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GENERAL MIXED PROBLEMS FOR THE KDV EQUATIONS ON BOUNDED INTERVALS

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Abstract. This article is concerned with initial-boundary value problems for the Korteweg-de Vries (KdV) equation on bounded intervals. For general linear boundary conditions and small initial data, we prove the existence and uniqueness of global regular solutions and its exponential decay, as $t \to \infty$.

1. Introduction

This work concerns the existence and uniqueness of global solutions for the KdV equation posed on a bounded interval with general linear boundary conditions. There is a number of papers dedicated to the initial value problem for the KdV equation due to various applications of those results in mechanics and physics such as the dynamics of long small-amplitude waves in various media, [2, 3, 4, 10, 14, 20, 24, 28]. On the other hand, last years appeared publications on solvability of initial-boundary value problems for dispersive equations (which included KdV and Kawahara equations) in bounded domains, [4, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 21, 22, 23]. In spite of the fact that there is not any clear physical interpretation for the problems in bounded intervals, their study is motivated by numerics.

Dispersive equations such as KdV and Kawahara equations have been developed for unbounded regions of wave propagations, however, if one is interested in implementing numerical schemes to calculate solutions in these regions, there arises the issue of cutting off a spatial domain approximating unbounded domains by bounded ones. In this occasion some boundary conditions are needed to specify the solution. Therefore, precise mathematical analysis of mixed problems in bounded domains for dispersive equations is welcome and attracts attention of specialists in this area, [4, 5, 6, 7, 8, 9, 11, 14, 15, 17, 19, 21, 22, 27, 28].

As a rule, simple boundary conditions at $x = 0$ such as $u = 0$ for the KdV equation or $u = u_x = 0$ for the Kawahara equation were imposed. Different kind of boundary conditions was considered in [9]. On the other hand, general initial-boundary value problems for odd-order evolution equations attracted little attention. We must mention a classical paper of Volevich and Gindikin [18], where general mixed problems for linear $(2b + 1)$-hyperbolic equations were studied by

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means of functional analysis methods. It is difficult to apply their method directly to nonlinear dispersive equations due to complexity of this theory. In [7], Bubnov considered general mixed problems for the KdV equation posed on a bounded interval and proved local in t solvability results. Here we study a mixed problem for the KdV equation in a bounded interval with general linear homogeneous conditions and prove the existence and uniqueness of global regular solutions as well as the exponential decay while $t \to \infty$ of the obtained solution with small initial data.

It has been shown in [21, 22] that the KdV equation is implicitly dissipative. This means that for small initial data the energy decays exponentially as $t \to +\infty$ without any additional damping terms in the equation. Moreover, the energy decays even for the modified KdV equation with a linear source term, [22] and for general dispersive equations, [17]. In the present paper we prove that this phenomenon takes place for general boundary conditions as well as smoothing of the initial data effect.

This article has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem, notations and definitions. The main results of the paper on well-posedness of the considered problem are also formulated in this section. In Section 3 we study a corresponding boundary value problem for a stationary equation. Section 4 is devoted to the mixed problem for a complete linear equation. In Section 5 local well-posedness of the original problem is established. Section 6 contains global existence result and the decay of small solutions while $t \to +\infty$. To prove our results, we use the semigroup theory to solve the linear problem, the Banach fixed point theorem for local in t existence and uniqueness results and, finally, a priori estimates, independent of t, for the nonlinear problem.

2. Formulation of the problem and main results

For a real $T > 0$, let Q_T be a bounded domain: $Q_T = \{(x,t) \in \mathbb{R}^2 : x \in (0,1), t \in (0,T)\}$. In Q_T we consider the KdV equation

$$u_t + D^3u + Du + uDu = 0 \tag{2.1}$$

subject to initial and boundary conditions:

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{2.2}$$

$$\begin{gathered} D^2u(0,t) = a_1Du(0,t) + a_0u(0,t), \quad D^2u(1,t) = b_0u(1,t), \\ Du(1,t) = c_0u(1,t), \quad t > 0, \end{gathered} \tag{2.3}$$

where the coefficients a_0, a_1, b_0 and c_0 are such that

$$2A_0 = -2a_0 - 2a_1^2 - 1 > 0, \quad 2B_0 = 2b_0 - 2c_0^2 + 1 > 0. \tag{2.4}$$

Remark 2.1. We call (2.3) general boundary conditions because they follow from the more general (formally) relations,

$$\sum_{i=0}^{2} a_iD^iu(0,t) = 0, \quad \sum_{i=0}^{2} b_iD^iu(1,t) = 0, \quad \sum_{i=0}^{2} c_iD^iu(1,t) = 0 \tag{2.5}$$

when the determinant $\Delta = \det \begin{pmatrix} b_2 & b_1 \\ c_2 & c_1 \end{pmatrix} \neq 0$, $a_2 \neq 0$. Explicitly, simple boundary conditions $u(0,t) = u(1,t) = u_x(1,t) = 0$ do not follow directly from (2.3) and represent a singular case of (2.5): when $\Delta = 0$, in order to determine $D^2u(1,t)$, $Du(1,t)$, we must have a homogeneous linear algebraic system which implies that

$u(1,t)=0$ and $D^2u(1,t)$, $Du(1,t)$ are free. If $a_2=0$, then to guarantee the first $\|u\|(t)_{L^2(0,1)}$ estimate that is crucial for solvability of (2.1)-(2.3) , see [5], we must put $a_1=0$ and $u(0,t)=0$ which leads to the simple boundary conditions.

Throughout this paper we adopt the usual notation $\|\cdot\|$ and $(\cdot,\cdot)$ for the norm and the inner product in $L^2(0,1)$ respectively and $D^j=\partial^j/\partial x^j$, $j\in\mathbb{N}$; $D=D^1$.

The main result of the article is the following theorem.

Theorem 2.2. *Let $u_0\in H^3(0,1)$, and conditions (2.4) hold. Then for all finite $T>0$ there exists a positive real constant $\gamma=\min\{\frac{1}{4},\frac{2B_0}{9},\frac{A_0}{2}\}$ such that if $\|u_0\|^2<\frac{\gamma^2}{192}$, problem (2.1)-(2.3) has a unique regular solution $u=u(x,t)$:*

$$u\in L^\infty\big(0,T;H^3(0,1)\big)\cap L^2\big(0,T;H^4(0,1)\big),$$
$$u_t\in L^\infty\big(0,T;L^2(0,1)\big)\cap L^2\big(0,T;H^1(0,1)\big)$$

which satisfies the inequality

$$\|u\|^2(t)\le 2\|u_0\|^2e^{-\chi t},$$

where $\chi=\frac{\gamma}{4(1+\gamma)}$.

3. Stationary problem

In this section our goal is to solve the boundary value problem

$$Lu\equiv D^3u+Du+\lambda u=f,\quad \text{in } (0,1); \tag{3.1}$$

$$D^2u(0)=a_1Du(0)+a_0u(0),\quad D^2u(1)=b_0u(1),\quad Du(1)=c_0u(1), \tag{3.2}$$

where $\lambda>0$, $f\in L^2(0,1)$; a_0, a_1, b_0 and c_0 satisfy (2.4).

We denote

$$U(u)\equiv\begin{pmatrix}1 & -a_1 & -a_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & -b_0\\ 0 & 0 & 0 & 0 & 1 & -c_0\end{pmatrix}\begin{pmatrix}D^2u(0)\\ Du(0)\\ u(0)\\ D^2u(1)\\ Du(1)\\ u(1)\end{pmatrix}.$$

Suppose initially $f\in C\big([0,1]\big)$. With the notation above consider the problem

$$Lu=f, \tag{3.3}$$
$$U(u)=0 \tag{3.4}$$

and the associated homogeneous problem

$$Lu=0, \tag{3.5}$$
$$U(u)=0. \tag{3.6}$$

It is known, [25], that problem (3.3)-(3.4) has a unique classical solution if and only if problem (3.5)-(3.6) has only the trivial solution.

Let u_1,u_2 be nontrivial solutions of (3.5)-(3.6) and $w=u_1-u_2$. Then

$$Lw=0, \tag{3.7}$$
$$U(w)=0. \tag{3.8}$$

Multiplying (3.7) by w and integrating over (0,1), we obtain

$$(D^3w+Dw,w)+\lambda\|w\|^2=0. \tag{3.9}$$

Integrating by parts and using the Young inequality, we have

$$
\begin{aligned}
&(D^3w + Dw, w)\\
&\geq \Big(b_0 - \frac{1}{2}c_0^2 + \frac{1}{2}\Big)w^2(1) + \Big(-a_0 - |a_1|^2 - \frac{1}{2}\Big)w^2(0) + \Big(\frac{1}{2} - \frac{1}{4}\Big)|Dw(0)|^2.
\end{aligned} \tag{3.10}
$$

It follows from (2.4)

$$
(D^3w + Dw, w) \geq 0. \tag{3.11}
$$

Returning to (3.9),

$$
\lambda\|w\|^2 \leq (D^3w + Dw, w) + \lambda\|w\|^2 = 0
$$

which implies $\lambda\|w\|^2 \leq 0$. Since $\lambda > 0$, then $w \equiv 0$ and $u_1 \equiv u_2$.

Therefore, (3.3)-(3.4) has a unique classical solution given by

$$
u(x) = \int_0^1 G(x,\xi)f(\xi)d\xi, \tag{3.12}
$$

where $G : [0,1] \times [0,1] \to \mathbb{R}$ is the Green function associated with problem (3.3)-(3.4), [25].

Theorem 3.1. *Let $f \in L^2(0,1)$. Then for all $\lambda > 0$ problem* (3.1)-(3.2) *admits a unique solution $u \in H^3(0,1)$ such that*

$$
\|u\|_{H^3(0,1)} \leq C\|f\|, \tag{3.13}
$$

where C is a positive constant independent of u and f.

Proof. Multiplying (3.1) by u and integrating over (0,1), we obtain

$$
(D^3u + Du, u) + \lambda\|u\|^2 = (f, u). \tag{3.14}
$$

Analogously to (3.10),

$$
(D^3u + Du, u) \geq 0.
$$

Returning to (3.14) and using the Cauchy inequality, we have

$$
\|u\| \leq \frac{1}{\lambda}\|f\|. \tag{3.15}
$$

On the other hand, multiplying (3.1) by $(1 + \gamma x)u$ and integrating over (0,1), we obtain

$$
(D^3u + Du, (1+\gamma x)u) + \lambda(u, (1+\gamma x)u) = (f, (1+\gamma x)u). \tag{3.16}
$$

We calculate

$$
\begin{aligned}
&(D^3u + Du, (1+\gamma x)u)\\
&= \Big((1+\gamma)b_0 - \gamma c_0 - \frac{(1+\gamma)}{2}c_0^2 + \frac{1+\gamma}{2}\Big)u^2(1) + \frac{1}{2}|Du(0)|^2\\
&\quad + \Big(-a_0 - \frac{1}{2}\Big)u^2(0) + (\gamma - a_1)Du(0)u(0) + \frac{3\gamma}{2}\|Du\|^2 - \frac{\gamma}{2}\|u\|^2
\end{aligned}
$$

Applying the Young inequality and (2.4), we obtain

$$
(D^3u + Du, (1+\gamma x)u) \geq B_0u^2(1) + \frac{A_0}{2}u^2(0) + \frac{1}{4}|Du(0)|^2 + \frac{3\gamma}{2}\|Du\|^2 - \frac{\gamma}{2}\|u\|^2. \tag{3.17}
$$

Moreover,

$$
\lambda(u, (1+\gamma x)u) \geq 0, \quad (f, (1+\gamma x)u) \leq \|f\|^2 + \|u\|^2.
$$

Returning to (3.16), we have

$$B_0u^2(1)+\frac{A_0}{2}u^2(0)+\frac{1}{4}|Du(0)|^2+\frac{3\gamma}{2}\|Du\|^2\le\|f\|^2+\Big(\frac{\gamma}{2}+1\Big)\|u\|^2.$$

It follows from (3.15) that

$$\|Du\|\le C_1\|f\|, \tag{3.18}$$

where $C_1=\big(\frac{2}{3\gamma}+\frac{2}{3\gamma\lambda^2}+\frac{1}{3\lambda^2}\big)^{1/2}>0$. Now, multiplying (3.1) by D^3u and integrating over $(0,1)$, we obtain

$$\|D^3u\|^2+(Du,D^3u)+\lambda(u,D^3u)=(f,D^3u).$$

Using the Cauchy-Schwartz inequality,

$$\|D^3u\|^2\le(\|f\|+\|Du\|+\lambda\|u\|)\|D^3u\|.$$

A consequence of (3.15), (3.18) reads

$$\|D^3u\|\le C_2\|f\|, \tag{3.19}$$

where $C_2=C_1+2>0$. From (3.15), (3.19) and according to the inequality of Ehrling, [1], we conclude that $u\in H^3(0,1)$, and

$$\|u\|_{H^3(0,1)}\le C\|f\|,$$

where $C>0$. Uniqueness of u follows from (3.15). In fact, such calculations must be performed for smooth solutions and the general case can be obtained using density arguments. □

4. Linear evolution problem

Consider the linear problem

$$u_t+D^3u+Du=f,\quad \text{in } Q_T, \tag{4.1}$$
$$u(x,0)=u_0(x),\quad x\in(0,1), \tag{4.2}$$
$$U(u)=0 \tag{4.3}$$

and define the linear operator A in $L^2(0,1)$:

$$Au:=D^3u+Du,\quad D(A):=\{u\in H^3(0,1):U(u)=0\}. \tag{4.4}$$

Theorem 4.1. *Let $u_0\in D(A)$. Then for every $T>0$, $f\in H^1\big(0,T,L^2(0,1)\big)$ problem (4.1)-(4.3) has a unique solution $u=u(x,t)$;*

$$u\in L^\infty\big(0,T;H^3(0,1)\big),\quad u_t\in L^\infty\big(0,T;L^2(0,1)\big)\cap L^2\big(0,T;H^1(0,1)\big).$$

Proof. To solve (4.1)-(4.3), we use the semigroups theory. For details, see [26, 29]. According to Theorem 3.1, for all $\lambda>0$ and $f\in L^2(0,1)$ there exists $u(x)$ such that $Lu=f$, hence, $R(A+\lambda I)=L^2(0,1)$. Moreover, by (3.11), $(Au,u)\ge0,\forall u\in D(A)$. It means that A is a m-accretive operator. By the Lumer-Phillips theorem, [26], A is the infinitesimal generator of a semigroup of contractions of class C_0. Therefore the following abstract Cauchy problem:

$$u_t+Au=f, \tag{4.5}$$
$$u(0)=u_0 \tag{4.6}$$

has a unique solution

$$u\in C\big([0,T],D(A)\big)\cap C^1\big([0,T],L^2(0,1)\big)$$

for all $f \in C([0,T], L^2(0,1))$ such that $f_t \in L^2(0,T,L^2(0,1))$ and all $u_0 \in D(A)$, see [29].

Using density arguments, we prove the following estimates.

Estimate I:. Multiplying (4.5) by u and integrating over $(0,1)$, we obtain

$$(u_t, u)(t) + (Au, u)(t) = (f, u)(t).$$

Applying (3.11) and the Schwartz inequality, we have

$$\frac{d}{dt}\|u\|^2(t) \le \|u\|^2(t) + \|f\|^2(t).$$

By the Gronwall lemma,

$$\|u\|^2(t) \le e^T\big(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\big). \tag{4.7}$$

Multiplying equation (4.5) by $(1+\gamma x)u$ and integrating over $(0,1)$, we obtain

$$(u_t, (1+\gamma x)u)(t) + (Au, (1+\gamma x)u)(t) = (f, (1+\gamma x)u)(t).$$

Using (3.17) and the Schwartz inequality,

$$\begin{aligned} &\frac{d}{dt}(1+\gamma x, u^2)(t) + 2B_0 u^2(1,t) + A_0 u^2(0,t) + \frac{1}{2}|Du(0,t)|^2 + 3\gamma\|Du\|^2(t) \\ &\le (1+\gamma x, u^2)(t) + (1+\gamma x, f^2)(t) + \gamma\|u\|^2(t). \end{aligned} \tag{4.8}$$

Taking into account (4.7), we have

$$\frac{d}{dt}(1+\gamma x, u^2)(t) \le (1+\gamma x, u^2)(t) + (1+\gamma x, f^2)(t) + \gamma e^T\big(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\big).$$

By the Gronwall lemma,

$$\begin{aligned} &(1+\gamma x, u^2)(t) \\ &\le e^T\Big[(1+\gamma x, u^2)(0) + \int_0^t (1+\gamma x, f^2)(s)ds + \gamma T e^T\big(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\big)\Big] \\ &\le (2e^T + \gamma T e^{2T})(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}). \end{aligned}$$

Returning to (4.8),

$$\begin{aligned} &\frac{d}{dt}(1+\gamma x, u^2)(t) + 2B_0 u^2(1,t) + A_0 u^2(0,t) + \frac{1}{2}|Du(0,t)|^2 + 3\gamma\|Du\|^2(t) \\ &\le ((2+\gamma)e^T + \gamma T e^{2T})\big(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\big) + 2\|f\|^2(t). \end{aligned}$$

Integration from 0 to t gives

$$\begin{aligned} &(1+\gamma x, u^2)(t) + \int_0^t [2B_0 u^2(1,s) + A_0 u^2(0,s) + \frac{1}{2}|Du(0,s)|^2 + 3\gamma\|Du\|^2(s)]ds \\ &\le ((2+\gamma)Te^T + \gamma T^2 e^{2T} + 2)\big(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\big). \end{aligned}$$

Denote $\alpha = \frac{1}{\min\{\frac{1}{2}, 2B_0, A_0, 3\gamma\}}$, then

$$\begin{aligned} &\|u\|^2(t) + \int_0^t [u^2(1,s) + u^2(0,s) + |Du(0,s)|^2 + \|Du\|^2(s)]ds \\ &\le ((2+\gamma)T\alpha e^T + \gamma T^2 \alpha e^{2T} + 2\alpha)\big(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}\big), \end{aligned}$$

whence

$$\begin{aligned} &\|u\|^2(t) + \int_0^t [u^2(1,s) + u^2(0,s) + |Du(0,s)|^2 + \|u\|^2_{H^1(0,1)}(s)]ds \\ &\le C_{1T}(\|u_0\|^2 + \|f\|^2_{L^2(Q_T)}), \end{aligned} \tag{4.9}$$

where $C_{1T} = ((2\alpha + \gamma\alpha + 1)Te^T + \gamma T^2\alpha e^{2T} + 2\alpha) > 0$.

Estimate II:. Differentiating (4.5) with respect to t, multiplying the result by $(1+\gamma x)u_t$, integrating over $(0,1)$ and acting as by proving (4.9), we have

$$\begin{aligned} &\|u_t\|^2(t) + \int_0^t [u_s^2(1,s) + u_s^2(0,s) + |Du_s(0,s)|^2 + \|u_s\|^2_{H^1(0,1)}(s)]ds \\ &\le C_{1T}\big(\|u_t\|^2(0) + \|f_t\|^2_{L^2(Q_T)}\big). \end{aligned}$$

Since

$$\|u_t\|^2(0) = \| - D^3u - Du + f\|^2(0) \le 3(\|D^3u_0\|^2 + \|Du_0\|^2 + \|f_0\|^2),$$

it follows that

$$\begin{aligned} &\|u_t\|^2(t) + \int_0^t [u_s^2(1,s) + u_s^2(0,s) + |Du_s(0,s)|^2 + \|u_s\|^2_{H^1(0,1)}(s)]ds \\ &\le C_{2T}\big(\|D^3u_0\|^2 + \|Du_0\|^2 + \|f_0\|^2 + \|f_t\|^2_{L^2(Q_T)}\big), \end{aligned} \tag{4.10}$$

where $C_{2T} = 3C_{1T}$. Returning to (4.1), we find

$$u \in L^\infty\big(0,T;H^3(0,1)\big), \quad u_t \in L^\infty\big(0,T;L^2(0,1)\big) \cap L^2\big(0,T;H^1(0,1)\big).$$

The proof is complete. □

5. Nonlinear evolution problem. Local solutions

In this section we prove the existence of local regular solutions to (2.1)-(2.3).

Theorem 5.1. *Let $u_0 \in H^3(0,1)$. Then there exists a real $T_0 > 0$ such that (2.1)-(2.3) has a unique regular solution $u(x,t)$ in Q_{T_0}:*

$$\begin{gathered} u \in L^\infty\big(0,T_0;H^3(0,1)\big) \cap L^2\big(0,T_0;H^4(0,1)\big), \\ u_t \in L^\infty\big(0,T_0;L^2(0,1)\big) \cap L^2\big(0,T_0;H^1(0,1)\big). \end{gathered}$$

We prove this theorem using the Banach fixed point theorem. Consider the spaces:

$$\begin{gathered} X = L^\infty\big(0,T;H^3(0,1)\big), \\ Y = L^\infty\big(0,T;L^2(0,1)\big) \cap L^2\big(0,T;H^1(0,1)\big); \end{gathered}$$

$$\begin{aligned} V = \big\{&v : [0,1]\times[0,T] \to \mathbb{R} : v \in X, v_t \in Y, \\ &D^2v(0,t) = a_1Dv(0,t) + a_0v(0,t),\ D^2v(1,t) = b_0v(1,t), \\ &Dv(1,t) = c_0v(1,t),\ v(x,0) = u_0(x)\big\} \end{aligned}$$

with the norm

$$\|v\|_V^2 = \operatorname{ess\,sup}_{t\in(0,T)}\{\|v\|^2(t) + \|v_t\|^2(t)\} + \int_0^T [\|Dv\|^2(t) + \|Dv_t\|^2(t)]dt. \tag{5.1}$$

The space V equipped with the norm (5.1) is a Banach space. Define the ball

$$B_R = \{v \in V : \|v\|_V \le 4R\},$$

where $R > 1$ is such that

$$\sum_{i=0}^{3}(\|D^i u_0\|^2 + \|u_0 Du_0\|^2) \le R^2. \tag{5.2}$$

For any $v \in B_R$ consider the linear problem

$$u_t + D^3 u + Du = -vDv, \quad \text{in } Q_T; \tag{5.3}$$

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{5.4}$$

$$\begin{gathered} D^2u(0,t) = a_1 Du(0,t) + a_0 u(0,t), \quad D^2u(1,t) = b_0 u(1,t), \\ Du(1,t) = c_0 u(1,t), \quad t > 0, \end{gathered} \tag{5.5}$$

where a_0, a_1, b_0 and c_0 satisfy (2.4).

It will be shown that $f(x,t) = -vDv \in H^1\big(0,T;L^2(0,1)\big)$. We have

$$\begin{aligned} &\|Dv\|^2(t) \\ &= \|Dv\|^2(0) + \int_0^t \frac{\partial}{\partial\tau}\|Dv\|^2(\tau)d\tau \le \|Du_0\|^2 + \int_0^t [\|Dv\|^2(\tau) + \|Dv_\tau\|^2(\tau)]d\tau. \end{aligned}$$

By (5.1)-(5.2),

$$\|Dv\|^2(t) \le R^2 + C_{0R},$$

where $C_{0R} = 16R^2$. Hence,

$$\|Dv\|^2(t) \le C_{1R} \quad \text{with } C_{1R} = 17R^2. \tag{5.6}$$

Moreover, by (5.1) and (5.6),

$$\operatorname{ess\,sup}_{x\in(0,1)} |v(x,t)|^2 \le 12\|v\|^2_{H^1(0,1)}(t) \le C_{2R}, \tag{5.7}$$

where $C_{2R} = 12(C_{0R} + C_{1R})$. Similarly,

$$\operatorname{ess\,sup}_{x\in(0,1)} |v_t(x,t)|^2 \le 12\|v_t\|^2_{H^1(0,1)}(t) \le 12(C_{0R} + \|Dv_t\|^2(t)). \tag{5.8}$$

By (5.7),

$$\begin{aligned} &\int_0^T\int_0^1 |vDv|^2 dx\,dt \\ &\le \operatorname{ess\,sup}_{t\in(0,T)} \Big(\operatorname{ess\,sup}_{x\in(0,1)} |v(x,t)|^2\Big) \int_0^T\int_0^1 |Dv|^2 dx\,dt < \infty. \end{aligned}$$

Moreover, by (5.8) and (5.7),

$$\int_0^T\int_0^1 |(vDv)_t|^2 dx\,dt \le 2\Big[\int_0^T\int_0^1 |v_t Dv|^2 dx\,dt + \int_0^T\int_0^1 |vDv_t|^2 dx\,dt\Big],$$

$$\int_0^T\int_0^1 |v_t Dv|^2 dx\,dt \le \int_0^T \Big(\operatorname{ess\,sup}_{x\in(0,1)} |v_t(x,t)|^2 \int_0^1 |Dv|^2 dx\Big) dt < \infty,$$

$$\begin{aligned} &\int_0^T\int_0^1 |vDv_t|^2 dx\,dt \\ &\le \operatorname{ess\,sup}_{t\in(0,T)} \Big(\operatorname{ess\,sup}_{x\in(0,1)} |v(x,t)|^2\Big) \int_0^T\int_0^1 |Dv_t|^2 dx\,dt < \infty. \end{aligned}$$

Hence, $f = -vDv \in H^1\big(0,T;L^2(0,1)\big)$. According to Theorem 4.1, there exists a unique solution $u(x,t)$ of (5.3)-(5.5):

$$u \in L^\infty\big(0,T;H^3(0,1)\big), \quad u_t \in L^\infty\big(0,T;L^2(0,1)\big) \cap L^2\big(0,T;H^1(0,1)\big).$$

and we may define an operator P, related to (5.3)-(5.5), such that $u = Pv$.

Lemma 5.2. *There is a real T_0: $0 < T_0 \le T \le 1$ such that the operator P maps B_R into B_R.*

Proof. We need the following estimates:

Estimate I:. Multiplying (5.3) by u and integrating over $(0,1)$, we have

$$(u_t, u)(t) + (Au, u)(t) = (-vDv, u)(t),$$

where $Au = D^3u + Du$. Using (3.11), (5.6), (5.7) and the Schwartz inequality,

$$\frac{d}{dt}\|u\|^2(t) \le \operatorname{ess\,sup}_{x\in(0,1)} |v(x,t)|^2 \|Dv\|^2(t) + \|u\|^2(t) \le C_{2R}C_{1R} + \|u\|^2(t). \quad (5.9)$$

By the Gronwall lemma,

$$\|u\|^2(t) \le e^T(\|u_0\|^2 + C_{2R}C_{1R}T). \quad (5.10)$$

Taking $0 < T_1 \le T$ such that $e^{T_1} \le 2$ and $C_{2R}C_{1R}T_1 \le R^2$, we obtain

$$\|u\|^2(t) \le 2(R^2 + R^2) = 4R^2, \quad t \in (0, T_1).$$

Multiplying (5.3) by $(1+\gamma x)u$ and integrating over $(0,1)$, we obtain

$$(u_t, (1+\gamma x)u)(t) + (Au, (1+\gamma x)u)(t) = (-vDv, (1+\gamma x)u)(t).$$

From inequalities (2.4), (3.17) and (5.9) it follows

$$\begin{aligned}
&\frac{d}{dt}(1+\gamma x, u^2)(t) + 2B_0u^2(1,t) + A_0u^2(0,t) \\
&+ \frac{1}{2}|Du(0,t)|^2 + 3\gamma\|Du\|^2(t) - \gamma\|u\|^2(t) \\
&\le C_{2R}C_{1R} + (1+\gamma x, u^2)(t).
\end{aligned}$$

By (5.10),

$$\frac{d}{dt}(1+\gamma x, u^2)(t) + \|Du\|^2(t) \le (4(\gamma+2)R^2 + 2C_{2R}C_{1R})\delta,$$

where $\delta = \frac{1}{3\gamma}$.

Integration from 0 to t gives

$$(1+\gamma x, u^2)(t) + \int_0^t \|Du\|^2(\tau)d\tau \le (1+\gamma x, u^2)(0) + (4(\gamma+2)R^2 + 2C_{2R}C_{1R})\delta T_1.$$

This implies

$$\|u\|^2(t) + \int_0^t \|Du\|^2(\tau)d\tau \le 2R^2 + (4(\gamma+2)R^2 + 2C_{2R}C_{1R})\delta T_1.$$

Taking $T_1 > 0$ such that $(4(\gamma+2)R^2 + 2C_{2R}C_{1R})\delta T_1 \le 6R^2$, we obtain

$$\|u\|^2(t) + \int_0^t \|Du\|^2(\tau)d\tau \le 8R^2. \quad (5.11)$$

Estimate II:. Differentiating (5.3) with respect to t, multiplying the result by u_t and integrating over $(0,1)$, we have

$$(u_{tt},u_t)(t)+(Au_t,u_t)(t)=(-v_tDv,u_t)(t)+(-vDv_t,u_t)(t). \tag{5.12}$$

We estimate

$$\begin{aligned}
I_1&=(-v_tDv,u_t)(t)\\
&\le \frac{\varepsilon^2}{2}\|v_tDv\|^2(t)+\frac{1}{2\varepsilon^2}\|u_t\|^2(t)\\
&\le \frac{\varepsilon^2}{2}\operatorname{ess\,sup}_{x\in(0,1)}|v_t(x,t)|^2\|Dv\|^2(t)+\frac{1}{2\varepsilon^2}\|u_t\|^2(t)\\
&\le \varepsilon^2 6(C_{0R}+\|Dv_t\|^2(t))\|Dv\|^2(t)+\frac{1}{2\varepsilon^2}\|u_t\|^2(t)\\
&\le \varepsilon^2 6C_{0R}C_{1R}+\varepsilon^2 6C_{1R}\|Dv_t\|^2(t)+\frac{1}{2\varepsilon^2}\|u_t\|(t),
\end{aligned}$$

and

$$\begin{aligned}
I_2&=(-vDv_t,u_t)(t)\\
&\le \frac{\varepsilon^2}{2}\|vDv_t\|^2(t)+\frac{1}{2\varepsilon^2}\|u_t\|^2(t)\\
&\le \frac{\varepsilon^2}{2}\operatorname{ess\,sup}_{x\in(0,1)}|v(x,t)|^2\|Dv_t\|^2(t)+\frac{1}{2\varepsilon^2}\|u_t\|^2(t)\\
&\le \frac{\varepsilon^2}{2}C_{2R}\|Dv_t\|^2(t)+\frac{1}{2\varepsilon^2}\|u_t\|^2(t),
\end{aligned}$$

where ε is an arbitrary positive number. Substituting I_1,I_2 into (5.12), we find

$$\frac{d}{dt}\|u_t\|^2(t)\le \varepsilon^2 12C_{0R}C_{1R}+\varepsilon^2(C_{2R}+12C_{1R}\|Dv_t\|^2(t)+\frac{2}{\varepsilon^2}\|u_t\|^2(t).$$

By the Gronwall lemma,

$$\begin{aligned}
\|u_t\|^2(t)&\le e^{2T}\varepsilon^2\big[\|u_t\|^2(0)+\varepsilon^2 12C_{0R}C_{1R}T+\varepsilon^2(C_{2R}+12C_{1R})\int_0^t\|Dv_t\|^2(\tau)d\tau\big]\\
&\le e^{2T}\varepsilon^2[3(\|u_0Du_0\|^2+\|D^3u_0\|^2+\|Du_0\|^2)+\varepsilon^2C_{3R}T+\varepsilon^2C_{4R}],
\end{aligned}$$

where $C_{3R}=12C_{0R}C_{1R}$ and $C_{4R}=(C_{2R}+12C_{1R})C_{0R}$. Taking $0<T_2\le T\le 1$, $\varepsilon>0$ such that $T_2<\frac{\varepsilon^2}{2}$, $e^{\frac{2T_2}{\varepsilon^2}}\le 2$ and $\varepsilon^2(C_{3R}+C_{4R})\le R^2$, we obtain

$$\|u_t\|^2(t)\le 2(3R^2+R^2)=8R^2,\quad t\in(0,T_2). \tag{5.13}$$

Differentiating (5.3) with respect to t, multiplying the result by $(1+\gamma x)u_t$ and integrating over $(0,1)$, we obtain

$$(u_{tt},(1+\gamma x)u_t)(t)+(Au_t,(1+\gamma x)u_t)(t)=((-vDv)_t,(1+\gamma x)u_t)(t)$$

which can be transformed, using (5.13), into

$$\frac{d}{dt}(1+\gamma x)u^2)(t)+\|Du_t\|^2(t)\le (8(\gamma+4)R^2+\varepsilon^2 2C_{3R})\delta+\varepsilon^2 2C_{4R}\delta\|Dv_t\|^2(t),$$

where $\delta=1/(3\gamma)$. Integration from 0 to t gives

$$(1+\gamma x,u_t^2)(t)+\int_0^t\|Du_\tau\|^2(\tau)d\tau\le(1+\gamma x,u_t^2)(0)+(8(\gamma+4)R^2+\varepsilon^2 2C_{3R})\delta T_2$$

$$+\varepsilon^2 2C_{4R}\delta\int_0^t \|Dv_\tau\|^2(\tau)$$

which implies

$$\|u_t\|^2(t)+\int_0^t \|Du_\tau\|^2(\tau)d\tau \le 6R^2+(8(\gamma+4)R^2+\varepsilon^2 2C_{3R})\delta T_2+\varepsilon^2 2C_{4R}C_{0R}\delta.$$

Taking $T_2<\frac{\varepsilon^2}{2}$ and $\varepsilon>0$ such that

$$(8(\gamma+4)R^2+\varepsilon^2 2C_{3R})\delta T_2+\varepsilon^2 2C_{4R}C_{0R}\delta \le R^2,$$

we obtain

$$\|u_t\|^2(t)+\int_0^t \|Du_\tau\|^2(\tau)d\tau \le 8R^2. \tag{5.14}$$

Putting $T_0=\min\{T_1,T_2\}$ and using (5.11), (5.14), we find

$$\|u\|_V \le 4R.$$

Lemma 5.3. *For $T_0>0$ sufficiently small the operator P is a contraction mapping in B_R.*

Proof. For $v_1,v_2\in B_R$ denote

$$u_i=Pv_i,\ i=1,2,\quad w=v_1-v_2\quad\text{and}\quad z=u_1-u_2$$

which satisfies the initial-boundary value problem

$$z_t+D^3z+Dz=-\frac{1}{2}D(v_1^2-v_2^2),\quad \text{in } Q_{T_0}; \tag{5.15}$$

$$z(x,0)=0,\quad x\in(0,1), \tag{5.16}$$

$$\begin{gathered}D^2z(0,t)=a_1Dz(0,t)+a_0z(0,t),\quad D^2z(1,t)=b_0z(1,t),\\ Dz(1,t)=c_0z(1,t),\quad t>0.\end{gathered} \tag{5.17}$$

Define the metric

$$\rho^2(v_1,v_2)=\rho^2(w)=\operatorname{ess\,sup}_{t\in(0,T_0)}\|w\|^2(t)+\int_0^{T_0}\|Dw\|^2(t)dt.$$

Multiplying (5.15) by z and integrating over $(0,1)$, we have

$$(z_t,z)(t)+(Az,z)(t)=(-\frac{1}{2}(v_1+v_2)Dw,z)(t)+(-\frac{1}{2}wD(v_1+v_2),z)(t). \tag{5.18}$$

We estimate

$$\begin{aligned}I_1&=(-\frac{1}{2}(v_1+v_2)Dw,z)(t)\\ &\le \frac{\varepsilon^2}{4}\|(v_1+v_2)Dw\|^2(t)+\frac{1}{2\varepsilon^2}\|z\|^2(t)\\ &\le \frac{\varepsilon^2}{4}\operatorname{ess\,sup}_{x\in(0,1)}|(v_1+v_2)(x,t)|^2\|Dw\|^2(t)+\frac{1}{2\varepsilon^2}\|z\|^2(t)\\ &\le \varepsilon^2 2C_{2R}\|Dw\|^2(t)+\frac{1}{2\varepsilon^2}\|z\|^2(t),\end{aligned}$$

and

$$\begin{aligned}I_2&=(-\frac{1}{2}wD(v_1+v_2),z)(t)\\ &\le \frac{\varepsilon^2}{4}\|wD(v_1+v_2)\|^2(t)+\frac{1}{2\varepsilon^2}\|z\|^2(t)\end{aligned}$$

$$\begin{aligned}&\le \frac{\varepsilon^2}{4}\operatorname{ess\,sup}_{x\in(0,1)}|w(x,t)|^2\|D(v_1+v_2)\|^2(t)+\frac{1}{2\varepsilon^2}\|z\|^2(t)\\&\le \varepsilon^2 12C_{1R}(\|w\|^2(t)+\|Dw\|^2(t))+\frac{1}{2\varepsilon^2}\|z\|^2(t).\end{aligned}$$

Substituting I_1, I_2 into (5.18), we obtain

$$\frac{d}{dt}\|z\|^2(t)\le \varepsilon^2 2C_{2R}\|Dw\|^2(t)+\varepsilon^2 24C_{1R}(\|w\|^2(t)+\|Dw\|^2(t))+\frac{2}{\varepsilon^2}\|z\|^2(t).$$

Define $C_{5R}=2\max\{2C_{2R},24C_{1R}\}$, then

$$\frac{d}{dt}\|z\|^2(t)\le \varepsilon^2 C_{5R}(\|w\|^2(t)+\|Dw\|^2(t))+\frac{2}{\varepsilon^2}\|z\|^2(t).$$

By the Gronwall lemma,

$$\begin{aligned}\|z\|^2(t)&\le e^{2T_0/\varepsilon^2}\varepsilon^2C_{5R}\int_0^t[\|w\|^2(\tau)+\|Dw\|^2(\tau)]d\tau\\&\le e^{2T_0/\varepsilon^2}\varepsilon^2C_{5R}\Big[t\ \operatorname{ess\,sup}_{t\in(0,T_0)}\|w\|^2(t)+\int_0^t\|Dw\|^2(\tau)d\tau\Big].\end{aligned}$$

Taking $0<T_0\le 1$ such that $e^{2T_0/\varepsilon^2}\le 2$, we find

$$\|z\|^2(t)\le \varepsilon^2 2C_{5R}\rho^2(w),\quad t\in(0,T_0). \tag{5.19}$$

Multiplying (5.15) by $(1+\gamma x)z$ and integrating over $(0,1)$, we obtain

$$(z_t,(1+\gamma x)z)(t)+(Az,(1+\gamma x)z)(t)=(-\frac{1}{2}D(v_1^2-v_2^2),(1+\gamma x)z)(t)$$

which may be transformed into

$$\begin{aligned}&\frac{d}{dt}(1+\gamma x,z^2)(t)+2B_0z^2(1,t)+A_0z^2(0,t)\\&+\frac12|Dz(0,t)|^2+3\gamma\|Dz\|^2(t)-\gamma\|z\|^2(t)\\&\le \varepsilon^2 4C_{2R}\|Dw\|^2(t)+\varepsilon^2 48C_{1R}(\|w\|^2(t)+\|Dw\|^2(t))+\frac{2}{\varepsilon^2}(1+\gamma x,z^2)(t).\end{aligned}$$

By (5.19),

$$\frac{d}{dt}(1+\gamma x,z^2)(t)+3\gamma\|Dz\|^2(t)\le(2\gamma\varepsilon^2+8)C_{5R}\rho^2(w)+\varepsilon^2 2C_{5R}(\|w\|^2(t)+\|Dw\|^2(t)).$$

Integration from 0 to t gives

$$(1+\gamma x,z^2)(t)+\int_0^t\|Dz\|^2(\tau)d\tau\le(2\gamma\varepsilon^2+8)\delta C_{5R}T_0\rho^2(w)+\varepsilon^2\delta 2C_{5R}\rho^2(w)$$

or

$$\|z\|^2(t)+\int_0^t\|Dz\|^2(\tau)d\tau\le((2\gamma\varepsilon^2+8)\delta C_{5R}T_0+\varepsilon^2\delta 2C_{5R})\rho^2(w).$$

Since $T_0<\varepsilon^2/2$, we take $\varepsilon>0$ such that $\varepsilon^2(\varepsilon^2\gamma\delta C_{5R}+6\delta C_{5R})\le\frac12$, whence,

$$\rho^2(z)\le\frac12\rho^2(w).$$

It implies that P is a contraction mapping in B_R. By the Banach fixed-point theorem, there exists a unique generalized solution $u=u(x,t)$ of problem (5.3)-(5.5) such that

$$u,u_t\in L^\infty\big(0,T_0;L^2(0,1)\big)\cap L^2\big(0,T_0;H^1(0,1)\big).$$

Returning to equation (2.1), we find that $u \in L^\infty\big(0,T_0;H^3(0,1)\big)$. Moreover, because $u_t \in L^2\big(0,T_0;H^1(0,1)\big)$, then

$$D^4u = -D^2u - (Du)^2 - uD^2u - Du_t.$$

Estimating

$$\begin{aligned}\int_0^{T_0}\int_0^1 |uD^2u|^2 dx\,dt &\le \int_0^{T_0}\Big(\operatorname{ess\,sup}_{x\in(0,1)}|u(x,t)|^2\int_0^1 |D^2u|^2dx\Big)dt\\ &\le 2\sqrt{3}\ \operatorname{ess\,sup}_{t\in(0,T_0)}\|D^2u\|^2(t)\int_0^{T_0}\|u\|^2_{H^1(0,1)}(t)dt<\infty,\end{aligned}$$

and

$$\begin{aligned}\int_0^{T_0}\int_0^1 |Du|^4 dx\,dt &\le \int_0^{T_0}\Big(\operatorname{ess\,sup}_{x\in(0,1)}|Du(x,t)|^2\int_0^1 |Du|^2dx\Big)dt\\ &\le 2\sqrt{3}\operatorname{ess\,sup}_{t\in(0,T_0)}\|Du\|^2(t)\int_0^{T_0}\|Du\|^2_{H^1(0,1)}(t)dt<\infty,\end{aligned}$$

we have $D^4u \in L^2\big(0,T_0;L^2(0,1)\big)$. Thus

$$u \in L^\infty\big(0,T_0,H^3(0,1)\big)\cap L^2\big(0,T_0;H^4(0,1)\big)$$

is a regular solution of problem (2.1)-(2.3). The proof is complete. □

6. Global solutions. Exponential decay

In this section we prove global solvability and exponential decay of small solutions as $t \to +\infty$ for the problem

$$u_t + D^3u + Du + uDu = 0, \quad x\in(0,1),\ t>0 \tag{6.1}$$

$$\begin{gathered}D^2u(0,t) = a_1Du(0,t) + a_0u(0,t), \quad D^2u(1,t) = b_0u(1,t),\\ Du(1,t) = c_0u(1,t), \quad t>0,\end{gathered} \tag{6.2}$$

$$u(x,0) = u_0(x), \quad x\in(0,1), \tag{6.3}$$

where the coefficients a_0, a_1, b_0, c_0 are real constants satisfying (2.4).

The existence of local regular solutions follows from Theorem 5.1. Hence, we need global in t a priori estimates of these solutions in order to prolong them for all $t>0$.

6.1. **Estimate I:.** Multiplying (6.1) by $2(1+\gamma x)u$, integrating the result by parts and taking into account (6.2), one gets

$$\begin{aligned}&\frac{d}{dt}(1+\gamma x,u^2)(t) + 3\gamma\|Du\|^2(t) - \gamma\|u\|^2(t) + 2((1+\gamma x)u^2,Du)(t)\\ &+ (1+2b_0-c_0^2+\gamma(1+2b_0-c_0)-2\gamma c_0)u^2(1,t)\\ &+ (-2a_0-1)u^2(0,t) + |Du(0,t)|^2 + 2(\gamma-a_1)u(0,t)Du(0,t) = 0.\end{aligned}$$

Since $\gamma = \min\{\frac14,\frac{2B_0}{9},A_0\}$, this equality can be reduced to the form:

$$\begin{aligned}&\frac{d}{dt}(1+\gamma x,u^2)(t) + 3\gamma\|Du\|^2(t) - \gamma\|u\|^2(t) + 2((1+\gamma x)u^2,Du)(t)\\ &+ 2B_0u^2(1,t) + A_0u^2(0,t) + \frac12|Du(0,t)|^2 \le 0.\end{aligned} \tag{6.4}$$

It is easy to see that

$$\|u\|^2(t) \le 2|u(1,t)|^2 + 2\|Du\|^2(t), \tag{6.5}$$

$$\operatorname{ess\,sup}_{x\in(0,1)} |u(x,t)| \le |u(1,t)| + \|Du\|(t). \tag{6.6}$$

Taking into account (6.5), (6.6), we estimate

$$\begin{aligned} I &= 2((1+\gamma x)u^2, Du)(t) \le 4(u^2, |Du|)(t) \le 4 \operatorname{ess\,sup}_{x\in(0,1)} |u(x,t)| \|u\|(t)\|Du\|(t) \\ &\le 4\|u\|(t)\|Du\|^2(t) + 4|u(1,t)|\|u\|(t)\|Du\|(t) \\ &\le 4\Big(\|u\|(t) + \frac{1}{2\gamma}\|u\|^2(t)\Big)\|Du\|^2(t) + 2\gamma|u(1,t)|^2. \end{aligned}$$

Substituting I into (6.4) and using (6.5), we obtain

$$\begin{aligned} &\frac{d}{dt}(1+\gamma x, u^2)(t) + \frac{\gamma}{2}\|Du\|^2(t) + \Big[\frac{\gamma}{2} - 4\Big(\|u\|(t) + \frac{1}{2\gamma}\|u\|^2(t)\Big)\Big]\|Du\|^2(t) \\ &+ (2B_0 - 4\gamma)u^2(1,t) + A_0u^2(0,t) + \frac{1}{2}|Du(0,t)|^2 \le 0 \end{aligned} \tag{6.7}$$

which we rewrite as

$$\begin{aligned} &\frac{d}{dt}(1+\gamma x, u^2)(t) + \frac{\gamma}{4}(\|u\|^2(t) - 2|u(1,t)|^2) + \left(\frac{\gamma}{2} - \frac{\gamma}{4} - \frac{18}{\gamma}\|u\|^2(t)\right)\|Du\|^2(t) \\ &+ (2B_0 - 4\gamma)u^2(1,t) + A_0u^2(0,t) + \frac{1}{2}|Du(0,t)|^2 \le 0, \end{aligned}$$

or, taking into account the values of γ,

$$\begin{aligned} &\frac{d}{dt}(1+\gamma x, u^2)(t) + \frac{\gamma}{4(1+\gamma)}(1+\gamma x, u^2)(t) \\ &+ \Big(\frac{\gamma}{4} - \frac{18}{\gamma}\|u\|^2(t)\Big)\|Du\|^2(t) + B_0u^2(1,t) + A_0u^2(0,t) + \frac{1}{2}|Du(0,t)|^2 \le 0. \end{aligned} \tag{6.8}$$

Whence,

$$\frac{d}{dt}(1+\gamma x, u^2)(t) + +\Big(\frac{\gamma}{4} - \frac{18}{\gamma}(1+\gamma x, u^2)(t)\Big)\|Du\|^2(t) \le 0.$$

From here it follows easily as in [17] that if

$$\frac{\gamma}{4} - \frac{18}{\gamma}(1+\gamma x, u_0^2) > 0, \tag{6.9}$$

then

$$\frac{\gamma}{4} - \frac{18}{\gamma}(1+\gamma x, u^2)(t) > 0 \quad \text{for all } t \in (0,T).$$

According to the conditions of Theorem 2.2, (6.9) is valid, whence, (6.8) becomes

$$\frac{d}{dt}(1+\gamma x, u^2)(t) + \frac{\gamma}{4(1+\gamma)}(1+\gamma x, u^2)(t) \le 0$$

and integration gives

$$(1+\gamma x, u^2)(t) \le (1+\gamma x, u_0^2)e^{-\chi t},$$

where $\chi = \frac{\gamma}{4(1+\gamma)}$. Finally,

$$\|u\|^2(t) \le 2\|u_0\|^2 e^{-\chi t}. \tag{6.10}$$

Returning to (6.7), we obtain

$$\int_0^t [u^2(1,\tau)+u^2(0,\tau)+|Du(0,\tau)|^2]d\tau+\|u\|^2(t)+\int_0^t\|Du\|^2(\tau)d\tau\le C,\quad t>0, \tag{6.11}$$

where the constant C does not depend on $t>0$.

Estimate II:. Differentiate (6.1)-(6.2) with respect to t, multiply the result by $2(1+\gamma x)u_t$ and integrate by parts as in Estimate I, to obtain

$$\begin{aligned} &\frac{d}{dt}(1+\gamma x,u_t^2)(t)+3\gamma\|Du_t\|^2(t)-\gamma\|u_t\|^2(t)+2B_0u_t^2(1,t)+A_0u_t^2(0,t)\\ &+\frac12|Du_t(0,t)|^2+2((1+\gamma x)u_t^2,Du)(t)+2((1+\gamma x)uu_t,Du_t)(t)\le 0. \end{aligned} \tag{6.12}$$

Taking into account (6.5), (6.6), we estimate

$$\begin{aligned} I_1&=2((1+\gamma x)uu_t,Du_t)(t)\le 4\operatorname{ess\,sup}_{x\in(0,1)}|u(x,t)|\|u_t\|(t)\|Du_t\|(t)\\ &\le 2\delta\|Du_t\|^2(t)+\frac4\delta(|u(1,t)|^2+\|Du\|^2(t))(1+\gamma x,u_t^2)(t), \end{aligned}$$

where δ is an arbitrary positive real number. Analogously,

$$\begin{aligned} I_2&=2((1+\gamma x)u_t^2,Du)(t)\le 4(|u_t(1,t)|+\|Du\|(t))\|u_t\|(t)\|Du\|(t)\\ &\le 4\delta(|u_t(1,t)|^2+\|Du_t\|^2(t))+\frac2\delta\|Du\|^2(t)(1+\gamma x,u_t^2)(t)\\ &\le 4\delta\|Du_t\|^2(t)+\Big(4\delta+\frac2\delta\|Du\|^2(t)\Big)(1+\gamma x,u_t^2)(t). \end{aligned}$$

Substituting I_1, I_2 into (6.12) and taking $\delta=\frac{\gamma}{12}$, we find

$$\begin{aligned} &\frac{d}{dt}(1+\gamma x,u_t^2)(t)+\frac\gamma2\|Du_t\|^2(t)+B_0u_t^2(1,t)+A_0u_t^2(0,t)+\frac12|Du_t(0,t)|^2\\ &\le C(1+\|Du\|^2(t)+u^2(1,t))(1+\gamma x,u_t^2)(t), \end{aligned} \tag{6.13}$$

where the constant C does not depend on t, u.

Due to (6.11), $u^2(1,t)+\|Du\|^2(t)\in L^1(0,T)$, hence, by the Gronwall lemma,

$$\begin{aligned} (1+\gamma x,u_t^2)(t)&\le\exp\{C\int_0^t[1+\|Du\|^2(\tau)+u^2(1,\tau)]d\tau\}(1+\gamma x,u_t^2)(0)\\ &\le C\|D^3u_0+Du_0+u_0Du_0\|^2 \end{aligned}$$

and

$$\|u_t\|^2(t)\le C\|u_0\|^4_{H^3(0,1)}. \tag{6.14}$$

Returning to (6.13), we obtain

$$\|u_t\|^2(t)+\int_0^t\|Du_\tau\|^2(\tau)d\tau+\int_0^t[u_\tau^2(1,\tau)+u_\tau^2(0,\tau)+|Du_\tau(0,\tau)|^2]d\tau\le C. \tag{6.15}$$

It remains to prove that

$$u\in L^\infty\big(0,T;H^3(0,1)\big)\cap L^2\big(0,T;H^4(0,1)\big).$$

First we estimate

$$\begin{aligned} \|uDu\|(t)&\le(|u(1,t)+\|Du\|(t))\|Du\|(t)\\ &\le\Big[|u(1,0)|+t^{1/2}\Big(\int_0^t|u_\tau(1,\tau)|^2d\tau\Big)^{1/2}+\|Du\|(t)\Big]\|Du\|(t) \end{aligned}$$

$$\le 4|u(1,0)|^2 + 3\|Du\|^2(t) + t\int_0^t |u_\tau(1,\tau)|^2 d\tau \in L^\infty(0,T).$$

On the other hand, due to (6.11), (6.15), $\|Du\|(t), \|Du_t\|(t) \in L^2(0,T)$, then $\|Du\|(t) \in L^\infty(0,T)$. We write (6.1) in the form,

$$D^3u + Du + u = u - u_t - uDu \in L^\infty\big(0,T;L^2(0,1)\big). \tag{6.16}$$

By Theorem 3.1, problem (6.16), (6.2) for all $t > 0$ fixed has a unique solution $u(x,t)$:

$$u \in L^\infty\big(0,T;H^3(0,1)\big). \tag{6.17}$$

Moreover, it is easy to see that

$$uDu \in L^2\big(0,T;H^1(0,1)\big), \quad u_t \in L^2\big(0,T;H^1(0,1)\big), \quad Du \in L^2\big(0,T;H^1(0,1)\big).$$

Hence $D^3u \in L^2\big(0,T;H^1(0,1)\big)$. Combining this and (6.10), (6.11), (6.15), (6.17), we complete the proof. □

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