &= \int_0^t S(t-s)(f(s, u(s), Tu(s)) + Mu(s))ds, \\ (Vu)(t) &= \sum_{0<t_k<t} S(t-t_k)I_k(u(t_k)), \quad u \in [v_0, w_0]. \end{aligned} \tag{3.2}$$

On the one hand, we prove that for any $0 < t \le a$, $Y(t) = \{(Wu)(t) : u \in [v_0, w_0]\}$ is precompact in E. For $0 < \epsilon < t$ and $u \in [v_0, w_0]$,

$$\begin{aligned}(W_\epsilon u)(t) &= \int_0^{t-\epsilon} S(t-s)(f(s,u(s),Tu(s)) + Mu(s))ds \\ &= S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)(f(s,u(s),Tu(s)) + Mu(s))ds.\end{aligned} \tag{3.3}$$

For any $u \in [v_0, w_0]$, by Assumption (H1), we have

$$\begin{aligned}f(t, v_0(t), Tv_0(t)) + Mv_0(t) &\le f(t, u(t), Tu(t)) + Mu(t) \\ &\le f(t, w_0(t), Tw_0(t)) + Mw_0(t).\end{aligned}$$

By the normality of the cone P, there exists $\overline{M_1} > 0$ such that

$$\|f(t, u(t), Tu(t)) + Mu(t)\| \le \overline{M_1}, \quad u \in [v_0, w_0].$$

By the compactness of $S(\epsilon)$, $Y_\epsilon(t) = \{(W_\epsilon u)(t) : u \in [v_0, w_0]\}$ is precompact in E. Since

$$\begin{aligned}\|(Wu)(t) - (W_\epsilon u)(t)\| &\le \int_{t-\epsilon}^{t} \|S(t-s)\| \cdot \|f(s,u(s),Tu(s)) + Mu(s)\|ds \\ &\le \overline{M}\ \overline{M_1}\epsilon,\end{aligned}$$

the set $Y(t)$ is totally bounded in E. Furthermore, $Y(t)$ is precompact in E.

On the other hand, for any $0 \le t_1 \le t_2 \le a$, we have

$$\begin{aligned}&\|(Wu)(t_2) - (Wu)(t_1)\| \\ &= \|\int_0^{t_1} (S(t_2-s) - S(t_1-s))(f(s,u(s),Tu(s)) + Mu(s))ds \\ &\quad + \int_{t_1}^{t_2} S(t_2-s)(f(s,u(s),Tu(s)) + Mu(s))ds\| \\ &\le \overline{M_1}\int_0^{t_1} \|S(t_2-s) - S(t_1-s)\|ds + \overline{M}\ \overline{M_1}(t_2-t_1) \\ &\le \overline{M_1}\int_0^{a} \|S(t_2-t_1+s) - S(s)\|ds + \overline{M}\ \overline{M_1}(t_2-t_1).\end{aligned} \tag{3.4}$$

The right side of (3.4) depends on $t_2 - t_1$, but is independen of u. As $T(\cdot)$ is compact, $S(\cdot)$ is also compact and therefore $S(t)$ is continuous in the uniform operator topology for $t > 0$. So, the right side of (3.4) tends to zero as $t_2 - t_1 \to 0$. Hence $W([v_0, w_0])$ is equicontinuous function of cluster in $C(J, E)$.

The same idea can be used to prove the compactness of V.

For $0 \le t \le a$, since $\{Qu(t) : u \in [v_0, w_0]\} = \{S(t)x_0 + (Wu)(t) + (Vu)(t) : u \in [v_0, w_0]\}$, and $Qu(0) = x_0$ is precompact in E. Hence, $Q([v_0, w_0])$ is precompact in $C(J, E)$ by the Arzela-Ascoli theorem. So $Q : [v_0, w_0] \to [v_0, w_0]$ is completely continuous. Hence, Q has minimal and maximal fixed points $\underline{u}$ and $\overline{u}$ in $[v_0, w_0]$, and therefore, they are the minimal and maximal mild solutions of the IVP(1.1) in $[v_0, w_0]$, respectively. ∎

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \ge 0)$ in E, $f \in C(J \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \dots, m$. If the IVP(1.1) has a lower solution $v_0 \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ and* *an upper solution $w_0 \in PC(J,E) \cap C^1(J',E) \cap C(J',E_1)$ with $v_0 \le w_0$, conditions* (H1) *and* (H2) *hold, and satisfy*

(H3) *There exist a constant $L > 0$ such that for all $t \in J$,*

$$\alpha(\{f(t,u_n,v_n)\}) \le L(\alpha(\{u_n\}) + \alpha(\{v_n\})),$$

and increasing or decreasing sequences $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [v_0(t), w_0(t)]$.

Then the IVP(1.1) *has minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Proof. From Theorem 3.1, we know that $Q : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator. Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1, 2, \dots. \tag{3.5}$$

Then from the monotonicity of Q, it follows that

$$v_0 \le v_1 \le v_2 \le \dots \le v_n \le \dots \le w_n \le \dots \le w_2 \le w_1 \le w_0. \tag{3.6}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J. For convenience, let$B = \{v_n : n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} : n \in \mathbb{N}\}$. Then $B = Q(B_0)$. Let $J'_1 = [0, t_1]$, $J'_k = (t_{k-1}, t_k]$, $k = 2, 3, \dots m+1$. From $B_0 = B \bigcup \{v_0\}$ it follows that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J$. Let $\varphi(t) := \alpha(B(t)), t \in J$, Going from J'_1 to J'_{m+1}interval by interval we show that $\varphi(t) \equiv 0$ in J.

For $t \in J$, there exists a J'_k such that $t \in J'_k$. By (1.2) and Lemma 2.1, we have that

$$\begin{aligned}
\alpha(T(B_0)(t)) &= \alpha\Big(\Big\{\int_0^t K(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\le \sum_{j=1}^{k-1} \alpha\Big(\Big\{\int_{t_{j-1}}^{t_j} K(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\quad + \alpha\Big(\Big\{\int_{t_{k-1}}^{t} K(t,s)v_{n-1}(s)ds : n \in \mathbb{N}\Big\}\Big) \\
&\le 2K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \alpha(B_0(s))ds + 2K_0 \int_{t_{k-1}}^{t} \alpha(B_0(s))ds \\
&= 2K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \varphi(s)ds + 2K_0 \int_{t_{k-1}}^{t} \varphi(s)ds \\
&= 2K_0 \int_0^t \varphi(s)ds,
\end{aligned}$$

and therefore,

$$\int_0^t \alpha(T(B_0)(s))ds \le 2aK_0 \int_0^t \varphi(s)ds. \tag{3.7}$$

For $t \in J_1'$, from (3.1), using Lemma 2.1, assumption (H3) and (3.7), we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(Q(B_0)(t)) \\
&= \alpha\Big(\Big\{S(t)x_0 + \int_0^t S(t-s)(f(s,v_{n-1}(s),Tv_{n-1}(s)) + Mv_{n-1}(s))ds\Big\}\Big) \\
&\le 2\overline{M}\int_0^t \alpha(\{f(s,v_{n-1}(s),Tv_{n-1}(s)) + Mv_{n-1}(s)\})ds \\
&\le 2\overline{M}\int_0^t (L(\alpha(B_0(s)) + \alpha(Q(B_0)(s))) + M\alpha(B_0(s)))ds \\
&\le 2\overline{M}(L+M+2aLK_0)\int_0^t \varphi(s)ds.
\end{aligned}$$

Hence by the Bellman inequality, $\varphi(t) \equiv 0$ in J_1'. In particular, $\alpha(B(t_1)) = \alpha(B_0(t_1)) = \varphi(t_1) = 0$, this implies that $B(t_1)$ and $B_0(t_1)$ are precompact in E. Thus $I_1(B_0(t_1))$ is precompact in E, and $\alpha(I_1(B_0(t_1))) = 0$.

Now, for $t \in J_2'$, by (3.1) and the above argument for $t \in J_1'$, we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(Q(B_0)(t)) \\
&= \alpha\Big(\Big\{S(t)x_0 + \int_0^t S(t-s)(f(s,v_{n-1}(s),Tv_{n-1}(s)) + Mv_{n-1}(s))ds \\
&\quad + S(t-t_1)I_1(v_{n-1}(t_1))\Big\}\Big) \\
&\le 2\overline{M}(L+M+2aLK_0)\int_0^t \varphi(s)ds \\
&= 2\overline{M}(L+M+2aLK_0)\int_{t_1}^t \varphi(s)ds.
\end{aligned}$$

Again by Bellman inequality, $\varphi(t) \equiv 0$ in J_2', from which we obtain that $\alpha(B_0(t_2)) = 0$ and $\alpha(I_2(B_0(t_2))) = 0$.

Continuing such a process interval by interval up to J_{m+1}', we can prove that $\varphi(t) \equiv 0$ in every J_k', $k = 1, 2, \dots, m+1$. Hence, for any $t \in J$, $\{v_n(t)\}$ is precompact, and $\{v_n(t)\}$ has a convergent subsequence. Combing this with the monotonicity (3.6), we easily prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in J$. Similarly, $\lim_{n\to\infty} w_n(t) = \overline{u}(t)$, $t \in J$.

Evidently $\{v_n(t)\} \in PC(J,E)$, so $\underline{u}(t)$ is bounded integrable in every J_k, $k = 1, 2, \dots, m+1$. Since for any $t \in J_k$,

$$\begin{aligned}
v_n(t) &= Qv_{n-1}(t) \\
&= S(t)x_0 + \int_0^t S(t-s)(f(s,v_{n-1}(s),Tv_{n-1}(s)) + Mv_{n-1}(s))ds \\
&\quad + \sum_{0<t_i<t} S(t-t_i)I_i(v_{n-1}(t_i)),
\end{aligned}$$

letting $n \to \infty$, by the Lebesgue dominated convergence theorem, for all $t \in J_k$, $k = 1, 2, \dots, m+1$, we have

$$\underline{u}(t) = S(t)x_0 + \int_0^t S(t-s)(f(s,\underline{u}(s),T\underline{u}(s)) + M\underline{u}(s))ds + \sum_{0<t_i<t} S(t-t_i)I_i(\underline{u}(t_i)),$$

and $\underline{u}(t) \in C(J_k, E)$, $k = 1, 2, \dots, m+1$. So, for $t \in J$, we have

$$\underline{u}(t) = S(t)x_0 + \int_0^t S(t-s)(f(s, \underline{u}(s), T\underline{u}(s)) + M\underline{u}(s))ds + \sum_{0<t_k<t} S(t-t_k)I_k(\underline{u}(t_k)).$$

Therefore, $\underline{u}(t) \in PC(J, E)$, and $\underline{u} = Q\underline{u}$. Similarly, $\overline{u}(t) \in PC(J, E)$, and $\overline{u} = Q\overline{u}$. Combing this with monotonicity (3.6), we see that $v_0 \leq \underline{u} \leq \overline{u} \leq w_0$. By the monotonicity of Q, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of Q in $[v_0, w_0]$. Therefore, $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of the IVP(1.1) in $[v_0, w_0]$, respectively. □

Corollary 3.3. *Let E be an ordered Banach space, whose positive cone P is regular, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ in E, $f \in C(J \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \dots, m$. If the IVP*(1.1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ with $v_0 \leq w_0$, and conditions* (H1) *and* (H2) *are satisfied, then the IVP*(1.1) *has minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Now we discuss the uniqueness of the mild solution to IVP(1.1) in $[v_0, w_0]$. If we replace the assumption (H3) by the assumption:

(H4) There exist positive constants $\overline{C}$ and $\overline{L}$ such that

$$f(t, u_2, v_2) - f(t, u_1, v_1) \leq \overline{C}(u_2 - u_1) + \overline{L}(v_2 - v_1),$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Tv_0(t) \leq v_1 \leq v_2 \leq Tw_0(t)$,

We have the following unique existence result.

Theorem 3.4. *Let E be an ordered Banach space, whose positive cone P is normal, $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ in E, $f \in C(J \times E \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \dots, m$. If the IVP*(1.1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$ with $v_0 \leq w_0$, such that conditions* (H1), (H2), (H4) *hold, then the IVP*(1.1) *has a unique mild solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Proof. We firstly prove that (H1) and (H4) imply (H3). For $t \in J$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [Tv_0(t), Tw_0(t)]$ be two increasing sequences. For $m, n \in \mathbb{N}$ with $m > n$, by (H1) and (H4),

$$\begin{aligned} \theta &\leq f(t, u_m, v_m) - f(t, u_n, v_n) + M(u_m - u_n) \\ &\leq (M + \overline{C})(u_m - u_n) + \overline{L}(v_n - v_m). \end{aligned}$$

By this and the normality of cone P, we have

$$\begin{aligned} &\|f(t, u_m, v_m) - f(t, u_n, v_n)\| \\ &\leq N\|(M + \overline{C})(u_m - u_n) + \overline{L}(v_n - v_m)\| + M\|u_m - u_n\| \\ &\leq (N(M + \overline{C}) + M)\|u_m - u_n\| + N\overline{L}\|v_n - v_m\|. \end{aligned}$$

From this inequality and the definition of the measure of noncompactness, it follows that

$$\begin{aligned}\alpha(\{f(t,u_n,v_n)\}) &\le (N(M+\overline{C})+M)\alpha(\{u_n\})+N\overline{L}\alpha(\{v_n\})\\ &\le L_1(\alpha(\{u_n\})+\alpha(\{v_n\})),\end{aligned}$$

where $L_1=\max\{(N(M+\overline{C})+M),N\overline{L}\}$. If $\{u_n\}$ and $\{v_n\}$ are two decreasing sequences, the above inequality is also valid. Hence (H3) holds. Therefore, by Theorem 3.2, the IVP(1.1) has minimal and maximal mild solutions $\underline{u}$ and$\overline{u}$ between v_0 and w_0. By the proof of Theorem 3.2, (3.5) and (3.6) are valid. Going from J_1' to J_{m+1}' interval by interval we show that $\underline{u}(t)\equiv\overline{u}(t)$ in every J_k'.

For $t\in J_1'$, by (3.1) and assumption (H4), we have

$$\begin{aligned}\theta &\le \overline{u}(t)-\underline{u}(t)=Q\overline{u}(t)-Q\underline{u}(t)\\ &=\int_0^t S(t-s)\big[f(s,\overline{u}(s),T\overline{u}(s))-f(s,\underline{u}(s),T\underline{u}(s))+M(\overline{u}(s)-\underline{u}(s))\big]ds\\ &\le \overline{M}(M+\overline{C}+a\overline{L}K_0)\int_0^t(\overline{u}(s)-\underline{u}(s))ds.\end{aligned}$$

From this and the normality of cone P it follows that

$$\|\overline{u}(t)-\underline{u}(t)\|\le N\overline{M}(M+\overline{C}+a\overline{L}K_0)\int_0^t\|\overline{u}(s)-\underline{u}(s)\|ds.$$

By this and Bellman inequality, we obtained that $\underline{u}(t)\equiv\overline{u}(t)$ in J_1'.

For $t\in J_2'$, since $I_1(\overline{u}(t_1))=I_1(\underline{u}(t_1))$, using (3.1) and completely the same argument as above for $t\in J_1'$, we can prove that

$$\begin{aligned}\|\overline{u}(t)-\underline{u}(t)\| &\le N\overline{M}(M+\overline{C}+a\overline{L}K_0)\int_0^t\|\overline{u}(s)-\underline{u}(s)\|ds\\ &=N\overline{M}(M+\overline{C}+a\overline{L}K_0)\int_{t_1}^t\|\overline{u}(s)-\underline{u}(s)\|ds.\end{aligned}$$

Again, by the Bellman inequality, we obtain that $\underline{u}(t)\equiv\overline{u}(t)$ in J_2'.

Continuing such a process interval by interval up to J_{m+1}', we see that $\underline{u}(t)\equiv\overline{u}(t)$ over the whole of J. Hence, $\widetilde{u}:=\underline{u}=\overline{u}$ is the unique mild solution of the IVP(1.1) in $[v_0,w_0]$, which can be obtained by the monotone iterative procedure (3.6) starting from v_0 or w_0. □

If lower solution and upper solutions for the IVP(1.1) do not exist, then we have the following result.

Theorem 3.5. *Let E be an ordered Banach space, whose positive cone P is normal, $A:D(A)\subset E\to E$ be a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t\ge 0)$ in E, $f\in C(J\times E\times E,E)$ and $I_k\in C(E,E)$, $k=1,2,\dots,m$. If there exist $a>0$, $x_0\in D(A)$, $x_0\ge\theta$, $y_k\in D(A)$, $y_k\ge\theta$, $k=1,2,\dots,m$, $h\in PC(J,E)\cap C^1(J',E)$ and $h(t)\ge\theta$, such that*

$$\begin{gathered}f(t,x,Tx)\le ax+h(t),\quad I_k(x)\le y_k, x\ge\theta;\\ f(t,x,Tx)\ge ax-h(t),\quad I_k(x)\ge -y_k, x\le\theta.\end{gathered}$$

Then we have:

(i) *If the C_0-semigroup $T(t)(t \geq 0)$ generated by $-A$ is compact in E, and conditions* (H1) *and* (H2) *are satisfied, then the IVP*(1.1) *has minimal and maximal mild solutions.*
(ii) *If conditions* (H1), (H2), (H3) *are satisfied, then the IVP*(1.1) *has minimal and maximal mild solutions.*
(iii) *If the positive cone P is regular, and conditions* (H1) *and* (H2) *are satisfied, then the IVP*(1.1) *has minimal and maximal mild solutions.*
(iv) *If conditions* (H1), (H2), (H4) *are satisfied, then the IVP*(1.1) *has a unique mild solution.*

Proof. Firstly, we consider the IVP of linear impulsive evolution equation in E

$$\begin{gathered} u'(t) + Au(t) - au(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \dots, m, \\ u(0) = x_0. \end{gathered} \tag{3.8}$$

Since $-A+aI$ generates a positive C_0-semigroup $S(t) = e^{at}T(t)(t \geq 0)$ in E. So, by [11, Chapter 4, Theorem 2.9] and Lemma 2.2, we know that IVP(3.8) has a unique positive classical solution $\widetilde{u} \in PC(J, E) \cap C^1(J', E) \cap C(J', E_1)$. Let $v_0 = -\widetilde{u}$, $w_0 = \widetilde{u}$, it is easy to see that v_0 and w_0 are lower solution and upper solution of the IVP(1.1) respectively. So, our conclusions (i), (ii), (iii) and (iv) follow from the Theorem 3.1, Theorem 3.2, Corollary 3.3 and Theorem 3.4 respectively. □

4. Applications

Consider the IVP of impulsive parabolic partial differential equation

$$\begin{gathered} \frac{\partial u}{\partial t} + A(x, D)u(t) = f(x, t, u(t), Tu(t)), \quad x \in \Omega, t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(x, t_k)), \quad x \in \Omega, k = 1, 2, \dots, m, \\ Bu = 0, \quad (x, t) \in \partial\Omega \times J, \\ u(x, 0) = \varphi(x), \quad x \in \Omega, \end{gathered} \tag{4.1}$$

where $J = [0, a]$, $0 < t_1 < t_2 < \dots < t_m < a$, integer $N \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$,

$$A(x, D) = -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial y_j} + \sum_{i=1}^{N} a_i(x)\frac{\partial}{\partial x_i} + a_0(x)$$

is a strongly elliptic operator of second order, coefficient functions $a_{ij}(x), a_i(x)$ and $a_0(x)$ are Hölder continuous in $\Omega, Bu = b_0(x)u + \delta\frac{\partial u}{\partial n}$ is a regular boundary operator on $\partial\Omega$, $f : \overline{\Omega} \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $I_k : \mathbb{R} \to \mathbb{R}$ are also continuous, $k = 1, 2, \dots, m$.

Let $E = L^p(\Omega)$ with $p > N + 2$, $P = \{u \in L^p(\Omega) : u(x) \geq 0, a.e.\ x \in \Omega\}$, and define the operator A as follows:

$$D(A) = \{u \in W^{2,p}(\Omega) : Bu = 0\}, \quad Au = A(x, D)u.$$

Then E is a Banach space, P is a regular cone of E, and $-A$ generates a positive and analytic C_0-semi-group $T(t)(t \geq 0)$ in E (see [8, 9, 11]). So, the problem (4.1) can be transformed into the IVP (1.1). To solve the IVP(4.1), we also need following assumptions:

(a) Let $f(x,t,0,0) \geq 0$, $I_k(0) \geq 0$, $\varphi(x) \geq 0$, $x \in \Omega$, and there exists a function $w = w(x,t) \in PC(J,E) \cap C^{2,1}(\overline{\Omega} \times J)$, such that

$$\begin{gathered}
\frac{\partial w}{\partial t} + A(x,D)w \geq f(x,t,w,Tw), \quad (x,t) \in \Omega \times J, t \neq t_k, \\
\Delta w|_{t=t_k} \geq I_k(w(x,t_k)), \quad x \in \Omega,\ k = 1,2,\dots,m, \\
Bw = 0, \quad (x,t) \in \partial\Omega \times J, \\
w(x,0) \geq \varphi(x), \quad x \in \Omega.
\end{gathered}$$

(b) There exists a constant $M > 0$ such that

$$f(x,t,x_2,y_2) - f(x,t,x_1,y_1) \geq -M(x_2 - x_1),$$

for any $t \in J$, and $0 \leq x_1 \leq x_2 \leq w(x,t), 0 \leq y_1 \leq y_2 \leq Tw(x,t)$.

(c) For any $u_1, u_2 \in [0, w(x,t)]$ with $u_1 \leq u_2$, we have

$$I_k(u_1(x,t_k)) \leq I_k(u_2(x,t_k)), \quad x \in \Omega,\ k = 1,2,\dots,m.$$

Assumption (a) implies that $v_0 \equiv 0$ and $w_0 \equiv w(x,t)$ are lower and upper solutions of the IVP(1.1) respectively, and from (b) and (c), it is easy to verify that conditions (H1) and (H2) are satisfied. So, from Corollary 3.3, we have the following result.

Theorem 4.1. *If the assumptions (a), (b) and (c) are satisfied, then the IVP*(4.1) *has minimal and maximal mild solutions between* 0 *and* $w(x,t)$*, which can be obtained by a monotone iterative procedure starting from* 0 *and* $w(x,t)$ *respectively.*

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