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EXISTENCE OF POSITIVE SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS AND SIGN CHANGING NONLINEARITIES

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Abstract. In this article, we show the existence of positive solutions for boundary value problems with integral boundary conditions and sign changing nonlinearities. By using a fixed point theorem in double cones, we obtain sufficient conditions for the existence of two positive solutions.

1. Introduction

The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can all be reduced to nonlocal problems with integral boundary conditions. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to [7, 10, 11] and the references therein. For more information about the general theory of integral equations and their relation with boundary value problems we refer to [1, 4].

Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention. To identify a few, we refer the reader to [2, 9, 12, 13, 14, 15] and references therein. But the corresponding theory for boundary-value problem with integral boundary conditions and sign changing nonlinearities of one-dimensional p-Laplacian is not investigated till now. By using the fixed point theorem in double cones, Guo in [8] discussed the existence of positive solutions for second-order three-point boundary-value problem

$$x'' + f(t,x) = 0, \quad 0 \le t \le 1,$$
$$x(0) - \beta x'(0) = 0, \quad x(1) = \alpha x(\eta),$$

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where f is allowed to change sign. Sufficient conditions are obtained by imposing growth conditions on f which ensure the existence of at least two positive solutions for the above boundary value problems. Meanwhile, they proved a fixed point theorem in double cones which generalize the fixed point theorem in a cones to some degree. By using a theorem similar to the one in [8], Cheung [3] proved the existence of two positive solutions for the problem

$$(\Phi_p(u'))' + h(t)f(t,u) = 0, \quad 0 < t < 1,$$

with each of the following two sets of boundary conditions

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i);$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = 0,$$

where $h : [0,1] \to \mathbb{R}^+$ and $f : [0,1] \times [0,\infty) \to \mathbb{R}$ are continuous functions.

Recently, by using the classical fixed-point index theorem for compact maps, Feng [5] established sufficient conditions for the existence of multiple positive solutions for the second-order impulsive differential equations, with p-Laplacian and integral boundary conditions,

$$\begin{gathered} -(\phi_p(u'(t)))' = f(t,u(t)), \quad t \neq t_k,\ t \in (0,1) \\ -\Delta u|_{t=t_k} = I_k(u(t_k)), \quad t = 1,2,\dots,n, \\ u'(0) = 0, \quad u(1) = \int_0^1 g(t)u(t)dt\,. \end{gathered}$$

Motivated by the above, we consider the existence of positive solutions for the boundary-value problem, with integral boundary conditions and sign changing nonlinearities of one-dimensional p-Laplacian,

$$\begin{gathered} (\varphi_p(u'))' + f(t,u) = 0, \quad 0 \le t \le 1, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ u(1) = \int_0^1 g(s)u(s)ds, \end{gathered} \tag{1.1}$$

where $a, b \in [0,+\infty)$, $a_i \in (0,+\infty)$, $i = 1,2,\dots,m-2$, $0 < \xi_1 < \dots < \xi_{m-2} < 1$, $m \ge 3$, $\varphi_p(u) = |u|^{p-2}u, p > 1$, $\varphi_q = (\varphi_p)^{-1}$, and $\frac{1}{q} + \frac{1}{p} = 1$.

We will assume the following conditions:

- (H1) $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous;
- (H2) $f(t,0) \ge 0$ ($f \not\equiv 0$) for $t \in [0,1]$;
- (H3) $a > \sum_{i=1}^{m-2} a_i$; g is non-negative, integrable, and $\sigma := \int_0^1 g(t)dt \in [0,1)$.

2. Preliminaries

For a cone K in a Banach space X with norm $\|\cdot\|$ and a constant $r > 0$, let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$. Suppose $\alpha : K \to [0,+\infty)$ is a continuously increasing functional, i.e., α is continuous and $\alpha(\lambda x) \le \lambda \alpha(x)$ for $\lambda \in (0,1)$. Denote $K(b) = \{x \in K : \alpha(x) < b\}$, $\partial K(b) = \{x \in K : \alpha(x) = b\}$ and $K_a(b) = \{x \in K : a < \|x\|, \alpha(x) < b\}$.

Lemma 2.1 ([8]). *Let X be a real Banach space with norm $\|\cdot\|$ and $K, K' \subset X$ two cones with $K' \subset K$. Suppose $T : K \to K$ and $T^* : K' \to K'$ are two completely continuous operators and $\alpha : K' \to \mathbb{R}^+$ a continuously increasing functional satisfying $\alpha(x) \le \|x\| \le M\alpha(x)$ for all x in K', where $M \ge 1$ is a constant. If there are constants $b > a > 0$ such that*

- (C1) $\|Tx\| < a$, *for* $x \in \partial K_a$;
- (C2) $\|T^*x\| < a$, *for* $x \in \partial K'_a$ *and* $\alpha(T^*x) > b$ *for* $x \in \partial K'(b)$;
- (C3) $Tx = T^*x$, *for* $x \in \partial K'_a(b) \cap \{u : T^*u = u\}$,

then T has at least two fixed points y_1 and y_2 in K such that

$$0 \le \|y_1\| < a < \|y_2\|, \quad \alpha(y_2) < b.$$

We also denote $C^+[0,1] = \{u \in C[0,1] : u(t) \ge 0, t \in [0,1]\}$.

Lemma 2.2. *For any $h \in C^+[0,1]$, the boundary-value problem*

$$\begin{gathered}(\varphi_p(u'))' + h(t) = 0, \quad 0 \le t \le 1, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ u(1) = \int_0^1 g(s)u(s)ds,\end{gathered} \tag{2.1}$$

has a unique solution of the form

$$u(t) = \frac{b\varphi_q(A_h) + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(A_h - \int_0^s h(\tau)d\tau)ds}{a - \sum_{i=1}^{m-2} a_i} + \int_0^t \varphi_q(A_h - \int_0^s h(\tau)d\tau)ds, \tag{2.2}$$

or

$$u(t) = \frac{\int_0^1 g(s)(\int_s^1 \varphi_q(\int_0^r h(\tau)d\tau - A_h)dr)ds}{1 - \int_0^1 g(s)ds} + \int_t^1 \varphi_q(\int_0^s h(\tau)d\tau - A_h)ds, \tag{2.3}$$

where A_h satisfies

$$\begin{aligned}b\varphi_q(A_h) &= \frac{(a - \sum_{i=1}^{m-2} a_i)\int_0^1 g(s)(\int_s^1 \varphi_q(\int_0^r h(\tau)d\tau - A_h)dr)ds}{1 - \int_0^1 g(s)ds} \\ &\quad + a\int_0^1 \varphi_q(\int_0^s h(\tau)d\tau - A_h)ds - \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \varphi_q(\int_0^s h(\tau)d\tau - A_h)ds.\end{aligned} \tag{2.4}$$

Proof. If u is a solution of (2.1), it is easy to see that

$$\varphi_p(u'(t)) - \varphi_p(u'(0)) = -\int_0^t h(s)ds.$$

Let $A_h = \varphi_p(u'(0))$. Integrating, we can have

$$u'(t) = \varphi_q(A_h - \int_0^t h(s)ds). \tag{2.5}$$

Integrating from t to 1, we have

$$u(t) = u(1) - \int_t^1 \varphi_q(A_h - \int_0^s h(\tau)d\tau)ds.$$

Substituting $u(1)=\int_0^1 g(s)u(s)ds$, we obtain

$$u(t)=\int_0^1 g(s)u(s)ds-\int_t^1 \varphi_q(A_h-\int_0^s h(\tau)d\tau)ds.$$

Using the above equation and $u(1)=\int_0^1 g(s)u(s)ds$, we have

$$\int_0^1 g(s)u(s)ds\int_0^1 g(s)ds-\int_0^1 g(s)(\int_s^1 \varphi_q(A_h-\int_0^r h(\tau)d\tau)dr)ds=\int_0^1 g(s)u(s)ds,$$

i.e.,

$$\int_0^1 g(s)u(s)ds=\frac{\int_0^1 g(s)(\int_s^1 \varphi_q(\int_0^r h(\tau)d\tau-A_h)dr)ds}{1-\int_0^1 g(s)ds}.$$

So

$$u(t)=\frac{\int_0^1 g(s)(\int_s^1 \varphi_q(\int_0^r h(\tau)d\tau-A_h)dr)ds}{1-\int_0^1 g(s)ds}+\int_t^1 \varphi_q(\int_0^s h(\tau)d\tau-A_h)ds. \quad (2.6)$$

Integrating (2.5) from 0 to t, we obtain

$$u(t)=u(0)+\int_0^t \varphi_q(A_h-\int_0^s h(\tau)d\tau)ds.$$

Using the boundary condition $au(0)-bu'(0)=\sum_{i=1}^{m-2}a_iu(\xi_i)$, we have

$$u(t)=\frac{b\varphi_q(A_h)+\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_q(A_h-\int_0^s h(\tau)d\tau)ds}{a-\sum_{i=1}^{m-2}a_i}+\int_0^t \varphi_q(A_h-\int_0^s h(\tau)d\tau)ds,$$

where A_h can be easily obtained from the boundary condition $au(0)-bu'(0)=\sum_{i=1}^{m-2}a_iu(\xi_i)$ and (2.6). □

Lemma 2.3. *Assume that* (H3) *holds. Then for each $h\in C^+[0,1]$, there exists a unique $A_h\in(0,\int_0^1 h(\tau)d\tau)$ satisfying* (2.2) *and* (2.3). *Moreover, there is a unique $\sigma_h\in(0,1)$ such that $A_h=\int_0^{\sigma_h}h(\tau)d\tau$.*

Proof. For $h\in C^+[0,1]$, define

$$\begin{aligned}H_h(\eta)&=b\varphi_q(\eta)-\frac{(a-\sum_{i=1}^{m-2}a_i)\int_0^1 g(s)(\int_s^1\varphi_q(\int_0^r h(\tau)d\tau-\eta)dr)ds}{1-\int_0^1 g(s)ds}\\&\quad-a\int_0^1\varphi_q(\int_0^s h(\tau)d\tau-\eta)ds+\sum_{i=1}^{m-2}a_i\int_{\xi_i}^1\varphi_q(\int_0^s h(\tau)d\tau-\eta)ds,\end{aligned}$$

where $\eta\in(-\infty,+\infty)$. Hence

$$\begin{aligned}H_h(\eta)&=(\sum_{i=1}^{m-2}a_i-a)\int_0^1\varphi_q(\int_0^s h(\tau)d\tau-\eta)ds-\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_q(\int_0^s h(\tau)d\tau-\eta)ds\\&\quad+b\varphi_q(\eta)-\frac{(a-\sum_{i=1}^{m-2}a_i)\int_0^1 g(s)(\int_s^1\varphi_q(\int_0^r h(\tau)d\tau-\eta)dr)ds}{1-\int_0^1 g(s)ds}.\end{aligned}$$

By the properties of the function $\varphi_q(\cdot)$, we know that $H_h(\eta)$ is strictly increasing function on $(-\infty,+\infty)$. We also easily know that $H_h(0)<0$, $H_h(\int_0^1 h(\tau)d\tau)>0$, so we can prove the lemma by the continuity of H_h. Namely, there is a unique $A_h\in(0,\int_0^1 h(\tau)d\tau)$ satisfying (2.2) and (2.3). Meanwhile, we could have that there is unique $\sigma_h \in (0,1)$ such that $A_h = \int_0^{\sigma_h} h(\tau)d\tau$ by the continuity of $\int_0^t h(\tau)d\tau$ on $[0,1]$. ☐

Lemma 2.4. *Assume that* (H3) *holds. If $h \in C^+[0,1]$, then the solution of boundary value problem* (2.1) *has the following properties:*

(i) *$u(t)$ is a concave function;*
(ii) *$u(t) \geq 0$, $t \in [0,1]$;*
(iii) *$u(\sigma_h) = \max_{0\leq t\leq 1} u(t)$, where σ_h is defined in lemma 2.3.*

Proof. Suppose that $u(t)$ is the solution of (2.1).

(i) From the fact that $(\varphi_p(u'))'(t) = -h(t) \leq 0$ we know that $\varphi_p(u')$ is non-increasing. It follows that $u'(t)$ is also non-increasing. Thus, $u(t)$ is concave down on $[0,1]$.

(ii) From (i), we know that the minimum of $u(t)$ is obtained at 0 or 1. So we only have to prove that $u(0) \geq 0$ and also $u(1) \geq 0$.

Firstly, we prove that $u(0) \geq 0$. If $u(0) < 0$ and $u'(0) \geq 0$, by the condition $au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, we have

$$au(0) - \sum_{i=1}^{m-2} a_i u(\xi_i) = bu'(0) \geq 0,$$

so

$$au(0) \geq \sum_{i=1}^{m-2} a_i u(\xi_i). \tag{2.7}$$

From $a > \sum_{i=1}^{m-2} a_i > 0$, we have

$$au(0) < \sum_{i=1}^{m-2} a_i u(0). \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\sum_{i=1}^{m-2} a_i u(0) > \sum_{i=1}^{m-2} a_i u(\xi_i).$$

Hence, there is a j, $1 \leq j \leq m-2$ such that $0 > u(0) > u(\xi_j)$. Then we obtain that $u(1) = \min_{t\in[0,1]} u(t) < 0$ and that there is $\eta_1 \leq \eta_2 \in [0,1]$ such that $u(t) \leq 0$ when $t \in [0,\eta_1] \cup [\eta_2, 1]$ and $u(t) \geq 0$ when $t \in [\eta_1, \eta_2]$.

From the boundary condition $u(1) = \int_0^1 g(s)u(s)ds$, we have

$$\begin{aligned} |u(1)| = -u(1) &= -\int_0^{\eta_1} g(s)u(s)ds - \int_{\eta_1}^{\eta_2} g(s)u(s)ds - \int_{\eta_2}^1 g(s)u(s)ds \\ &\leq \int_0^{\eta_1} g(s)|u(s)|ds + \int_{\eta_1}^{\eta_2} g(s)|u(s)|ds + \int_{\eta_2}^1 g(s)|u(s)|ds \\ &= |u(1)| \int_0^1 g(s)ds. \end{aligned} \tag{2.9}$$

We have $\int_0^1 g(s)ds \geq 1$ which is contradiction to with (H3), so $u(0) \geq 0$.

If $u(0) < 0$ and $u'(0) < 0$, we can prove that $u(0) \geq 0$ by using the same way as in (2.9). Similarly, we can prove $u(1) \geq 0$. Therefore, we obtain that $u(t) \geq 0$ for $t \in [0,1]$.

(iii) From the boundary conditions, we can easily show that the maximum of $u(t)$ can not be at 0 or 1. If $u(0)$ is the maximum, then $u'(0) \le 0$ which is contradicted with boundary condition $au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$. If $u(1)$ is the maximum, it is contradicts the boundary condition $u(1) = \int_0^1 g(s)u(s)ds$. □

From Lemmas 2.2 and 2.3, we have $u'(t) = \varphi_q(\int_t^{\sigma_h} h(\tau)d\tau)$. By the concavity of $u(t)$, we know $u(\sigma_h) = \max_{0\le t\le 1} u(t)$.

Lemma 2.5. *If $u \in C[0,1]$ and $u(t) \ge 0$ is a concave function, then for any $\delta \in (0, \frac{1}{2})$, we have*

$$\min_{\delta\le t\le 1-\delta} u(t) \ge \delta\|u\|.$$

Proof. Let $u(\sigma) = \|u\|$. We discuss three cases:

(i) If $\sigma < \delta$, then $\min_{\delta\le t\le 1-\delta} u(t) = u(1-\delta)$. By the concavity of $u(t)$, we have

$$\frac{u(\sigma) - u(1)}{1-\sigma} \le \frac{u(1-\delta) - u(1)}{\delta};$$

i.e.,

$$\frac{u(1-\delta)}{\delta} \ge \frac{u(\sigma)}{1-\sigma} + (\frac{1}{\delta} - \frac{1}{1-\sigma})u(1).$$

For $\sigma < \delta < 1-\delta$, we have $1-\sigma > \delta$. So

$$u(1-\delta) \ge \frac{\delta}{1-\sigma} u(\sigma) \ge \delta u(\sigma).$$

(ii) If $\sigma > 1-\delta$, then $\min_{\delta\le t\le 1-\delta} u(t) = u(\delta)$ and $\sigma > \delta$. By the concavity of $u(t)$, we have

$$\frac{u(\sigma) - u(0)}{\sigma} \le \frac{u(\delta) - u(0)}{\delta};$$

i.e.,

$$\frac{u(\delta)}{\delta} \ge \frac{u(\sigma)}{\sigma} + (\frac{1}{\delta} - \frac{1}{\sigma})u(0).$$

So

$$u(\delta) \ge \frac{\delta}{\sigma} u(\sigma) \ge \delta u(\sigma).$$

(iii) For $\delta \le \sigma \le 1-\delta$, if $\min_{\delta\le t\le 1-\delta} u(t) = u(1-\delta)$, the proof is the same as (i); if $\min_{\delta\le t\le 1-\delta} u(t) = u(\delta)$, the proof is the same as (ii).

From the three cases above, the proof is complete. □

Let $X = C[0,1]$ and $\|u\| = \max_{0\le t\le 1} |u(t)|$, denote $K = \{u \in X : u(t) \ge 0, t \in [0,1]\}$ and $K' = \{u \in K : u(t)$ is a concave function on $[0,1]\}$. Obviously, $K, K' \subset X$ are two cones of X with $K' \subset K$. For $u \in K$, from Lemma 2.5 we know $\min_{t\in[\frac{1}{k}, 1-\frac{1}{k}]} u(t) \ge \frac{1}{k}\|u\|$ with $k > \max\{\frac{1}{\xi_1}, \frac{2}{1-\xi_{m-2}}\}$, and we can show $1 - \frac{1}{k} > \xi_1$.

Denote

$$\alpha(u) = \min_{t\in[\frac{1}{k}, 1-\frac{1}{k}]} u(t), \quad \text{for } u \in K'.$$

We have $\alpha(u) \le \|u\| \le k\alpha(u)$. Define $T : K \to K$ by

$$Tu(t) = \begin{cases} \Big[\frac{b\varphi_q(\int_0^{\sigma_u} f(\tau,u(\tau))d\tau)+\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_s^{\sigma_u} f(\tau,u(\tau))d\tau)ds}{a-\sum_{i=1}^{m-2} a_i} \\ + \int_0^t \varphi_q(\int_s^{\sigma_u} f(\tau,u(\tau))d\tau)ds\Big]^+, & 0 \le t \le \sigma_u, \\ \Big[\frac{\int_0^1 g(s)(\int_s^1 \varphi_q(\int_{\sigma_u}^r f(\tau,u(\tau))d\tau)dr)ds}{1-\int_0^1 g(s)ds} \\ + \int_t^1 \varphi_q(\int_{\sigma_u}^s f(\tau,u(\tau))d\tau)ds\Big]^+, & \sigma_u \le t \le 1, \end{cases}$$

where $B^+ = \max\{B, 0\}$ and σ_u is defined in Lemma 2.3.

Define $A : K \to X$ by

$$Au(t) = \begin{cases} \frac{b\varphi_q(\int_0^{\sigma_u} f(\tau,u(\tau))d\tau)+\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_s^{\sigma_u} f(\tau,u(\tau))d\tau)ds}{a-\sum_{i=1}^{m-2} a_i} \\ + \int_0^t \varphi_q(\int_s^{\sigma_u} f(\tau,u(\tau))d\tau)ds, & 0 \le t \le \sigma_u, \\ \frac{\int_0^1 g(s)(\int_s^1 \varphi_q(\int_{\sigma_u}^r f(\tau,u(\tau))d\tau)dr)ds}{1-\int_0^1 g(s)ds} \\ + \int_t^1 \varphi_q(\int_{\sigma_u}^s f(\tau,u(\tau))d\tau)ds, & \sigma_u \le t \le 1. \end{cases}$$

For $u \in X$, denote $\theta : X \to K$ while $(\theta u)(t) = \max\{u(t), 0\}$, then $T = \theta \circ A$. For $u \in K'$, define $T^* : K' \to K'$ by

$$T^*u(t) = \begin{cases} \frac{b\varphi_q(\int_0^{\sigma_u} f^+(\tau,u(\tau))d\tau)+\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_s^{\sigma_u} f^+(\tau,u(\tau))d\tau)ds}{a-\sum_{i=1}^{m-2} a_i} \\ + \int_0^t \varphi_q(\int_s^{\sigma_u} f^+(\tau,u(\tau))d\tau)ds, & 0 \le t \le \sigma_u, \\ \frac{\int_0^1 g(s)(\int_s^1 \varphi_q(\int_{\sigma_u}^r f^+(\tau,u(\tau))d\tau)dr)ds}{1-\int_0^1 g(s)ds} \\ + \int_t^1 \varphi_q(\int_{\sigma_u}^s f^+(\tau,u(\tau))d\tau)ds, & \sigma_u \le t \le 1. \end{cases}$$

Lemma 2.6. *$T^* : K' \to K'$ is completely continuous.*

Proof. Obviously, we know $T^*(u) \ge 0$. By $[\varphi_p((T^*u)'(t))]' = -f^+(t, u(t)) \le 0$, it is easy to show $\varphi_p((T^*u)'(t))$ is non-increasing. Then $(T^*u)'(t)$ is also monotone non-increasing, i.e., $(T^*u)(t)$ is concave function. Thus $T^* : K' \to K'$. We can have T^* is completely continuous from Ascoli-Arzela theorem and concavity of f^+. □

From Lemma 2.2, we have the following lemma.

Lemma 2.7. *A function $u(t)$ is a solution of boundary value problem* (1.1) *if and only if $u(t)$ is a fixed point of the operator A.*

Lemma 2.8 ([8])**.** *If $A : K \to X$ is completely continuous, then $T = \theta \circ A : K \to K$ is also completely continuous.*

Lemma 2.9. *If u is a fixed point of operator T, then u is also a fixed point of operator A.*

Proof. Let u be a fixed point of operator of T. Obviously, if $Au(t) \ge 0$, $t \in [0, 1]$, then $u(t)$ is a fixed point of operator A.

We prove that if $Tu(t) = u(t)$, then $Au(t) \ge 0$ for $t \in [0, 1]$.

Suppose this is not true, then there is a $t_0 \in (0, 1)$ such that $Au(t_0) < 0 = u(t_0)$. Let (t_1, t_2) be the maximal interval which contain t_0 and such that $Au(t) < 0$, $t \in (t_1, t_2)$. It follows $[t_1, t_2] \ne [0, 1]$ from (H2).

If $t_2 < 1$, we have $u(t) = 0$ for $t \in [t_1, t_2]$, $Au(t) < 0$ for $t \in (t_1, t_2)$ and $Au(t_2) = 0$. Thus $(Au)'(t_2) \geq 0$. From (H2), we know $[\varphi_p((Au)'(t))]' = -f(t, 0) \leq 0$, so $t_1 = 0$.

On the other hand, from the fact that $Au(t) < 0$ for $t \in [0, t_2)$ and $Au(t_2) = 0, (Au)'(t_2) \geq 0$ and $(Au)''(t) \leq 0$, we can obtain $(Au)'(0) > 0$. Without loss of generality, we suppose that there exists $i = k$ such that $u(\xi_i) \leq 0$ for $1 \leq i \leq k$ and $u(\xi_i) > 0$ for $k < i \leq m-2$. Since $(Au)'(0) > 0$, $a > \sum_{i=1}^{m-2} a_i$ and $aAu(0) - bAu'(0) = \sum_{i=1}^{m-2} a_i Au(\xi_i) < 0$, we have

$$\begin{aligned}
|aAu(0) - bAu'(0)| > a|Au(0)| &> \sum_{i=1}^{k} a_i |Au(0)| + \sum_{i=k+1}^{m-2} a_i |Au(0)| \\
&> \sum_{i=1}^{k} a_i |Au(\xi_i)| - \sum_{i=k+1}^{m-2} a_i Au(\xi_i) \\
&= -(\sum_{i=1}^{m-2} a_i Au(\xi_i)) \\
&= |\sum_{i=1}^{m-2} a_i Au(\xi_i)|,
\end{aligned}$$

a contradiction. So $t_2 = 1$.

If $t_1 > 0$, then we have $Au(t) = 0$ for $t \in [t_1, t_2]$, $Au(t) < 0$ for $t \in (t_1, 1)$ and $Au(t_1) = 0$. Thus $(Au)'(t_1) \leq 0$. We have $[\varphi_p((Au)'(t))]' = -f(t, 0) \leq 0$ by (H2). This implies $Au(t) < 0$ for $t \in (t_1, 1]$ and $Au(1) = \min_{t \in [t_1, 1]} Au(t)$.

We can prove that $Au(t) \geq 0$ for $t \in [0, t_1]$. If there exists a $t_3 \notin [t_1, 1]$ such that $Au(t_3) < 0$ and there is a maximal interval $[t_4, t_5]$ which contains t_3 such that $Au(t) < 0$ for $t \in (t_4, t_5)$. Obviously $[t_4, t_5) \cap [t_1, 1] = \emptyset$, so $1 \notin (t_4, t_5)$; i.e., $t_5 < 1$, this is a contradiction with the above discussion. Thus we can show $Au(t) \geq 0$ for $t \in [0, t_1]$.

For $Au(1) < 0$, we have

$$Au(1) = \int_0^1 g(s) Au(s) ds.$$

Then

$$\begin{aligned}
|Au(1)| = -Au(1) &= -\int_0^{t_1} g(s) Au(s) ds - \int_{t_1}^1 g(s) Au(s) ds \\
&\leq \int_0^{t_1} g(s) |Au(1)| ds + \int_{t_1}^1 g(s) |Au(1)| ds \\
&= |Au(1)| \int_0^1 g(s) ds.
\end{aligned}$$

So $|Au(1)| \leq |Au(1)| \int_0^1 g(s) ds$ which is a contradiction with (H_3). Thus $t_1 = 0$.

The above also contradicts $[t_1, t_2] \neq [0, 1]$. Thus the proof is complete. $\square$

3. Main result

Denote

$$t^*=\frac{\xi_{m-2}+1}{2},\quad M=\frac{1}{q}(\frac{1-\xi_{m-2}}{2}-\frac{1}{k})^q,\quad N=\frac{1}{\max\big\{\frac{a+b}{a-\sum_{i=1}^{m-2}a_i},\frac{1}{1-\int_0^1 g(s)ds}\big\}},$$

$$\begin{aligned}N_0=\min\Big\{&\frac{b+a_1\xi_1}{(b+a\xi_1)q}(\xi_1-\frac{1}{k})^q,\ \frac{1}{q}\int_{\frac{1}{k}}^{\xi_1} g(s)((\xi_1-\frac{1}{k})^q-(\xi_1-s)^q)ds,\\ &\frac{b+a_1\xi_1}{(b+a\xi_1)q}(1-\frac{1}{k}-\xi_1)^q,\ \frac{1}{q}\int_{\xi_1}^{1-\frac{1}{k}} g(s)((1-\xi_1-\frac{1}{k})^q-(s-\xi_1)^q)ds\Big\}.\end{aligned}$$

It is easy to see that $N_0>0$ and $\frac{1}{k}<t^*<1-\frac{1}{k}$.

Theorem 3.1. *Suppose* (H1)–(H3) *hold, that there exist nonnegative constants* c_1,c_2,c_3 *such that*

$$0<c_1<\max\{\frac{N_0}{N},1\}c_2<\frac{1}{k}c_3<c_3$$

and that f satisfies the following growth conditions:

- (H4) $f(t,u)\geq 0$ *for* $(t,u)\in[0,1]\times[c_1,c_3]$;
- (H5) $f(t,u)<\varphi_p(\frac{c_2}{N})$ *for* $(t,u)\in[0,1]\times[0,c_2]$;
- (H6) $f(t,u)\geq\varphi_p(\frac{c_3}{M})$ *for* $(t,u)\in[\frac{1}{k},1-\frac{1}{k}]\times[\frac{1}{k}c_3,c_3]$;
- (H7) $f(t,u)\geq\varphi_p(\frac{c_1}{N_0})$ *for* $(t,u)\in[\frac{1}{k},1-\frac{1}{k}]\times[\frac{1}{k}c_2,c_3]$.

Then (1.1) *has at least two positive solutions u_1 and u_2 such that*

$$0<\|u_1\|<c_2\leq\|u_2\|,\quad \alpha(u_2)<\frac{1}{k}c_3.$$

Proof. For any $u\in\partial K_{c_2}$, from (H_5) we have

$$\|Tu\|=\max_{0\leq t\leq 1}|Tu(t)|=Tu(\bar t).$$

If $\bar t<\sigma_u$, we have

$$\begin{aligned}Tu(\bar t)&=\Big[\frac{b\varphi_q(\int_0^{\sigma_u}f(\tau,u(\tau))d\tau)+\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_q(\int_s^{\sigma_u}f(\tau,u(\tau))d\tau)ds}{a-\sum_{i=1}^{m-2}a_i}\\ &\quad+\int_0^{\bar t}\varphi_q(\int_s^{\sigma_u}f(\tau,u(\tau))d\tau)ds\Big]^+\\ &<\frac{b\varphi_q(\int_0^1\varphi_p(\frac{c_2}{N})d\tau)+\sum_{i=1}^{m-2}a_i\int_0^1\varphi_q(\int_0^1\varphi_p(\frac{c_2}{N})d\tau)ds}{a-\sum_{i=1}^{m-2}a_i}\\ &\quad+\int_0^1\varphi_q(\int_0^1\varphi_p(\frac{c_2}{N})d\tau)ds\\ &=\frac{c_2}{N}\frac{a+b}{a-\sum_{i=1}^{m-2}a_i}\leq c_2,\end{aligned}$$

and if $\bar t>\sigma_u$, we have

$$Tu(\bar t)=\Big[\frac{\int_0^1 g(s)(\int_s^1\varphi_q(\int_{\sigma_u}^r f(\tau,u(\tau))d\tau)dr)ds}{1-\int_0^1 g(s)ds}+\int_{\bar t}^1\varphi_q(\int_{\sigma_u}^s f(\tau,u(\tau))d\tau)ds\Big]^+$$

$$
\begin{aligned}
&< \frac{\int_0^1 g(s)(\int_0^1 \varphi_q(\int_0^1 \varphi_p(\frac{c_2}{N})d\tau)dr)ds}{1-\int_0^1 g(s)ds} + \int_0^1 \varphi_q(\int_0^1 \varphi_p(\frac{c_2}{N})d\tau)ds \\
&= \frac{c_2}{N}\frac{1}{1-\int_0^1 g(s)ds} \le c_2.
\end{aligned}
$$

From the above, we have $\|Tu\| < c_2$, So (C1) of Lemma 2.1 is satisfied.

For $u \in \partial K'_{c_2}$; i.e., $u \in K'$ and $\|u\| = c_2$. From Lemma 2.4 and (H5), we can prove that $\|T^*u\| < c_2$ using the same way as the Tu above.

For $u \in \partial K'(\frac{1}{k}c_3)$; i.e., $u \in K'$ and $\alpha(u) = \frac{1}{k}c_3$. So $\frac{1}{k}c_3 \le u(t) \le c_3$ for $\frac{1}{k} \le t \le 1-\frac{1}{k}$. From Lemmas 2.6, 2.7 and (H6), we have

$$
\begin{aligned}
\alpha(T^*u) &= \min_{t\in[\frac{1}{k},1-\frac{1}{k}]} T^*u(t) \ge \frac{1}{k}\|T^*u\| = \frac{1}{k}T^*u(\sigma_u) \\
&= \frac{1}{k}\Big(\frac{b\varphi_q(\int_0^{\sigma_u} f^+(\tau,u(\tau))d\tau) + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_s^{\sigma_u} f^+(\tau,u(\tau))d\tau)ds}{a-\sum_{i=1}^{m-2} a_i} \\
&\quad + \int_0^{\sigma_u} \varphi_q(\int_s^{\sigma_u} f^+(\tau,u(\tau))d\tau)ds\Big) \\
&= \frac{1}{k}\Big(\frac{\int_0^1 g(s)(\int_s^1 \varphi_q(\int_{\sigma_u}^r f^+(\tau,u(\tau))d\tau)dr)ds}{1-\int_0^1 g(s)ds} \\
&\quad + \int_{\sigma_u}^1 \varphi_q(\int_{\sigma_u}^s f^+(\tau,u(\tau))d\tau)ds\Big) \\
&\ge \frac{1}{k}\min\Big\{\int_0^{t^*} \varphi_q(\int_s^{t^*} f^+(\tau,u(\tau))d\tau)ds, \int_{t^*}^1 \varphi_q(\int_{t^*}^s f^+(\tau,u(\tau))d\tau)ds\Big\} \\
&\ge \frac{1}{k}\min\Big\{\int_{\frac{1}{k}}^{t^*} \varphi_q(\int_s^{t^*} \varphi_p(\frac{c_3}{M})d\tau)ds, \int_{t^*}^{1-\frac{1}{k}} \varphi_q(\int_{t^*}^s \varphi_p(\frac{c_3}{M})d\tau)ds\Big\} \\
&\ge \frac{c_3}{kM}\min\frac{1}{q}\{(\frac{1+\xi_{m-2}}{2}-\frac{1}{k})^q, (\frac{1-\xi_{m-2}}{2}-\frac{1}{k})^q\} \\
&= \frac{1}{k}c_3.
\end{aligned}
$$

Finally, we show that (C3) of Lemma 2.1 is also satisfied.

Let $u \in \partial K'_{c_2}(\frac{1}{k}c_3) \cap \{u : u = T^*u\}$, then $u = T^*u$, $u \in K'$, $\|u\| > c_2$ and $\alpha(u) \le \frac{1}{k}c_3$.

We know $\|u\| \le k\alpha(u) \le c_3$ and $\alpha(u) \ge \frac{1}{k}\|u\| > \frac{1}{k}c_2$ by Lemma 2.5. By $T^* : K' \to K'$ and the definition of T^*, we can get

$$
\min_{0\le t\le 1} u(t) = \min\{u(0), u(1)\} \ge 0.
$$

From the boundary condition $au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, we have

$$
\begin{aligned}
au(0) &= bu'(0) + \sum_{i=1}^{m-2} a_i u(\xi_i) \\
&\ge bu'(0) + a_1 u(\xi_1) \\
&\ge bu'(\xi) + a_1 u(\xi_1)
\end{aligned}
$$

$$= \frac{b}{\xi_1}(u(\xi_1) - u(0)) + a_1 u(\xi_1),$$

where $\xi \in (0, \xi_1)$. So we can obtain

$$u(0) \geq \frac{b + a_1\xi_1}{b + a\xi_1} u(\xi_1).$$

From the definition T^* and the use of condition (H7), we have

$$\begin{aligned} u(\xi_1) &> \int_0^{\xi_1} \varphi_q(\int_s^{\sigma_u} f^+(\tau, u(\tau))d\tau)ds > \int_{\frac{1}{k}}^{\xi_1} \varphi_q(\int_s^{\xi_1} f^+(\tau, u(\tau))d\tau)ds \\ &> \frac{c_1}{qN_0}(\xi_1 - \frac{1}{k})^q, \quad \text{if } \sigma_u > \xi_1, \end{aligned}$$

and

$$\begin{aligned} u(\xi_1) &> \int_{\xi_1}^{1-\frac{1}{k}} \varphi_q(\int_{\sigma_u}^{s} f^+(\tau, u(\tau))d\tau)ds > \int_{\xi_1}^{1-\frac{1}{k}} \varphi_q(\int_{\xi_1}^{s} f^+(\tau, u(\tau))d\tau)ds \\ &> \frac{c_1}{qN_0}(1 - \frac{1}{k} - \xi_1)^q, \quad \text{if } \sigma_u \leq \xi_1. \end{aligned}$$

Hence,

$$u(0) \geq \frac{b + a_1\xi_1}{b + a_1\xi} \cdot \frac{c_1}{qN_0} \min\{(\xi_1 - \frac{1}{k})^q, (1 - \frac{1}{k} - \xi_1)^q\} \geq c_1.$$

From the boundary condition $u(1) = \int_0^1 g(s)u(s)ds$, we have

$$\begin{aligned} u(1) &> \int_0^{\xi_1} g(s)u(s)ds > \int_{\frac{1}{k}}^{\xi_1} g(s)(\int_{\frac{1}{k}}^{\xi_1} \varphi_q(\int_r^{\xi_1} f^+(\tau, u(\tau))d\tau)dr)ds \\ &> \frac{c_1}{qN_0}(\int_{\frac{1}{k}}^{\xi_1} g(s)((\xi_1 - \frac{1}{k})^q - (\xi_1 - s)^q)ds), \quad \text{if } \sigma_u > \xi_1, \end{aligned}$$

and

$$\begin{aligned} u(1) &> \int_{\xi_1}^{1-\frac{1}{k}} g(s)u(s)ds > \int_{\xi_1}^{1-\frac{1}{k}} g(s)(\int_s^{1-\frac{1}{k}} \varphi_q(\int_{\xi_1}^{r} f^+(\tau, u(\tau))d\tau)dr)ds \\ &> \frac{c_1}{qN_0}(\int_{\xi_1}^{1-\frac{1}{k}} g(s)((1 - \xi_1 - \frac{1}{k})^q - (s - \xi_1)^q)ds), \quad \text{if } \sigma_u \leq \xi_1. \end{aligned}$$

So, $u(1) \geq c_1$. Therefore, for $u \in \partial K'_{c_2}(\frac{1}{k}c_3) \cap \{u : u = T^*u\}$, we have

$$c_1 \leq u(t) \leq \|u\| \leq k\alpha(u) = c_3.$$

It follows $f(t, u(t)) \geq 0$, $t \in [0, 1]$ from (H4). Thus, $T^*u = Tu$. So the condition of Lemma 2.1 is satisfied. Then by Lemma 2.7, we know that operator T has two fixed points u_1 and u_2 satisfying

$$0 < \|u_1\| < c_2 \leq \|u_2\|, \quad \alpha(u_2) < \frac{1}{k}c_3.$$

The proof is complete. $\square$

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