Electronic Journal of Differential Equations, Vol. 2010(2010), No. 161, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ANALYSIS OF A QUADRATIC SYSTEM OBTAINED FROM A SCALAR THIRD ORDER DIFFERENTIAL EQUATION

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Abstract. In this article, we study the nonlinear dynamics of a quadratic system in the three dimensional space which can be obtained from a scalar third order differential equation. More precisely, we study the stability and bifurcations which occur in a parameter dependent quadratic system in the three dimensional space. We present an analytical study of codimension one, two and three Hopf bifurcations, generic Bogdanov-Takens and fold-Hopf bifurcations.

1. Introduction

In this paper we study the stability and bifurcations in the dynamics of the third order differential equation

$$x''' + f(x)x'' + g(x)x' + h(x) = 0, \tag{1.1}$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$ are

$$f(x) = a_1x + a_0, \quad g(x) = b_1x + b_0, \quad h(x) = c_2x^2 + c_1x + c_0, \tag{1.2}$$

with $a_1, a_0, b_1, b_0, c_2, c_1, c_0 \in \mathbb{R}$, $c_2 \neq 0$.

By defining of the variables $y = x'$ and $z = x''$, differential equation (1.1) can be written as the system of nonlinear differential equations

$$\begin{aligned} x' &= y,\\ y' &= z,\\ z' &= -\big((a_1x + a_0)z + (b_1x + b_0)y + c_2x^2 + c_1x + c_0\big), \end{aligned} \tag{1.3}$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables and $(a_0, a_1, b_0, b_1, c_0, c_1, c_2) \in \mathbb{R}^7$, $c_2 \neq 0$, are real parameters.

The choice of real affine functions to f and g and a quadratic function to h imply that the vector field that defines (1.3),

$$F(x, y, z) = \big(y, z, -\big((a_1x + a_0)z + (b_1x + b_0)y + c_2x^2 + c_1x + c_0\big)\big), \tag{1.4}$$

is a quadratic vector field. So, system (1.3) is a quadratic system of differential equations in $\mathbb{R}^3$.

2000 *Mathematics Subject Classification.* 70K50, 70K20, 34C60.
Key words and phrases. Bifurcation; Hopf bifurcation; Bogdanov-Takens bifurcation; fold-Hopf bifurcation; limit cycle; stability; quadratic system.

Submitted February 15, 2010. Published November 10, 2010.

Quadratic systems in $\mathbb{R}^3$ are some of the simplest systems after linear ones and have been extensively studied in the last five decades. Examples of such systems are the Lorenz system [12], the Chen system [2], the Liu system [10], the Rössler system [16], the Rikitake system [15], the Lü system [13], the Genesio system [5] among several others.

An interesting problem related to quadratic systems defined in $\mathbb{R}^n$ is the determination of the number of their limit cycles. In $\mathbb{R}^2$ this number is finite [3, 6]. For quadratic systems in $\mathbb{R}^n$, $n \geq 3$ the scenario is very different. Recently Ferragut, Llibre and Pantazi [4] provided an example of quadratic vector field in $\mathbb{R}^3$ and an analytical proof that it has infinitely many limit cycles.

As far as we know, differential equation (1.1), or equivalently system (1.3), was analyzed in two particular cases:

(a) When $a_1 = b_1 = c_0 = 0$, $c_1 = 1$ and $c_2 = -1$ differential equation (1.1) is a feedback control system of Lur'e type. The Hopf bifurcations of codimension one of the equivalent system (1.3) were studied in [8];
(b) When $a_1 = b_1 = c_0 = 0$ and $c_2 = -1$ differential equation (1.1) is an extension of the above feedback control system of Lur'e type. The equivalent system (1.3) was studied in [5] from the chaotic behavior point of view and in [20] were studied its Hopf bifurcations of codimension one and homoclinic connections.

On the other hand, differential equation (1.1), or equivalently system (1.3), can be seen as a particular case of a more general quadratic third order differential equation [7]. In [7] the authors studied oscillations that appear from codimension one Hopf bifurcations. The study was made using an approach based on harmonic balance techniques. However there exist more degenerate cases that were not analyzed by them.

Despite the simplicity, system (1.3) has a rich local dynamical behavior presenting several degenerate bifurcations. The study carried out in the present paper may contribute to understand analytically the stability and some bifurcations of system (1.3). For this purpose the paper is organized as follows. After some general results the linear analysis of the equilibria of system (1.3) is presented in Section 2. A brief review of the methods used to study Hopf, Bogdanov-Takens and fold-Hopf bifurcations are presented in Section 3. These methods are used in Section 4 to prove the main results of this paper. More specifically, in subsections 4.1 and 4.2 we study all the possible Hopf bifurcations (generic and degenerate ones) which occur in the equilibria of system (1.3). An application of these results is made in subsection 4.3 for a particular case of system (1.3). In subsection 4.4 we present the study of a Bogdanov-Takens bifurcation which occurs at an equilibrium point of system (1.3) for a suitable choice of the parameters. This study leads to the existence of homoclinic connections and global bifurcations in system (1.3). Other global bifurcations in system (1.3) can be determined by the existence of a fold-Hopf bifurcation at an equilibrium point for a suitable choice of the parameters. The study of this bifurcation is presented in subsection 4.5. In Section 5 we make some concluding comments.

2. Linear analysis of system (1.3)

The equilibria of system (1.3) are $E_* = (x_*, 0, 0)$, where x_* is a real zero of the function h, that is $h(x_*) = 0$. By assumption h is a quadratic function, so it may have 0, 1 or 2 real zeros. This implies that system (1.3) has 0, 1 or 2 equilibrium points. The local behavior of the flow of system (1.3) is trivial when there is no equilibrium point. Nevertheless the global behavior of the flow can be very interesting with the study, for example, of large amplitude limit cycles, that is limit cycles out of compact parts of $\mathbb{R}^3$ [11]. In this paper we only study the cases with 1 or 2 equilibria.

Suppose that system (1.3) has only one equilibrium point. Without loss of generality, we can consider $h(x) = x^2$, that is $c_2 = 1$ and $c_1 = c_0 = 0$. This implies that the equilibrium point E_* is at the origin. The linear part of system (1.3) at the origin has the form

$$A = DF(E_*) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b_0 & -a_0 \end{pmatrix},$$

and its characteristic polynomial is

$$p(\lambda) = -\lambda\left(\lambda^2 + a_0\lambda + b_0\right). \tag{2.1}$$

It follows that one eigenvalue is $\lambda_1 = 0$ and this implies that the origin is a non-hyperbolic equilibrium point. A more detailed study of the stability of this equilibrium point is presented in subsections 4.4 and 4.5.

Now suppose that system (1.3) has two equilibrium points. Thus the function h has the form $h(x) = c_2(x - x_0)(x - x_1)$, $c_2 \neq 0$. By the following change of coordinates and a reparametrization in time

$$x = X, \quad y = c_2^{1/3}Y, \quad z = c_2^{2/3}Z, \quad t = c_2^{1/3}\tau,$$

system (1.3) can be written with a function h of the form $h(x) = (x - x_0)(x - x_1)$. Without loss of generality, we can consider $x_0 = 0$ and $x_1 = -1$. It follows that system (1.3) has the equilibria $E_0 = (0, 0, 0)$ and $E_1 = (-1, 0, 0)$ and can be written as

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -\big((a_1x + a_0)z + (b_1x + b_0)y + x(x + 1)\big), \end{aligned} \tag{2.2}$$

where $(x, y, z) \in \mathbb{R}^3$ are the state variables and $(a_0, b_0, a_1, b_1) \in \mathbb{R}^4$ are real parameters.

A useful tool for the linear analysis of an equilibrium point is the following Routh-Hurwitz stability criterion whose proof can be found in [14, p. 58].

Lemma 2.1. *The polynomial $L(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3$ with real coefficients has all roots with negative real parts if and only if the numbers p_1, p_2, p_3 are positive and the inequality $p_1p_2 > p_3$ is satisfied.*

2.1. **Linear analysis at E_0.** In this subsection we study the stability of the equilibrium $E_0 = (0, 0, 0)$ of system (2.2) from the linear point of view. Consider the set of parameters

$$\mathcal{W} = \{(a_0, b_0, a_1, b_1) \in \mathbb{R}^4\}.$$

We have the following proposition.

Proposition 2.2. *Define the following subsets of $\mathcal{W}$:*

$$\mathcal{W}_1 = \{(a_0, b_0, a_1, b_1) \in \mathcal{W} : a_0 \leq 0\} \cup \{(a_0, b_0, a_1, b_1) \in \mathcal{W} : b_0 \leq 0\},$$

$$\mathcal{W}_2 = \{(a_0, b_0, a_1, b_1) \in \mathcal{W} : a_0 > 0,\ b_0 > 0,\ a_0 b_0 < 1\},$$
$$\mathcal{W}_3 = \{(a_0, b_0, a_1, b_1) \in \mathcal{W} : a_0 > 0,\ b_0 > 0,\ a_0 b_0 > 1\}.$$

Then the following statements hold:

(1) *If* $(a_0, b_0, a_1, b_1) \in \mathcal{W}_1$ *then the equilibrium* E_0 *is unstable;*
(2) *If* $(a_0, b_0, a_1, b_1) \in \mathcal{W}_2$ *then the equilibrium* E_0 *is unstable;*
(3) *If* $(a_0, b_0, a_1, b_1) \in \mathcal{W}_3$ *then the equilibrium* E_0 *is locally asymptotically stable.*

Proof. The characteristic polynomial of the Jacobian matrix of system (2.2) at E_0 is

$$p(\lambda) = \lambda^3 + a_0\lambda^2 + b_0\lambda + 1.$$

If $(a_0, b_0, a_1, b_1) \in \mathcal{W}_1$ then the coefficients a_0 and b_0 of $p(\lambda)$ are non-positive. From Lemma 2.1 it follows that the equilibrium E_0 is unstable. This proves item 1 of the proposition. From Lemma 2.1 the equilibrium E_0 is locally asymptotically stable if the coefficients of the characteristic polynomial satisfy

$$a_0 > 0, \quad b_0 > 0, \quad a_0 b_0 > 1. \tag{2.3}$$

So if $(a_0, b_0, a_1, b_1) \in \mathcal{W}_2$ then E_0 is unstable and if $(a_0, b_0, a_1, b_1) \in \mathcal{W}_3$ then E_0 is locally asymptotically stable. This proves item 2 and 3 of the proposition. □

Define the set

$$\mathcal{H}_0 = \{(a_0, b_0, a_1, b_1) \in \mathcal{W} : a_0 > 0,\ b_0 > 0,\ a_0 b_0 = 1\}. \tag{2.4}$$

Thus $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{H}_0$. If $(a_0, b_0, a_1, b_1) \in \mathcal{H}_0$ then the equilibrium E_0 is non-hyperbolic, that is the Jacobian matrix of system (2.2) at E_0 has one negative real eigenvalue and a pair of purely imaginary eigenvalues

$$\lambda_1 = -\frac{1}{b_0} < 0, \quad \lambda_{2,3} = \pm i\sqrt{b_0}.$$

The set $\mathcal{H}_0$ is called the Hopf hypersurface of the equilibrium E_0. From the Center Manifold Theorem, at a Hopf point a two dimensional center manifold is well-defined, it is invariant under the flow generated by (2.2) and can be continued with arbitrary high class of differentiability to nearby parameter values (see [8, p. 152]). This center manifold is attracting since $\lambda_1 < 0$. So it is enough to study the stability of E_0 for the flow restricted to the family of parameter-dependent continuations of the center manifold. A detailed analysis of this case will be presented in subsection 4.1.

2.2. **Linear analysis at E_1.** In this subsection, we study the stability of the equilibrium $E_1 = (-1, 0, 0)$ of system (2.2) from the linear point of view.

The characteristic polynomial of the Jacobian matrix of system (2.2) at E_1 is

$$p(\lambda) = \lambda^3 + (a_0 - a_1)\lambda^2 + (b_0 - b_1)\lambda - 1.$$

The coefficient -1 of $p(\lambda)$ is negative. From Lemma 2.1 it follows that the equilibrium E_1 is unstable for all parameters $(a_0, b_0, a_1, b_1) \in \mathcal{W}$.

Define the set

$$\mathcal{H}_1 = \{(a_0, b_0, a_1, b_1) \in \mathcal{W} : (a_0 - a_1) < 0,\ (a_0 - a_1)(b_0 - b_1) = -1\}. \tag{2.5}$$

If $(a_0, b_0, a_1, b_1) \in \mathcal{H}_1$ then the Jacobian matrix of system (2.2) at E_1 has eigenvalues

$$\lambda_1 = (a_1 - a_0) > 0, \quad \lambda_{2,3} = \pm i\frac{1}{\sqrt{a_1 - a_0}}.$$

The set $\mathcal{H}_1$ is called the Hopf hypersurface of the equilibrium E_1. From the Center Manifold Theorem, at a Hopf point a two dimensional center manifold is well-defined, it is invariant under the flow generated by (2.2) and can be continued with arbitrary high class of differentiability to nearby parameter values (see [8, p. 152]). This center manifold is repelling since $\lambda_1 > 0$. We are interested in the study the stability of E_1 for the flow restricted to the family of parameter-dependent continuations of the center manifold. A detailed analysis of this case will be presented in subsection 4.2.

3. Generalities on Hopf, Bogdanov-Takens and fold-Hopf bifurcations

3.1. **Hopf bifurcations.** In this subsection we present a review of the projection method described in [8] for the calculation of the first and second Lyapunov coefficients associated to Hopf bifurcations, denoted by l_1 and l_2 respectively. This method was extended to the calculation of the third and fourth Lyapunov coefficients in [17] and [18], respectively.

Consider the differential equation

$$x' = f(x, \zeta), \tag{3.1}$$

where $x \in \mathbb{R}^3$ and $\zeta \in \mathbb{R}^n$ are respectively vectors representing phase variables and control parameters. Assume that f is of class C^∞ in $\mathbb{R}^3 \times \mathbb{R}^n$. Suppose that (3.1) has an equilibrium point $x = x_0$ at $\zeta = \zeta_0$ and, denoting the variable $x - x_0$ also by x, write

$$F(x) = f(x, \zeta_0) \tag{3.2}$$

as

$$\begin{aligned} F(x) = Ax &+ \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \frac{1}{24}D(x,x,x,x) + \frac{1}{120}E(x,x,x,x,x) \\ &+ \frac{1}{720}K(x,x,x,x,x,x) + \frac{1}{5040}L(x,x,x,x,x,x,x) + O(\|x\|^8), \end{aligned} \tag{3.3}$$

where $A = f_x(0, \zeta_0)$ and, for $i = 1, 2, 3$,

$$B_i(x,y) = \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}\Big|_{\xi=0} x_j y_k, \quad C_i(x,y,z) = \sum_{j,k,l=1}^{3} \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}\Big|_{\xi=0} x_j y_k z_l,$$

and so on for D_i, E_i, K_i and L_i.

Suppose that $(x_0, \zeta_0) = (0, \zeta_0)$ is an equilibrium point of (3.1) where the Jacobian matrix A has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and the other eigenvalues $\lambda_1 \neq 0$. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$. By this it is meant the largest subspace invariant by A on which the eigenvalues are $\lambda_{2,3}$.

Let $p, q \in \mathbb{C}^3$ be vectors such that

$$Aq = i\omega_0 q, \quad A^T p = -i\omega_0 p, \quad \langle p, q \rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1, \tag{3.4}$$

where A^T is the transpose of the matrix A. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle p, y \rangle \in \mathbb{C}$. The two dimensional center manifold associated to the eigenvalues $\lambda_{2,3} = \pm i\omega_0$ can be parameterized by the variables w

and $\bar{w}$ by means of an immersion of the form $x = H(w, \bar{w})$, where $H : \mathbb{C}^2 \to \mathbb{R}^3$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \le j+k \le 7} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^8), \tag{3.5}$$

with $h_{jk} \in \mathbb{C}^3$ and $h_{jk} = \bar{h}_{kj}$. Substituting this expression into (3.1) we obtain the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})), \tag{3.6}$$

where F is given by (3.2). The complex vectors h_{ij} are obtained solving the system of linear equations defined by the coefficients of (3.6), taking into account the coefficients of F (see Remark 3.1 of [17], p. 27), so that system (3.6), on the chart w for a central manifold, writes as

$$w' = i\omega_0 w + \frac{1}{2} G_{21} w|w|^2 + \frac{1}{12} G_{32} w|w|^4 + \frac{1}{144} G_{43} w|w|^6 + O(|w|^8),$$

with $G_{jk} \in \mathbb{C}$.

The *first Lyapunov coefficient* l_1 is

$$l_1 = \frac{1}{2} \operatorname{Re} G_{21}, \tag{3.7}$$

where $G_{21} = \langle p, \mathcal{H}_{21} \rangle$ and $\mathcal{H}_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})$.

The *second Lyapunov coefficient* is

$$l_2 = \frac{1}{12} \operatorname{Re} G_{32}, \tag{3.8}$$

where $G_{32} = \langle p, \mathcal{H}_{32} \rangle$ and

$$\begin{aligned}
\mathcal{H}_{32} = {} & 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) + 2B(\bar{q}, h_{31}) \\
& + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) + 6C(q, \bar{q}, h_{21}) \\
& + 6C(\bar{q}, h_{20}, h_{11}) + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) + 6D(q, q, \bar{q}, h_{11}) \\
& + 3D(q, \bar{q}, \bar{q}, h_{20}) + E(q, q, q, \bar{q}, \bar{q}) - 6G_{21} h_{21} - 3\bar{G}_{21} h_{21},
\end{aligned}$$

The *third Lyapunov coefficient* is

$$l_3 = \frac{1}{144} \operatorname{Re} G_{43}, \tag{3.9}$$

where $G_{43} = \langle p, \mathcal{H}_{43} \rangle$. The expression for $\mathcal{H}_{43}$ is too large to be put in print and can be found in [17, eq. (44)].

A *Hopf point of codimension one* is an equilibrium point (x_0, ζ_0) such that linear part of the vector field f has eigenvalues λ_2 and $\lambda_3 = \overline{\lambda}$ with $\lambda = \lambda(\zeta) = \gamma(\zeta) + i\eta(\zeta)$, $\gamma(\zeta_0) = 0$, $\eta(\zeta_0) = \omega_0 > 0$, the other eigenvalue $\lambda_1 \neq 0$ and the first Lyapunov coefficient, $l_1(\zeta_0)$, is different from zero. A *transversal Hopf point of codimension one* is a Hopf point of codimension one for which the complex eigenvalues depending on the parameters cross the imaginary axis with nonzero derivative. As $l_1 < 0$ ($l_1 > 0$) one family of stable (unstable) periodic orbits can be found on the center manifold and its continuation, shrinking to the Hopf point.

Hopf point of codimension 2 is an equilibrium point (x_0, ζ_0) of f that satisfies the definition of Hopf point of codimension one, except that $l_1(\zeta_0) = 0$, and an additional condition that the second Lyapunov coefficient, $l_2(\zeta_0)$, is nonzero. This point is *transversal* if the sets $\gamma^{-1}(0)$ and $l_1^{-1}(0)$ have transversal intersection, or equivalently, if the map $\zeta \mapsto (\gamma(\zeta), l_1(\zeta))$ is regular at $\zeta = \zeta_0$. The bifurcation diagrams for $l_2 \neq 0$ can be found in [8, p. 313], and in [19].

A *Hopf point of codimension 3* is a Hopf point of codimension 2 where l_2 vanishes but $l_3 \neq 0$. A Hopf point of codimension 3 is called *transversal* if the sets $\gamma^{-1}(0)$, $l_1^{-1}(0)$ and $l_2^{-1}(0)$ have transversal intersections. The bifurcation diagram for $l_3 \neq 0$ can be found in [17] and in Takens [19].

3.2. **Bogdanov-Takens bifurcations.** In this subsection we present an approach based on [8, p. 321], and [9] for the Bogdanov-Takens bifurcation. Consider a system $x' = f(x, \alpha)$, $x \in \mathbb{R}^3$, $\alpha \in \mathbb{R}^n$ and assume that f is of class C^∞ in $\mathbb{R}^3 \times \mathbb{R}^n$. Suppose that for $\alpha = \alpha_0$ there is an equilibrium point $x = x_0$ such that the Jacobian matrix A of f at x_0 has a double zero eigenvalue; that is, $\lambda_{2,3} = 0$ and the other eigenvalue $\lambda_1 \neq 0$. Denoting the variable $x - x_0$ also by x we consider

$$F(x) = f(x, \alpha_0) = Ax + \frac{1}{2}B(x, x) + O(\|x\|^3),$$

where, for $i = 1, 2, 3$,

$$B_i(x, y) = \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}\Big|_{\xi=0} x_j y_k.$$

Let $q_0, q_1, p_0, p_1 \in \mathbb{R}^3$ be vectors such that $Aq_0 = 0$, $Aq_1 = q_0$, $A^T p_1 = 0$, $A^T p_0 = p_1$, where A^T is the transpose of the matrix A, satisfying the conditions $\langle q_0, p_1\rangle = 0$, $\langle q_1, p_0\rangle = 0$, $\langle q_0, p_0\rangle = 1$ and $\langle q_1, p_1\rangle = 1$. Write the polynomial characteristic of the Jacobian matrix of f at (x, α) as $p(\lambda) = \lambda^3 + R(x, \alpha)\lambda^2 + T(x, \alpha)\lambda + D(x, \alpha)$ and assume that the following conditions hold:

(BT1) The Jacobian matrix satisfies $A \neq 0$;

(BT2)
$$a(\alpha_0) = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle \neq 0; \tag{3.10}$$

(BT3)
$$b(\alpha_0) = \langle p_0, B(q_0, q_0)\rangle + \langle p_1, B(q_0, q_1)\rangle \neq 0; \tag{3.11}$$

(BT4) The map $G : (x, \alpha) \to (f(x, \alpha), T(x, \alpha), D(x, \alpha))$ is regular at (x_0, α_0).

Under the above assumptions the system undergoes a Bogdanov-Takens bifurcation at x_0 for parameters at α_0. The bifurcation diagram of the Bogdanov-Takens bifurcation can be found in [8, p. 322]. The assumption (BT4) is called transversality condition for the Bogdanov-Takens bifurcation while the assumptions (BT1)-(BT3) are the non-degenerescence conditions.

Define $s = \operatorname{sign} a(\alpha_0)b(\alpha_0) = \pm 1$. If $s = -1$ ($s = 1$, resp.) then the limit cycle bifurcating from the Hopf point or from the homoclinic loop is attracting (repelling, resp.).

3.3. **Fold-Hopf bifurcations.** In this subsection a review of the fold-Hopf bifurcation is presented based on [8] and [9]. This kind of bifurcation is also called zero-Hopf bifurcation.

Consider the differential equation (3.1), where $x \in \mathbb{R}^3$ and $\zeta \in \mathbb{R}^n$ are respectively vectors representing phase variables and control parameters. Assume that f is of class C^∞ in $\mathbb{R}^3 \times \mathbb{R}^n$. Suppose that (3.1) has an equilibrium point $x = x_0$ at $\zeta = \zeta_0 = 0$. Denoting the variable $x - x_0$ also by x, we can write (3.2) as

$$F(x) = f(x, 0)$$

where

$$F(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x\|^4),$$

$A = f_x(0,0)$ and, for $i = 1,2,3$,

$$B_i(x,y) = \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial\xi_j \partial\xi_k}\Big|_{\xi=0} x_j y_k, \quad C_i(x,y,z) = \sum_{j,k,l=1}^{3} \frac{\partial^3 F_i(\xi)}{\partial\xi_j \partial\xi_k \partial\xi_l}\Big|_{\xi=0} x_j y_k z_l.$$

Suppose that $(x_0,\zeta_0) = (0,0)$ is an equilibrium point of (3.1) where the Jacobian matrix A has a zero eigenvalue $\lambda_1 = 0$ and a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$. Let $p_0, q_0 \in \mathbb{R}^3$ be vectors such that

$$Aq_0 = 0, \quad A^T p_0 = 0, \quad \langle p_0, q_0\rangle = 1, \tag{3.12}$$

and let $p_1, q_1 \in \mathbb{C}^3$ be vectors such that

$$Aq_1 = i\omega_0 q_1, \quad A^T p_1 = -i\omega_0 p_1, \quad \langle p_1, q_1\rangle = 1, \tag{3.13}$$

where A^T is the transpose of the matrix A. From the above assumptions, it follows that

$$\langle p_1, q_0\rangle = \langle p_0, q_1\rangle = 0.$$

Consider the complex numbers

$$G_{200} = \langle p_0, B(q_0,q_0)\rangle, \tag{3.14}$$

$$G_{110} = \langle p_1, B(q_0,q_1)\rangle, \tag{3.15}$$

$$G_{011} = \langle p_0, B(q_1,\bar{q}_1)\rangle, \tag{3.16}$$

the complex vectors, in $\mathbb{C}^3$,

$$h_{020} = (2i\omega_0 I_3 - A)^{-1} B(q_1,q_1), \tag{3.17}$$

h_{200} the solution of

$$\begin{pmatrix} A & q_0 \\ p_0 & 0 \end{pmatrix} \begin{pmatrix} h_{200} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_0,q_0) + \langle p_0, B(q_0,q_0)\rangle q_0 \\ 0 \end{pmatrix}, \tag{3.18}$$

h_{011} the solution of

$$\begin{pmatrix} A & q_0 \\ p_0 & 0 \end{pmatrix} \begin{pmatrix} h_{011} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_1,\bar{q}_1) + \langle p_0, B(q_1,\bar{q}_1)\rangle q_0 \\ 0 \end{pmatrix}, \tag{3.19}$$

and the vector h_{110} which is solution of

$$\begin{pmatrix} i\omega_0 I_3 - A & q_1 \\ \bar{p}_1 & 0 \end{pmatrix} \begin{pmatrix} h_{110} \\ s \end{pmatrix} = \begin{pmatrix} B(q_0,q_1) - \langle p_1, B(q_0,q_1)\rangle q_1 \\ 0 \end{pmatrix}. \tag{3.20}$$

From the above complex vectors define the complex numbers

$$G_{300} = \langle p_0, C(q_0,q_0,q_0) + 3B(q_0,h_{200})\rangle, \tag{3.21}$$

$$G_{111} = \langle p_0, C(q_0,q_1,\bar{q}_1) + B(q_0,h_{011}) + B(q_1,\bar{h}_{110}) + B(\bar{q}_1,h_{110})\rangle, \tag{3.22}$$

$$G_{210} = \langle p_1, C(q_0,q_0,q_1) + 2B(q_0,h_{110}) + B(q_1,h_{200})\rangle, \tag{3.23}$$

$$G_{021} = \langle p_1, C(q_1,q_1,\bar{q}_1) + 2B(q_1,h_{011}) + B(\bar{q}_1,h_{020})\rangle. \tag{3.24}$$

The theorem about the fold-Hopf bifurcation states that if

(FH1) $b(0)c(0)e(0) \neq 0$,

(FH2) The map $G : (x,\zeta) \mapsto (f(x,\zeta), \operatorname{Tr}(f_x(x,\zeta)), \det(f_x(x,\zeta)))$ is regular at $(x_0,\zeta_0) = (0,0)$,

then (3.1) is locally orbitally smoothly equivalent near the origin to the complex normal form

$$\xi' = \beta_1 + b(\beta)\xi^2 + c(\beta)|\chi|^2 + O(\|(\xi,\chi)\|^4),$$
$$\chi' = (\beta_2 + i\omega(\beta))\chi + d(\beta)\xi\chi + e(\beta)\xi^2\chi + O(\|(\xi,\chi)\|^4),$$

where $\beta = (\beta_1,\beta_2)$, $\omega(0) = \omega_0$,

$$b(0) = \frac{G_{200}}{2}, \quad c(0) = G_{011}, \quad d(0) = G_{110} - i\omega_0 \frac{G_{300}}{3G_{200}} \tag{3.25}$$

and

$$e(0) = \frac{1}{2}\operatorname{Re}\Big(G_{210} + G_{110}\Big(\frac{\operatorname{Re} G_{021}}{G_{011}} - \frac{G_{300}}{G_{200}} + \frac{G_{111}}{G_{011}}\Big) - \frac{G_{021}G_{200}}{2G_{011}}\Big). \tag{3.26}$$

In general the O-terms cannot be truncated. See [8, p. 336.], Depending upon the coefficients $b(0)$, $c(0)$, $d(0)$ and $e(0)$ the system can have two-dimensional invariant tori and even chaotic motions. Define

$$s = \operatorname{sign} b(0)c(0), \quad \theta(0) = \frac{\operatorname{Re} d(0)}{G_{200}}. \tag{3.27}$$

For example, if $s = 1$ and $\theta(0) < 0$ then the system exhibits Hopf bifurcations and torus "heteroclinic destruction" (see [8, p. 341]), giving rise to chaotic invariant sets. The bifurcation diagrams for the fold-Hopf bifurcation can be found in [8, pp. 339–343].

4. Bifurcation analysis of system (1.3)

4.1. Hopf bifurcation analysis at E_0.

In this subsection we study the Hopf bifurcations that occur at the equilibrium E_0 for parameters in the set $\mathcal{H}_0$ defined in (2.4). Define the critical parameter

$$a_{0_c} = \frac{1}{b_0} > 0.$$

Theorem 4.1. *Consider system* (2.2). *The first Lyapunov coefficient at E_0 for parameter values in $\mathcal{H}_0$ is*

$$l_1(a_{0_c}, b_0, a_1, b_1) = \frac{N(a_{0_c}, b_0, a_1, b_1)}{2(b_0 + 5b_0^4 + 4b_0^7)}, \tag{4.1}$$

where

$$N(a_{0_c}, b_0, a_1, b_1) = b_1 - b_0\Big(2 + b_0\Big(16b_0^2 + a_1^2 b_0(-3 + 8b_0^3) - 10b_0b_1 + b_1^2 + a_1\big(1 + 12b_0^2(-2b_0 + b_1)\big)\Big)\Big).$$

If $\zeta_0 = (a_{0_c}, b_0, a_1, b_1) \in \mathcal{H}_0$ is such that $l_1(\zeta_0) \neq 0$ then system (2.2) *has a transversal Hopf point at E_0 for the parameter vector ζ_0.*

Proof. For parameters on the Hopf hypersurface $\mathcal{H}_0$ (2.4), the eigenvalues of the Jacobian matrix of system (2.2) at E_0 are

$$\lambda_1 = -\frac{1}{b_0}, \quad \lambda_{2,3} = \pm i\omega_0, \quad \omega_0 = \sqrt{b_0}, \quad b_0 > 0,$$

the eigenvectors q and p defined in (3.4) are

$$q = \Big(-\frac{1}{b_0}, \frac{-i}{\sqrt{b_0}}, 1\Big), \quad p = \Big(\frac{-ib_0}{2(b_0^{3/2}+i)}, \frac{-i\sqrt{b_0}}{2}, \frac{b_0^{3/2}}{2(b_0^{3/2}+i)}\Big)$$

and the multilinear symmetric functions B and C write as

$$B(x,y) = (0,0,-a_1(x_1y_3+x_3y_1) - b_1(x_1y_2+x_2y_1) - 2x_1y_1)\,, \quad C(x,y,z) = (0,0,0)\,.$$

The complex vectors h_{11} and h_{20} are

$$h_{11} = \Big(\frac{2(-1+a_1b_0)}{b_0^2}, 0, 0\Big),$$

$$h_{20} = \Big(\frac{2(-i+ia_1b_0+\sqrt{b_0}b_1)}{3b_0^2(-i+2b_0^{3/2})}, \frac{4(1-a_1b_0+i\sqrt{b_0}b_1)}{3b_0^{3/2}(-i+2b_0^{3/2})}, \frac{8(i-ia_1b_0-\sqrt{b_0}b_1)}{3b_0(-i+2b_0^{3/2})}\Big).$$

The complex number G_{21} defined in (3.7) has the form

$$\begin{aligned} G_{21} = &\Big(a_1^2b_0^2(i-12b_0^{2/3}) - i\big(-5i+\sqrt{b0}(12b_0-b_1)\big)(-2i+\sqrt{b_0}b_1) \\ &+ a_1b_0(-11i+36b_0^{2/3}+12ib_0^2b_1)\Big)/\Big(-3b_0^{2/3}-9ib_0^3+6b_0^{9/2}\Big). \end{aligned}$$

Performing the calculations in (3.7), the first Lyapunov coefficient is given by (4.1).

It remains only to verify the transversality condition of the Hopf bifurcation. In order to do so, consider the family of differential equations (2.2) regarded as dependent on the parameter a_0. The real part, $\gamma = \gamma(a_0)$, of the pair of complex eigenvalues at the critical parameter $a_0 = a_{0_c}$ verifies

$$\gamma'(a_{0_c}) = \mathrm{Re}\langle p, \frac{dA}{da_0}\Big|_{a_0=a_{0_c}} q\rangle = -\frac{b_0^3}{2(b_0^3+1)} < 0,$$

since $b_0 > 0$. In the above expression A is the Jacobian matrix of system (2.2) at E_0. Therefore, the transversality condition at the Hopf point holds. □

The sign of the first Lyapunov coefficient (4.1) is determined by the sign of the numerator of (4.1), $N(a_{0_c}, b_0, a_1, b_1)$, since the denominator is positive.

If $\zeta_0 = (a_{0_c}, b_0, a_1, b_1) \in \mathcal{H}_0$ is such that $l_1(\zeta_0) \neq 0$ then system (2.2) has a transversal Hopf point at E_0 for the parameter vector ζ_0. More specifically, if $\zeta_0 = (a_{0_c}, b_0, a_1, b_1) \in \mathcal{H}_0$ is such that $l_1(\zeta_0) < 0$ then the Hopf point at E_0 is asymptotically stable (weak attracting focus for the flow of system (2.2) restricted to the center manifold) and for a suitable ζ close to ζ_0 there exists a stable limit cycle near the unstable equilibrium point E_0; if $\zeta_0 = (a_{0_c}, b_0, a_1, b_1) \in \mathcal{H}_0$ is such that $l_1(\zeta_0) > 0$ then the Hopf point at E_0 is unstable (weak repelling focus for the flow of system (2.2) restricted to the center manifold) and for a suitable ζ close to ζ_0 there exists an unstable limit cycle near the asymptotically stable equilibrium point E_0.

In the rest of this subsection we study the stability of the equilibrium E_0 with the restriction $a_1 = 0$. This makes the analysis of the sign as well as the analysis of the zero set of the first Lyapunov coefficient (4.1) more simple. See Remark 4.3. Define the following subset $\mathcal{H}_{00}$ of the Hopf hypersurface $\mathcal{H}_0$

$$\mathcal{H}_{00} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_0 : a_1 = 0\}.$$

Corollary 4.2. *Consider system* (2.2) *with parameter values in* $\mathcal{H}_{00}$. *If either*

$$b_1 = b_{11} = \frac{1+8b_0^3}{b_0^2} \quad or \quad b_1 = b_{12} = 2b_0,$$

then the first Lyapunov coefficient at E_0 *vanishes; that is,*

$$l_1(a_{0_c}, b_0, 0, b_{11}) = l_1(a_{0_c}, b_0, 0, b_{12}) = 0.$$

Proof. Substituting $a_1 = 0$ into the expression of G_{21} in the proof of Theorem 4.1 results

$$\begin{aligned} G_{21} = &-\frac{(2b_0 - b_1)(1+8b_0^3 - b_0^2 b_1)}{b_0 + 5b_0^4 + 4b_0^7} \\ &+ i\frac{-10 + b_0(-52b_0^2 + 3b_0 b_1(1-8b_0^3) + b_1^2(-1+2b_0^3))}{3b_0^{3/2}(1+5b_0^3+4b_0^6)}. \end{aligned}$$

If $b_1 = b_{11}$ then the second parenthesis in the numerator of the real part of G_{21} vanishes. Then $l_1(a_{0_c}, b_0, 0, b_{11}) = 0$. On the other hand, if $b_1 = b_{12}$ then the first parenthesis in the numerator of the real part of G_{21} vanishes. Then $l_1(a_{0_c}, b_0, 0, b_{12}) = 0$. □

From Corollary 4.2 the first Lyapunov coefficient vanishes on the curves

$$\mathcal{L}_1 = \big\{(a_0, b_0, b_1) \in \mathcal{H}_{00} : a_0 = \frac{1}{b_0}, \quad b_1 = \frac{1+8b_0^3}{b_0^2}\big\}$$

and

$$\mathcal{L}_2 = \big\{(a_0, b_0, b_1) \in \mathcal{H}_{00} : a_0 = \frac{1}{b_0}, \quad b_1 = 2b_0\big\}.$$

See Figure 1. It is simple to see that the curves $\mathcal{L}_1$ and $\mathcal{L}_2$ have no intersection and divide the Hopf surface $\mathcal{H}_{00}$ into three connected components

$$\begin{gathered} \mathcal{H}_{01} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_{00} : b_1 > \frac{1+8b_0^3}{b_0^2}\}, \\ \mathcal{H}_{02} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_{00} : 2b_0 < b_1 < \frac{1+8b_0^3}{b_0^2}\}, \\ \mathcal{H}_{03} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_{00} : b_1 < 2b_0\}, \end{gathered}$$

where the sign of the first Lyapunov coefficient at E_0 is fixed: $l_1(a_{0_c}, b_0, 0, b_1) > 0$ on $\mathcal{H}_{02}$ and $l_1(a_{0_c}, b_0, 0, b_1) < 0$ on $\mathcal{H}_{01} \cup \mathcal{H}_{03}$. See Figure 1. The bifurcation diagram for $l_1 < 0$ can be found in [8, p. 161].

Remark 4.3. It is well known that the first Lyapunov coefficient is a continuous function of the parameters. Thus if $\zeta_{00} = (a_{0_c}, b_0, 0, b_1) \in \mathcal{H}_{01}$ then there exists a neighborhood $\mathcal{V}_{\zeta_{00}}$ of ζ_{00} in the Hopf hypersurface $\mathcal{H}_0$ such that $l_1(\zeta_0) < 0$ for all $\zeta_0 \in \mathcal{V}_{\zeta_{00}}$, since $l_1(\zeta_{00}) < 0$. Analogous conclusions hold for the other subsets $\mathcal{H}_{02}$ and $\mathcal{H}_{03}$.

In the next theorem we give the stability of the equilibrium E_0 for parameters in the curve $\mathcal{L}_1$.

Theorem 4.4. *Consider system* (2.2) *with parameter values in* $\mathcal{L}_1$. *Then the second Lyapunov coefficient at* E_0 *is*

$$l_2(a_{0_c}, b_0, 0, b_{11}) = -\frac{9 + 121b_0^3 + 570b_0^6 + 1008b_0^9}{3b_0^5(1+14b_0^3+49b_0^6+36b_0^9)}. \tag{4.2}$$

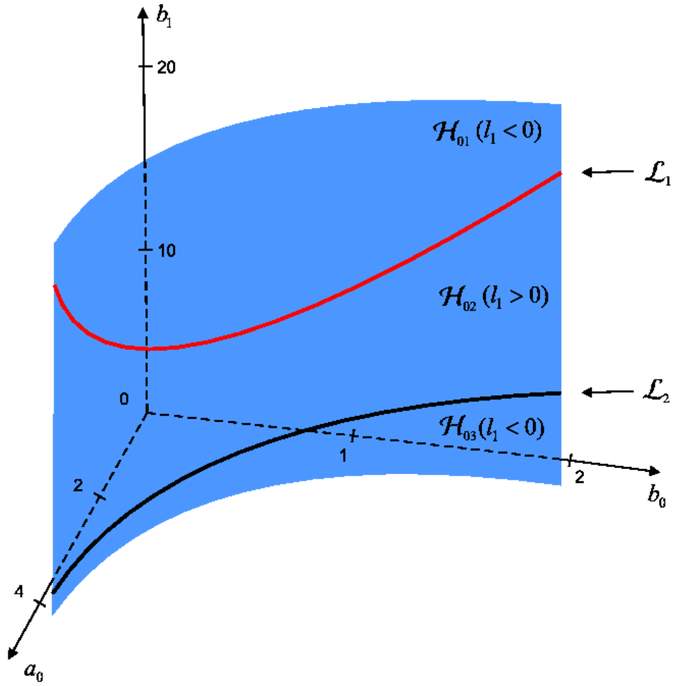

FIGURE 1. The Hopf surface $\mathcal{H}_{00} = \mathcal{H}_0 \cap \{a_1 = 0\}$ for E_0, the sets $\mathcal{H}_{01}$, $\mathcal{H}_{02}$ and $\mathcal{H}_{03}$ and the curves $\mathcal{L}_1$ and $\mathcal{L}_2$

As $b_0 > 0$ *then* $l_2(a_{0_c}, b_0, 0, b_{11}) < 0$ *and system* (2.2) *has a transversal Hopf point of codimension 2 at* E_0 *which is a stable equilibrium point. The bifurcation diagram of system* (2.2) *at a typical point on the curve* $\mathcal{L}_1$ *can be found in* [8, p. 313].

Proof. By Corollary 4.2, for parameters in $\mathcal{L}_1$, $l_1(a_{0_c}, b_0, 0, b_{11}) = 0$. Due to the quadratic nature of the system, the multilinear symmetric functions D and E are

$$D(x, y, z, w) = (0, 0, 0), \quad E(x, y, z, w, r) = (0, 0, 0).$$

The complex vectors h_{11}, h_{20}, h_{21}, h_{22}, h_{30} and h_{31} are

$$h_{11} = \Big(-\frac{2}{b_0^2}, 0, 0\Big),$$

$$h_{20} = \Big(\frac{16b_0^3 - 2ib_0^{3/2} + 2}{6b_0^5 - 3ib_0^{7/2}}, \frac{4\big(8ib_0^3 + b_0^{3/2} + i\big)}{6b_0^{9/2} - 3ib_0^3}, \frac{-64b_0^3 + 8ib_0^{3/2} - 8}{6b_0^4 - 3ib_0^{5/2}}\Big),$$

$$h_{21} = \Big(\frac{i - 3b_0^{3/2} + 16ib_0^3 - 48b_0^{9/2}}{6b_0^6(-i + b_0^{3/2})}, \frac{1 - ib_0^{3/2} + 16b_0^3 - 16ib_o^{9/2}}{6(-ib_0^{11/2} + b_0^7)},$$
$$- \frac{-3i + b_0^{3/2} - 48ib_0^3 + 16b_0^{9/2}}{6b_0^5(-i + b_0^{3/2})}\Big),$$

$$h_{22} = \Big(-\frac{4(5 + 141b_0^3 + 714b_0^6 + 848b_0^9)}{9b_0^7(1 + 5b_0^3 + 4b_0^6)}, 0, 0\Big),$$

$$
\begin{aligned}
h_{30} = \Big(& \frac{3 - 5ib_0^{3/2} + 46b_0^3 - 40ib_0^{9/2} + 192b_0^6}{4b_0^6 + 20ib_0^{15/2} - 24b_0^9}, \\
& - \frac{3i(3 - 5ib_0^{3/2} + 46b_0^3 - 40ib_0^{9/2} + 192b_0^6)}{4b_0^{11/2}(-1 - 5ib_0^{3/2} + 6b_0^3)}, \\
& \frac{9(3 - 5ib_0^{3/2} + 46b_0^3 - 40ib_0^{9/2} + 192b_0^6)}{4b_0^5(-1 - 5ib_0^{3/2} + 6b_0^3)} \Big)
\end{aligned}
$$

and

$$
\begin{aligned}
h_{31} = \Big(& \Big(7i - 65b_0^{3/2} + 133ib_0^3 - 1897b_0^{9/2} - 2050ib_0^6 - 12056b_0^{15/2} \\
& - 31744ib_0^9 + 21504b_0^{21/2}\Big) / \Big(18b_0^{17/2}(i - 2b_0^{3/2})^2(-1 - 4ib_0^{3/2} + 3b_0^3)\Big), \\
& - \Big(1 + 35ib_0^{3/2} + 19b_0^3 + 1099ib_0^{9/2} - 1774b_0^6 + 7256ib_0^{15/2} \\
& - 22720b_0^9 - 16896ib_0^{21/2}\Big) / \Big(9b_0^8(i - 2b_0^{3/2})^2(-1 - 4ib_0^{3/2} + 3b_0^3)\Big), \\
& \Big(2(5i + 5b_0^{3/2} + 95ib_0^3 + 301b_0^{9/2} + 1498ib_0^6 + 2456b_0^{15/2} + 13696ib0^9 \\
& - 12288b_0^{21/2})\Big) / \Big(9b_0^{15/2}(i - 2b_0^{3/2})^2(-1 - 4ib_0^{3/2} + 3b_0^3)\Big)\Big),
\end{aligned}
$$

respectively. From (3.8),

$$
\begin{aligned}
G_{32} = & -\frac{4(9 + 121b_0^3 + 570b_0^6 + 1008b_0^9)}{b_0^5(1 + 14b_0^3 + 49b_0^6 + 36b_0^9)} \\
& - i\frac{(17 + 1214b_0^3 + 21105b_0^6 + 155492b_0^9 + 463040b_0^{12} + 377856b_0^{15})}{36b_0^{19/2}(1 + 14b_0^3 + 49b_0^6 + 36b_0^9)}.
\end{aligned}
$$

Thus, the second Lyapunov coefficient (3.8) is

$$
l_2(a_{0_c}, b_0, 0, b_{11}) = \frac{1}{12}\operatorname{Re} G_{32} = -\frac{9 + 121b_0^3 + 570b_0^6 + 1008b_0^9}{3b_0^5(1 + 14b_0^3 + 49b_0^6 + 36b_0^9)}.
$$

The proof is complete. □

From Theorem 4.4, the sign of the second Lyapunov coefficient at E_0 is always negative on $\mathcal{L}_1$. Thus the equilibrium E_0 is a weak attracting focus (for the flow of system (2.2) restricted to the center manifold) and there are two limit cycles, one stable and the other unstable, near the equilibrium E_0 for suitable value of the parameters. See the pertinent bifurcation diagram in [8, p. 313].

In the next theorem we study the stability of the equilibrium E_0 for parameters in the curve $\mathcal{L}_2$.

Theorem 4.5. *Consider system* (2.2) *with parameter values in* $\mathcal{L}_2$. *See Figure 1. Then the second and third Lyapunov coefficients at* E_0 *vanish; that is,*

$$
l_2(a_{0_c}, b_0, 0, b_{12}) = l_3(a_{0_c}, b_0, 0, b_{12}) = 0.
$$

Proof. By Corollary 4.2, for parameters in $\mathcal{L}_2$, $l_1(a_{0_c}, b_0, 0, b_{12}) = 0$. Due to the quadratic nature of the system, the multilinear symmetric functions D, E, K and L are

$$
D(x, y, z, w) = E(x, y, z, w, r) = K(x, y, z, w, r, s) = L(x, y, z, w, r, s, t) = (0, 0, 0).
$$

The complex vectors h_{11}, h_{20}, h_{21}, h_{22}, h_{30} and h_{31} are

$$h_{11} = \Big(-\frac{2}{b_0^2}, 0, 0\Big), \quad h_{20} = \Big(\frac{2}{3b_0^2}, \frac{4i}{3b_0^{3/2}}, -\frac{8}{3b_0}\Big),$$

$$h_{21} = \Big(-\frac{5(-i+3b_0^{3/2})}{3b_0^3(-i+b_0^{3/2})}, \frac{5(1-ib_0^{3/2})}{3(-ib_0^{5/2}+b_0^4)}, -\frac{5(-3i+ib_0^{3/2})}{3b_0^2(-i+b_0^{3/2})}\Big),$$

$$h_{22} = \Big(-\frac{16(17+32b_0^3)}{9(b_0^4+b_0^7)}, 0, 0\Big), \quad h_{30} = \Big(-\frac{1}{2b_0^3}, \frac{3i}{2b_0^{5/2}}, \frac{9}{2b_0^2}\Big),$$

$$h_{31} = \Big(\frac{89i - 149b_0^{3/2}}{9ib_0^4 - 9b_0^{11/2}}, \frac{118+238ib_0^{3/2}}{9(-ib_0^{7/2}+b_0^5)}, -\frac{4(-29i+89b_0^{3/2})}{9b_0^3(-i+b_0^{3/2})}\Big),$$

respectively. From the above results the complex number G_{32} (3.8) can be written as

$$G_{32} = -\frac{5i(157+277b_0^3)}{9b_0^{7/2}(1+b_0^3)}.$$

By the above expression of G_{32}, $l_2(a_{0_c}, b_0, 0, b_{12}) = \operatorname{Re} G_{32}/12 = 0$.

The complex vectors h_{32}, h_{33}, h_{40}, h_{41} and h_{42} are, respectively,

$$\begin{aligned} h_{32} = \Big(&-\frac{5(-187i + 561b_0^{3/2} - 187ib_0^3 + 1041b_0^{9/2})}{18b_0^5(-i+b_0^{3/2})^2(i+b_0^{3/2})}, \\ &-\frac{5i(187+120ib_0^{3/2}+307b_0^3)}{18b_0^{9/2}(-i+b_0^{3/2})^2}, -\frac{5(-441i+307(b_0^{3/2}-3ib_0^3+b_0^{9/2}))}{18b_0^4(-i+b_0^{3/2})^2(i+b_0^{3/2})}\Big), \end{aligned}$$

$$h_{33} = \Big(-\frac{33137+114154b_0^3+109817b_0^6}{18b_0^6(1+b_0^3)^2}, 0, 0\Big),$$

$$h_{40} = \Big(\frac{4}{9b_0^4}, \frac{16i}{9b_0^{7/2}}, -\frac{64}{9b_0^3}\Big),$$

$$h_{41} = \Big(\frac{109i - 169b_0^{3/2}}{6b_0^5(-i+b_0^{3/2})}, \frac{89+149ib_0^{3/2}}{2b_0^{9/2}(i-b_0^{3/2})}, \frac{9(-23i+43b_0^{3/2})}{2b_0^4(-i+b_0^{3/2})}\Big),$$

$$h_{42} = (h_{42_1}, h_{42_2}, h_{42_3}),$$

where

$$h_{42_1} = \frac{2(-8001i + 14701b_0^{3/2} - 9761ib_0^3 + 22461b_0^{9/2})}{27b_0^6(-i+b_0^{3/2})^2(i+b_0^{3/2})},$$

$$h_{42_2} = \frac{4(4651 + 10151b_0^{3/2} + 5211b_0^3 + 16711b_0^{9/2})}{27b_0^{11/2}(-i+b_0^{3/2})^2(i+b_0^{3/2})},$$

$$h_{42_3} = \frac{8(1901i - 6201b_0^{3/2} + 1261ib_0^3 - 11561b_0^{9/2})}{27b_0^5(-i+b_0^{3/2})^2(i+b_0^{3/2})}.$$

Substituting the above results into the expression of the complex number G_{43} (3.9) and making the simplifications it follows that

$$G_{43} = -\frac{5i(13099+43838b_0^3+40339b_0^6)}{9b_0^{11/2}(1+b_0^3)^2},$$

and, by (3.9), $l_3(a_{0_c}, b_0, 0, b_{12}) = \frac{1}{144}\operatorname{Re} G_{43} = 0$. □

Based on the above theorem we have the following question.

Question 4.6. Consider system (2.2) with parameters in $\mathcal{L}_2$. Is the equilibrium E_0 a center for the flow of system (2.2) restricted to the center manifold?

This question is related with the planar center-focus problem. In his seminal paper Bautin [1] solves the center-focus problem for quadratic systems in the plane: If the three first Lyapunov coefficients are zero at the equilibrium point then it is a center. It is not known an extension of the Bautin's theorem for quadratic systems in $\mathbb{R}^3$.

We have calculated the following Lyapunov coefficient, l_4, at E_0 for parameters in $\mathcal{L}_2$ and it vanishes too. These calculations are not presented here. Based on this information and Theorem 4.5 we have the following question.

Question 4.7. How many limit cycles can bifurcate from E_0 for a suitable perturbation of a parameter vector in $\mathcal{L}_2$?

4.2. **Hopf bifurcation analysis at E_1.** In this subsection we study the Hopf bifurcations that occur at the equilibrium E_1 for parameters in the set $\mathcal{H}_1$ defined in (2.5). Define the critical parameter

$$b_{0_c} = \frac{1}{a_1 - a_0} + b_1.$$

Theorem 4.8. *Consider system* (2.2)*. The first Lyapunov coefficient at E_1 for parameter values in $\mathcal{H}_1$ is*

$$l_1(a_0, b_{0_c}, a_1, b_1) = \frac{D(a_0, b_{0_c}, a_1, b_1)}{2(-4 + (a_0 - a_1)^3)(-1 + (a_0 - a_1)^3)}, \tag{4.3}$$

where

$$\begin{aligned} &D(a_0, b_{0_c}, a_1, b_1)\\ &= a_0(a_0 - a_1)\big(2a_0(-8 + a_0^3) + 8a_1 - 11a_0^3 a_1 + 21a_0^2 a_1^2 - 17a_0 a_1^3 + 5a_1^4\big)\\ &\quad - (a_0 - a_1)^3\big((a_0 - a_1)^4 - 2a_1 - 10a_0\big)b_1 - (a_0 - a_1)^5 b_1^2. \end{aligned}$$

If $\zeta_1 = (a_0, b_{0_c}, a_1, b_1) \in \mathcal{H}_1$ is such that $l_1(\zeta_1) \neq 0$ then system (2.2) *has a transversal Hopf point at E_1 for the parameter vector ζ_1.*

Proof. For parameters on the Hopf hypersurface $\mathcal{H}_1$ we have

$$\begin{gathered} \lambda_1 = a_1 - a_0, \quad \lambda_{2,3} = \pm i\omega_0, \quad \omega_0 = \frac{1}{\sqrt{a_1 - a_0}}, \quad a_1 - a_0 > 0,\\ q = (a_0 - a_1, -i\sqrt{a_1 - a_0}, 1),\\ p = \Big(\frac{\sqrt{a_1 - a_0}}{2(-i - (a_1 - a_0)^{3/2})}, \frac{-i}{2\sqrt{a_1 - a_0}}, \frac{-i}{2(-i - (a_1 - a_0)^{3/2})}\Big),\\ B(x, y) = (0, 0, -a_1(x_1 y_3 + x_3 y_1) - b_1(x_1 y_2 + x_2 y_1) - 2x_1 y_1),\\ C(x, y, z) = (0, 0, 0). \end{gathered}$$

The complex vectors h_{11} and h_{20} are

$$h_{11} = \Big(2a_0(a_0 - a_1), 0, 0\Big),$$

$$h_{20} = \Big(- \frac{2(a_0-a_1)^2(a_0(\sqrt{a_1-a_0}+ib_1)-ia_1b_1)}{6i-3(a_1-a_0)^{3/2}}, \\ - \frac{4(a_0-a_1)^2(ia_0+b_1\sqrt{a_1-a_0})}{6i-3(a_1-a_0)^{3/2}}, \\ - \frac{8(a_0-a_1)(a_0(\sqrt{a_1-a_0}+ib_1)-ia_1b_1}{6i-3(a_1-a_0)^{3/2}}\Big).$$

Substituting the above expressions into (3.7) and making the simplifications, results that the complex number G_{21} is

$$G_{21} = \frac{D^*(a_0,b_{0_c},a_1,b_1)}{3(2+a_0^3-3a_0^2a_1-a_1^3+3ia_1\sqrt{a_1-a_0}+3a_0(a_1^2-i\sqrt{a_1-a_0}))},$$

where

$$\begin{aligned} &D^*(a_0,b_{0_c},a_1,b_1) \\ &= (a_0-a_1)\Big(a_0^3(10i\sqrt{a_1-a_0}-3b_1)+a_1^2b_1(3a_1-ib_1\sqrt{a_1-a_0}) \\ &\quad + a_0^2(-24-19ia_1\sqrt{a_1-a_0}+9a_1b_1-ib_1^2\sqrt{a_1-a_0}) \\ &\quad + a_0\big(9ia_1^2(\sqrt{a_1-a_0}+ib_1)+12ib_1\sqrt{a_1-a_0}+2a_1(6+ib_1^2\sqrt{a_1-a_0})\big)\Big). \end{aligned}$$

Performing the calculations in (3.7), the first Lyapunov coefficient is given by (4.3).

It remains only to verify the transversality condition of the Hopf bifurcation. In order to do so, consider the family of differential equations (2.2) regarded as dependent on the parameter b_0. The real part, $\gamma = \gamma(b_0)$, of the pair of complex eigenvalues at the critical parameter $b_0 = b_{0_c}$ verifies

$$\gamma'(b_{0_c}) = \operatorname{Re}\big\langle p, \frac{dA}{db_0}\Big|_{b_0=b_{0_c}} q\big\rangle = \frac{(a_1-a_0)^2}{2\big((a_1-a_0)^3+1\big)} > 0,$$

since $a_1 - a_0 > 0$. In the above expression A is the Jacobian matrix of system (2.2) at E_1. Therefore, the transversality condition at the Hopf point holds. □

Note that the sign of the first Lyapunov coefficient (4.3) in Theorem 4.8 is determined by the sign of the function $D(a_0, b_{0_c}, a_1, b_1)$, the numerator of l_1, since the denominator is positive.

In the rest of this subsection we study the stability of the equilibrium E_1 with the restriction $a_0 = 0$. This makes the analysis of the sign as well as the analysis of the zero set of the first Lyapunov coefficient (4.3) simpler. See Remark 4.3. Define the following subset of the Hopf hypersurface $\mathcal{H}_1$ for E_1

$$\mathcal{H}_{10} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_1 : a_0 = 0\}.$$

Corollary 4.9. *Consider system (2.2) with parameter values in $\mathcal{H}_{10}$. Then the first Lyapunov coefficient at E_1 is*

$$l_1(0, b_{0_c}, a_1, b_1) = \frac{a_1^4b_1(-2+a_1^3+a_1b_1)}{2(4+5a_1^3+a_1^6)}.$$

If either

$$b_1 = b_{13} = 0, \quad \text{or} \quad b_1 = b_{14} = \frac{2-a_1^3}{a_1},$$

then the first Lyapunov coefficient at E_1 vanishes; that is,

$$l_1(0, b_{0_c}, a_1, b_{13}) = l_1(0, b_{0_c}, a_1, b_{14}) = 0.$$

Proof. Substituting $a_0 = 0$ into the expression of G_{21} in the proof of Theorem 4.8 results

$$G_{21} = \frac{a_1^4 b_1(-2 + a_1^3 + a_1 b_1)}{4 + 5a_1^3 + a_1^6} + i\frac{a_1^{7/2} b_1(2b_1 + a_1^2(9 - a_1 b_1))}{3(4 + 5a_1^3 + a_1^6)}.$$

If $b_1 = b_{13}$, then the numerator of the real part of G_{21} vanishes. Then the first Lyapunov coefficient $l_1(0, b_{0_c}, a_1, b_{13}) = 0$. On the other hand, if $b_1 = b_{14}$ then the parenthesis in the numerator of the real part of G_{21} vanishes. Then $l_1(0, b_{0_c}, a_1, b_{14}) = 0$. □

From Corollary 4.9 the first Lyapunov coefficient vanishes on the curves

$$\mathcal{L}_3 = \{(b_0, a_1, b_1) \in \mathcal{H}_{10} : b_0 = \frac{1}{a_1}, \quad b_1 = 0\},$$

$$\mathcal{L}_4 = \{(b_0, a_1, b_1) \in \mathcal{H}_{10} : b_0 = \frac{3 - a_1^3}{a_1}, \quad b_1 = \frac{2 - a_1^3}{a_1}\}.$$

See Figure 2. These curves have only one intersection point $P_1 = \big((\sqrt[3]{2})^{-1}, \sqrt[3]{2}, 0\big)$ and divide the Hopf surface $\mathcal{H}_{10}$ into four connected components

$$\mathcal{H}_{11} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_{10} : b_1 > 0, \quad b_0 > \frac{3 - a_1^3}{a_1}\},$$

$$\mathcal{H}_{12} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_{10} : b_1 > 0, \quad b_0 < \frac{3 - a_1^3}{a_1}\},$$

$$\mathcal{H}_{13} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_{10} : b_1 < 0, \quad b_0 < \frac{3 - a_1^3}{a_1}\},$$

$$\mathcal{H}_{14} = \{(a_0, b_0, a_1, b_1) \in \mathcal{H}_{10} : b_1 < 0, \quad b_0 > \frac{3 - a_1^3}{a_1}\},$$

where the first Lyapunov coefficient at E_1 has fixed sign: $l_1(0, b_{0_c}, a_1, b_1) > 0$ on $\mathcal{H}_{11} \cup \mathcal{H}_{13}$ and $l_1(0, b_{0_c}, a_1, b_1) < 0$ on $\mathcal{H}_{12} \cup \mathcal{H}_{14}$. See Figure 2.

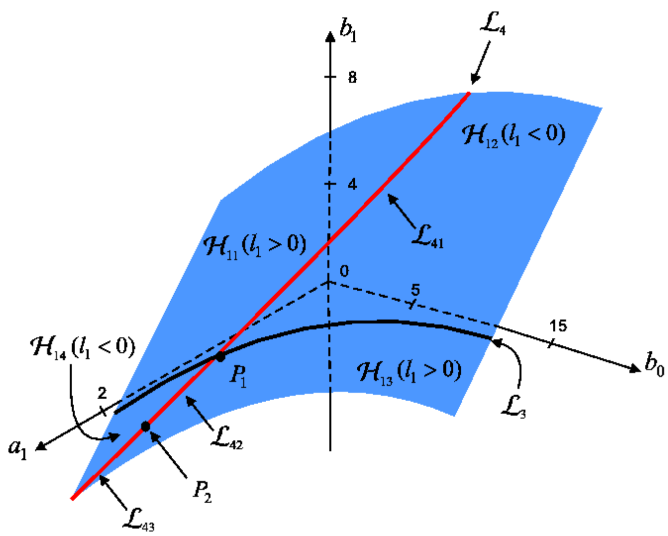

FIGURE 2. The Hopf surface $\mathcal{H}_{10} = \mathcal{H}_1 \cap \{a_0 = 0\}$ for E_1, the sets $\mathcal{H}_{11}$, $\mathcal{H}_{12}$, $\mathcal{H}_{13}$, $\mathcal{H}_{14}$ and the curves $\mathcal{L}_3$ and $\mathcal{L}_4$.

In the next theorem we give the stability of the equilibrium E_1 for parameters in the curve $\mathcal{L}_3$.

Theorem 4.10. *Consider system* (2.2) *with parameter values in* $\mathcal{L}_3$. *Then the second and third Lyapunov coefficients at* E_1 *vanish; that is,*

$$l_2(0, b_{0_c}, a_1, b_{13}) = l_3(0, b_{0_c}, a_1, b_{13}) = 0.$$

Proof. By Corollary 4.9, $l_1(0, b_{0_c}, a_1, b_{13}) = 0$. Due to the quadratic nature of the system the multilinear symmetric functions D, E, K and L satisfy

$$D(x, y, z, w) = E(x, y, z, w, r) = K(x, y, z, w, r, s) = L(x, y, z, w, r, s, t) = (0, 0, 0).$$

For $a_0 = 0$ and $b_1 = b_{13} = 0$ all the complex vectors h_{11}, h_{20}, h_{21}, h_{22}, h_{30}, h_{31}, h_{32}, h_{33}, h_{40}, h_{41} and h_{42} are the zero vector. Therefore, from (3.8) and (3.9), $G_{32} = G_{43} = 0$ and we have $l_2(0, b_{0_c}, a_1, b_{13}) = l_3(0, b_{0_c}, a_1, b_{13}) = 0$. □

Based on the above theorem we have a question analogous to Question 4.6 about the stability of the equilibrium point E_1 for the flow of system (2.2) restricted to the center manifold. Moreover, we can formulate a similar question to Question 4.7 about the number of limit cycles that can bifurcate from E_1 for a suitable perturbation of the parameters.

In the next three theorems we study the stability of the equilibrium E_1 for parameters in the curve $\mathcal{L}_4$.

Theorem 4.11. *Consider system* (2.2) *with parameter values in* $\mathcal{L}_4$. *Then the second Lyapunov coefficient at* E_1 *is*

$$l_2(0, b_{0_c}, a_1, b_{14}) = -\frac{2a_1^5(a_1^3 - 2)(a_1^6 + 22a_1^3 - 105)}{3(36 + a_1^3(7 + a_1^3)^2)}. \tag{4.4}$$

Proof. By Corollary 4.9, $l_1(0, b_{0_c}, a_1, b_{14}) = 0$. Due to the quadratic nature of the system the multilinear symmetric functions D, E, K and L satisfy

$$D(x, y, z, w) = E(x, y, z, w, r) = (0, 0, 0).$$

The complex vectors h_{11}, h_{20}, h_{21}, h_{22}, h_{30} and h_{31} are

$$h_{11} = (0, 0, 0), \quad h_{20} = \Big(\frac{2ia_1^2(a_1^3 - 2)}{3(a_1^{3/2} - 2i)}, -\frac{4a_1^{3/2}(a_1^3 - 2)}{3(a_1^{3/2} - 2i)}, -\frac{8ia_1(a_1^3 - 2)}{3(a_1^{3/2} - 2i)}\Big),$$

$$h_{21} = \Big(-\frac{a_1^3(a_1^{3/2} - 3i)(a_1^3 - 2)}{6(a_1^{3/2} - i)}, \frac{i(-2ia_1^{5/2} - 2a_1^4 + ia_1^{11/2} + a_1^7)}{6(a_1^{3/2} - i)}, -\frac{(a_1^2(3a_1^{3/2} - i)(a_1^3 - 2)}{6(a_1^{3/2} - i)}\Big),$$

$$h_{22} = \Big(\frac{16a_1^4(a_1^3 - 2)}{4 + a_1^3}, 0, 0\Big),$$

$$h_{30} = \Big(\frac{(3a_1^3(a_1^3 - 2)(a_1^3 - 2 - ia_1^{3/2})}{4(-6 - 5ia_1^{3/2} + a_1^3)}, \frac{9a_1^{5/2}(ia_1^3 - 2i + a_1^{3/2})(a_1^3 - 2)}{4(-6 - 5ia_1^{3/2} + a_1^3)}, -\frac{27a_1^2(a_1^3 - 2)(-2 - ia_1^{3/2} + a_1^3)}{4(a_1^3 - 6 - 5ia_1^{3/2})}\Big)$$

and

$$h_{31} = \Big(\frac{(a_1^4(a_1^3-2)(372+370ia_1^{3/2}-150a_1^3-127ia_1^{9/2}+54a_1^6+7ia_1^{15/2})}{18(a_1^{3/2}-2i)^2(-3-4ia_1^{3/2}+a_1^3)},$$
$$-\frac{(a_1^{7/2}(a_1^3-2)(-300i+238a_1^{3/2}+42ia_1^3-49a_1^{9/2}-18ia_1^6+a_1^{15/2})}{9(a_1^{3/2}-2i)^2(-3-4ia_1^{3/2}+a_1^3)},$$
$$\frac{2a_1^3(a_1^3-2)(-228-106ia_1^{3/2}-66a_1^3-29ia_1^{9/2}+18a_1^6+5ia_1^{15/2})}{9(a_1^{3/2}-2i)^2(-3-4ia_1^{3/2}+a_1^3)}\Big),$$

respectively. Substituting the above expressions into (3.8) and making the simplifications it follows that

$$G_{32} = -\frac{8a_1^5(a_1^3-2)(a_1^6+22a_1^3-105)}{36+a_1^3(7+a_1^3)^2}$$
$$- i\frac{(a_1^{7/2}(a_1^3-2)(-20232+17714a_1^3+93a_1^6+180a_1^9+17a_1^{12})}{36(36+a_1^3(7+a_1^3)^2)}.$$

From the expression of G_{32} and (3.8) we have

$$l_2(0,b_{0_c},a_1,b_{14}) = \frac{1}{12}\operatorname{Re} G_{32} = -\frac{2a_1^5(a_1^3-2)(a_1^6+22a_1^3-105)}{3(36+a_1^3(7+a_1^3)^2)}.$$

The proof is complete. □

Remark 4.12. When $a_0 = 0$ we have $a_1 > 0$, since $a_1 - a_0 > 0$ in $\mathcal{H}_1$. So $l_2(0,b_{0_c},a_1,b_{14}) = 0$ if and only if $a_1 = a_{11} = \sqrt[3]{2}$ or $a_1 = a_{12} = \sqrt[3]{\sqrt{226}-11}$.

From Theorem 4.11 and Remark 4.12 it follows that the sets

$$\mathcal{L}_{41} = \{(b_0,a_1,b_1) \in \mathcal{L}_4 : 0 < a_1 < \sqrt[3]{2}\},$$
$$\mathcal{L}_{42} = \{(b_0,a_1,b_1) \in \mathcal{L}_4 : \sqrt[3]{2} < a_1 < \sqrt[3]{\sqrt{226}-11}\},$$
$$\mathcal{L}_{43} = \{(b_0,a_1,b_1) \in \mathcal{L}_4 : a_1 > \sqrt[3]{\sqrt{226}-11}\}$$

are arcs of the curve $\mathcal{L}_4$ where the second Lyapunov coefficient at E_1 is nonzero. More specifically, $l_2(0,b_{0_c},a_1,b_1) < 0$ on $\mathcal{L}_{41} \cup \mathcal{L}_{43}$ and $l_2(0,b_{0_c},a_1,b_1) > 0$ on $\mathcal{L}_{42}$. See Figure 2. At the points

$$P_1 = \Big((\sqrt[3]{2})^{-1}, \sqrt[3]{2}, 0\Big),$$
$$P_2 = \Big(\frac{\sqrt{226}-14}{\sqrt[3]{\sqrt{226}-11}}, \frac{13-\sqrt{226}}{\sqrt[3]{\sqrt{226}-11}}, \sqrt[3]{\sqrt{226}-11}\Big)$$

the second Lyapunov coefficient at E_1 vanishes.

From Theorem 4.11 it follows that the sign of the second Lyapunov coefficient at E_1 is negative on $\mathcal{L}_{41} \cup \mathcal{L}_{43}$. Thus the equilibrium E_1 is a weak attracting focus (for the flow of system (2.2) restricted to the center manifold) and there are two limit cycles, one stable and the other unstable, near the equilibrium E_1 for suitable values of the parameters. On the other hand, the sign of the second Lyapunov coefficient at E_1 is positive on $\mathcal{L}_{42}$. Thus the equilibrium E_1 is a weak repelling focus (for the flow of system (2.2) restricted to the center manifold) and there are two limit cycles, one unstable and the other stable, near the equilibrium E_1 for suitable values of the parameters. See the pertinent bifurcation diagrams in [8, p. 313].

In the next two theorems we study the stability of the equilibrium E_1 for the parameters at P_1 and P_2, respectively.

Theorem 4.13. *Consider system* (2.2) *with parameter values at* P_1*. Then the second and third Lyapunov coefficients at* E_1 *vanish, that is*

$$l_2(P_1) = l_3(P_1) = 0.$$

Proof. Substituting $a_1 = a_{11} = \sqrt[3]{2}$ into (4.4) results $l_2(P_1) = 0$. The calculations to find $l_3(P_1)$ follow the same steps presented in the proof of Theorem 4.10 and will be omitted here. □

Theorem 4.14. *Consider system* (2.2) *with the parameter values at* P_2*. Then the second and third Lyapunov coefficients at* E_1 *are* $l_2(P_2) = 0$ *and*

$$l_3(P_2) = \frac{1728\left(\sqrt{226}-11\right)^{7/3}\left(1775502296303\sqrt{226}-26691643307570\right)}{144\left(430054-28843\sqrt{226}\right)^2\left(72+\sqrt{226}\right)} > 0.$$

Proof. Substituting $a_1 = a_{12}$ into expression (4.4) results $l_2(P_2) = 0$. The value of $l_3(P_2)$ is obtained following the same steps as presented in the proof of Theorem 4.5 and will be omitted here. The value of $l_3(P_2)$ is approximately $2.528833 > 0$ with five decimal round-off coordinates. □

From Theorem 4.14 it follows that the equilibrium E_1 is a weak repelling focus for the flow of system (2.2) restricted to the center manifold and there are three limit cycles, one stable and two unstable, near the equilibrium E_1 for suitable values of the parameters. See the pertinent bifurcation diagram in [17, 19].

4.3. **Genesio system.** Consider the system of quadratic differential equations

$$\begin{aligned} x' &= y,\\ y' &= z,\\ z' &= cz + by + ax + x^2, \end{aligned} \tag{4.5}$$

where (x, y, z) are the state variables and $a < 0$, $b < 0$, $c < 0$ are parameters. System (4.5) is called Genesio system and was studied in [5] from the point of view of its chaotic behavior. In [20] the Hopf bifurcations of system (4.5) were analyzed, but there are errors in the signs of the first Lyapunov coefficient.

System (4.5) can be obtained from system (2.2) taking the following parameters values

$$a_1 = b_1 = 0, \quad a_0 = \frac{c}{\sqrt[3]{a}}, \quad b_0 = -\frac{b}{\sqrt[3]{a^2}}$$

and performing the following change of coordinates and a reparametrization in time

$$x = \frac{X}{a}, \quad y = -\frac{Y}{\sqrt[3]{a^4}}, \quad z = \frac{Z}{\sqrt[3]{a^5}}, \quad t = -\sqrt[3]{a}\tau.$$

Therefore, all the calculations and results obtained in subsections 4.1 and 4.2 for system (2.2) can be applied to system (4.5). In what follows we will concentrate our attention only in the Hopf bifurcations of system (4.5).

It is simple to see that system (4.5) has a Hopf point at $\mathcal{E}_0 = (0,0,0)$ for parameters on the surface

$$\mathcal{H} = \{a = a_c = -bc, b < 0, c < 0\}.$$

By the above change of coordinates and reparametrization in time, in order to study the Hopf point at $\mathcal{E}_0 = (0,0,0)$ for parameters in $\mathcal{H}$ of system (4.5) it is sufficient to study the Hopf point at $E_0 = (0,0,0)$ for parameters in $\mathcal{H}_{00}$ of system (2.2).

The following corollary gives the corrected sign of the first Lyapunov coefficient at $\mathcal{E}_0$ for parameters in $\mathcal{H}$.

Corollary 4.15. *Consider system* (4.5) *with parameters in* $\mathcal{H}$. *Then the first Lyapunov coefficient at* $\mathcal{E}_0$ *is negative and system* (4.5) *has a transversal Hopf point at* $\mathcal{E}_0$ *for all parameters in* $\mathcal{H}$. *More specifically, the Hopf point at* $\mathcal{E}_0$ *is stable (weak attracting focus) and for each* $a < a_c$, *but close to* a_c, *there exists a stable limit cycle near the unstable equilibrium point* $\mathcal{E}_0$.

Proof. It is sufficient to study the sign of the first Lyapunov coefficient at E_0 for parameters in $\mathcal{H}_{00}$ of system (2.2). Now, the expression

$$l_1(a_{0_c}, b_0) = -\frac{1 + 8b_0^3}{1 + 5b_0^3 + 4b_0^6} \tag{4.6}$$

of this first Lyapunov coefficient follows directly from the general expression (4.1) obtained in Theorem 4.1 taking into account $a_1 = b_1 = 0$. The transversality condition is also a consequence of Theorem 4.1. As $b_0 > 0$ then $l_1(a_{0_c}, b_0) < 0$ and system (2.2) has a transversal Hopf point at E_0 for all critical parameters. The corollary is proved. □

4.4. **Bogdanov-Takens bifurcation analysis at E_*.** In this subsection we analyze the Bogdanov-Takens bifurcation at the equilibrium point $E_* = (0,0,0)$ of system (1.3) when the quadratic function h has only one real zero. Without loss of generality, we consider $h(x) = x^2 + c_0$ at $c_0 = 0$. Thus system (1.3) has the form

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -\big((a_1x + a_0)z + (b_1x + b_0)y + x^2 + c_0\big). \end{aligned} \tag{4.7}$$

We have the following theorem.

Theorem 4.16. *System* (4.7) *undergoes a Bogdanov-Takens bifurcation at equilibrium point* $E_* = (0,0,0)$ *for parameter values* $b_0 = c_0 = 0$, $a_0 \neq 0$, $b_1 \neq 2/a_0$ *and* $a_1 \in \mathbb{R}$.

Proof. It is simple to see that $E_* = (0,0,0)$ is the only equilibrium point of system (4.7) when $c_0 = 0$. Take the parameter values $b_0 = c_0 = 0$, $a_0 \neq 0$. The Jacobian matrix of system (4.7) at E_* is written as

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -a_0 \end{pmatrix}$$

and its characteristic polynomial is $p(\lambda) = \lambda^2(\lambda + a_0)$. Thus we have the following eigenvalues $\lambda_1 = -a_0 \neq 0$ and $\lambda_{2,3} = 0$. Consider the vectors

$$q_0 = \big(\frac{1}{a_0}, 0, 0\big), \quad q_1 = \big(0, \frac{1}{a_0}, 0\big), \quad p_0 = \big(a_0, 0, -\frac{1}{a_0}\big), \quad p_1 = \big(0, a_0, 1\big).$$

It follows that

$$Aq_0 = 0, \quad Aq_1 = q_0, \quad A^T p_1 = 0, \quad A^T p_0 = p_1,$$
$$\langle q_1, p_1\rangle = \langle q_0, p_0\rangle = 1, \quad \langle q_1, p_0\rangle = \langle q_0, p_1\rangle = 0.$$

The bilinear symmetric function is written as

$$B(x, y) = (0, 0, -a_1(x_1y_3 + x_3y_1) - b_1(x_1y_2 + x_2y_1) - 2x_1y_1).$$

From (3.10) and (3.11) and the previous calculations we have

$$a = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle = \frac{-1}{a_0^2} \neq 0,$$
$$b = \langle p_0, B(q_0, q_0)\rangle + \langle p_1, B(q_0, q_1)\rangle = \frac{2 - a_0b_1}{a_0^3} \neq 0,$$

since $b_1 \neq 2/a_0$. Therefore, conditions (BT1), (BT2) and (BT3) are satisfied. See subsection 3.2. It remains to prove the transversality condition (BT4). Define the map

$$G : (x, y, z, b_0, c_0) \mapsto (f_1, f_2, f_3, T, D)(x, y, z, b_0, c_0).$$

The transversality condition (BT4) is satisfied if the map G is regular at $(0, 0, 0, 0, 0)$. Now, the determinant of the derivative of G at $(0, 0, 0, 0, 0)$ is

$$\det DG(0, 0, 0, 0, 0) = 2 \neq 0,$$

proving the regularity of G at $(0, 0, 0, 0, 0)$. The theorem is proved. □

The number a is negative and, from the assumption $b_1 \neq 2/a_0$, it follows that $b \neq 0$. Therefore, the sign s of the product ab is determined by the sign of $b_1 - 2/a_0$. Therefore it is possible to choose parameters for which $s = 1$ or $s = -1$. Recall that the sign s determines the stability of the limit cycle that bifurcates from the Hopf point or from the homoclinic loop. See subsection 3.2.

4.5. **Fold-Hopf bifurcation analysis at E_*.** In this subsection we analyze the fold-Hopf bifurcation at the equilibrium point $E_* = (0, 0, 0)$ of system (1.3) when the quadratic function h has only one real zero. Without loss of generality, we consider $h(x) = x^2 + c_0$ at $c_0 = 0$. Thus system (1.3) has the form presented in (4.7). We have the following theorem.

Theorem 4.17. *System* (4.7) *undergoes a fold-Hopf bifurcation at the equilibrium point* $E_* = (0, 0, 0)$ *for parameter values*

$$a_0 = c_0 = 0, \quad b_0 > 0, \quad b_1 \neq 0, \quad a_1 \notin \Big\{\frac{2}{b_0}, \frac{1}{b_0}, \frac{9}{10b_0}, 0\Big\}.$$

Proof. It is easy to see that $E_* = (0, 0, 0)$ is the only equilibrium point of system (4.7) when $c_0 = 0$. Take the parameter values $a_0 = c_0 = 0$, $b_0 > 0$. The Jacobian matrix of system (4.7) at E_* is written as

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b_0 & 0 \end{pmatrix}$$

and its characteristic polynomial is $p(\lambda) = \lambda(\lambda^2 + b_0)$. Thus we have the following eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 = \sqrt{b_0}$. Consider the vectors

$$q_0 = \big(\frac{1}{b_0}, 0, 0\big), \quad q_1 = \big(1, i\sqrt{b_0}, -b_0\big), \quad p_0 = \big(b_0, 0, 1\big), \quad p_1 = \big(0, \frac{i}{2\sqrt{b_0}}, \frac{-1}{2b_0}\big).$$

It follows that

$$Aq_0 = 0, \quad Aq_1 = i\omega_0 q_1, \quad A^T p_0 = 0, \quad A^T p_1 = -i\omega_0 p_1,$$
$$\langle p_0, q_0\rangle = \langle p_1, q_1\rangle = 1, \quad \langle p_1, q_0\rangle = \langle p_0, q_1\rangle = 0.$$

The multilinear symmetric functions B and C are written as

$$B(x,y) = (0, 0, -a_1(x_1y_3 + x_3y_1) - b_1(x_1y_2 + x_2y_1) - 2x_1y_1),$$
$$C(x,y,z) = (0,0,0).$$

Performing the calculations, the numbers G_{200}, G_{110} and G_{011} defined in (3.14), (3.15), (3.16), respectively, are

$$G_{200} = \frac{-2}{b_0^2}, \quad G_{110} = \frac{2 - a_1b_0 + ib_1\sqrt{b_0}}{2b_0^2}, \quad G_{011} = 2a_1b_0 - 2.$$

From (3.17), (3.18), (3.19) and (3.20), the complex vectors h_{200}, h_{020}, h_{110} and h_{011} can be written as

$$h_{200} = \Big(0, -\frac{2}{b_0^3}, 0\Big),$$
$$h_{020} = \Big(\frac{ia_1b_0 + b_1\sqrt{b_0} - i}{3b_0^{3/2}}, \frac{-2a_1b_0 + 2ib_1\sqrt{b_0} + 2}{3b_0}, -\frac{4i\left(a_1b_0 - ib_1\sqrt{b_0} - 1\right)}{3\sqrt{b_0}}\Big),$$
$$h_{110} = \Big(-\frac{3i\left(a_1b_0 - ib_1\sqrt{b_0} - 2\right)}{4b_0^{5/2}}, \frac{a_1b_0 - ib_1\sqrt{b_0} - 2}{4b_0^2}, -\frac{i\left(a_1b_0 - ib_1\sqrt{b_0} - 2\right)}{4b_0^{3/2}}\Big),$$
$$h_{011} = \Big(0, 2a_1 - \frac{2}{b_0}, 0\Big).$$

Performing the calculations of the numbers G_{300} (3.21), G_{111} (3.22), G_{210} (3.23) and G_{021} (3.24), respectively, we have

$$G_{300} = \frac{6b_1}{b_0^4}, \quad G_{111} = \frac{(3 - 2a_1b_0)b_1}{b_0^2},$$
$$G_{210} = -\frac{i(a_1^2b_0^2 + b_1^2b_0 + 4a_1b_0 - 12ib_1\sqrt{b_0} - 12)}{4b_0^{9/2}},$$
$$G_{021} = -\frac{i\big(5a_1^2b_0^2 - b_1^2b_0 - 9ib_1\sqrt{b_0} + a_1(6ib_0^{3/2}b_1 - 7b_0) + 2\big)}{6b_0^{5/2}}.$$

Therefore, the numbers $b(0)$, $c(0)$, $d(0)$ defined in (3.25) are

$$b(0) = -\frac{1}{b_0^2}, \quad c(0) = 2(a_1b_0 - 1), \quad d(0) = \frac{-a_1b_0 + 3ib_1\sqrt{b_0} + 2}{2b_0^2},$$

while the number $e(0)$ defined in (3.26) can be written as

$$e(0) = \frac{a_1(9 - 10a_1b_0)b_1}{16b_0^3(a_1b_0 - 1)}.$$

The number $b(0)$ is negative and, from the assumption $a_1 \neq 1/b_0$, it follows that $c(0) \neq 0$. Therefore, the sign s of the product $b(0)c(0)$ is determined by the sign of $a_1b_0 - 1$. On the other hand, from our assumptions it follows that $e(0) \neq 0$ and its sign can be determined easily if we fix some parameters. So (FH1) is satisfied.

It remains to prove the transversality condition (FH2) which is equivalent to the nonvanishing of $\det DG(x, y, z, a_0, c_0)$ evaluated at $(x, y, z, a_0, c_0) = (0, 0, 0, 0, 0)$, where the map G is defined by

$$G(x, y, z, a_0, c_0) = (f(x, y, z, a_0, c_0), \operatorname{Tr}(f_x(x, y, z, a_0, c_0)), \det(f_x(x, y, z, a_0, c_0))).$$

By simple calculations it follows that $\det DG(0, 0, 0, 0, 0) = 2 \neq 0$. Finally, from (3.27) we have

$$\theta(0) = \frac{1}{4}(a_1 b_0 - 2) \neq 0.$$

The proof is complete. □

It is possible to choose parameters so that $s = 1$ and $\theta(0) < 0$. For example, taking $0 < a_1 < 1/b_0$, $b_0 > 0$, it follows that $0 < a_1 < 2/b_0$ and therefore $s = 1$ and $\theta(0) < 0$. Thus a nontrivial invariant set bifurcates from the equilibrium under variation of the parameters. See [8, pp. 341–343].

5. Concluding remarks

This paper starts with the stability analysis which accounts for the characterization, in the space of parameters, of the structural as well as Lyapunov stability of the equilibria of system (1.3). It continues, after a suitable choice of parameters, with recounting the extension of the analysis to the first order, codimension one stable points, happening on the complement of the curves $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ (see Figures 1 and 2) in the critical surfaces $\mathcal{H}_{00}$ and $\mathcal{H}_{10}$ where the criterium of Lyapunov holds based on the calculation of the first Lyapunov coefficient. Here the bifurcation analysis at the equilibrium points of system (2.2) is pushed forward to the calculation of the second and third Lyapunov coefficients which make possible the determination of the Lyapunov as well as higher order structural stability at the equilibrium points E_0 and E_1. See Theorems 4.4, 4.5, 4.10, 4.11, 4.13, 4.14.

With the analytic data provided in the analysis performed here, the bifurcation diagrams can be established along the points of the curves $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ where the first Lyapunov coefficient vanishes. These bifurcation diagrams provide a qualitative synthesis of the dynamical conclusions achieved here at the parameter values where system (2.2) achieves most complex equilibrium points.

Concerning with the vanishing of the Lyapunov coefficients in a quadratic system (see Theorems 4.5 and 4.10) a question about the stability of the equilibria E_0 and E_1 is formulated. See Question 4.6. Another question (see Question 4.7) about the number of small limit cycles that can bifurcate from the equilibria E_0 and E_1, for a suitable perturbation of the parameters, is also presented.

Two other codimension 2 bifurcations are also analyzed: Bogdanov-Takens and fold-Hopf bifurcations. See Theorems 4.16 and 4.17. With the analytic data provided here, the bifurcation diagrams can be established leading to the existence of global bifurcations such as homoclinic ones. There is also the possibility of torus bifurcation.

Finally, we would like to stress that although this work ultimately focuses a quadratic three dimensional system of differential equations (1.3), the method of analysis and calculations explained in Section 4 can be adapted to the study of other polynomial systems. A cubic three dimensional system analogous to (1.3) will be the subject of a future work.

Acknowledgements. This work was done under the project APQ-01511-09 from FAPEMIG .

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