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# PERSISTENCE OF SOLUTIONS TO NONLINEAR EVOLUTION EQUATIONS IN WEIGHTED SOBOLEV SPACES

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Abstract. In this article, we prove that the initial value problem associated with the Korteweg-de Vries equation is well-posed in weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, for $s \geq 2\theta \geq 2$ and the initial value problem associated with the nonlinear Schrödinger equation is well-posed in weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, for $s \geq \theta \geq 1$. Persistence property has been proved by approximation of the solutions and using a priori estimates.

## 1. Introduction

In this paper we consider the initial value problem (IVP) for the Korteweg-de Vries (KdV) equation

$$\begin{gathered} \partial_t u + u_{xxx} + a(u)u_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{gathered} \tag{1.1}$$

where $u$ a real-valued function and $a \in C^\infty(\mathbb{R},\mathbb{R})$ is a real function.

And the initial value problem for the nonlinear Schrödinger (NLS) equation

$$\begin{gathered} \partial_t u = i(\Delta u - F(u)) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(x,0) = u_0(x), \end{gathered} \tag{1.2}$$

where $u$ a complex-valued function and $F$ satisfies:

- (F1) $F \in C^{[s]+1}(\mathbb{C},\mathbb{C})$ with $F(0) = 0$.
- (F2) If $s \leq n/2$ and if $F(\eta)$ is a polynomial in $\eta$ and $\bar{\eta}$, then $\deg(F) = k \leq \chi(s) := 1 + 4/(n-2\sigma)$, $-\infty \leq \sigma \leq n/2$. If $s \leq n/2$ and if $F(\eta)$ is not a polynomial, then
$$|D^i F(\eta)| \leq c|\eta|^{k-i}, \quad i = 0,1,\dots,[s]+1, \quad \text{as } |\eta| \to \infty, \tag{1.3}$$
where $[s]+1 \leq k \leq \chi(s)$.

The above conditions on $a$ and $F$ guarantee the well-posedness for (1.1) and (1.2) in the usual Sobolev spaces $H^s$, $s \geq 2$ and $H^s$, $s \geq 1$ respectively, see [4, 7]. We are mainly concerned with the question of the persistence property in weighted Sobolev spaces. The aim of this work is to use Lemmas proved in [10, Lemmas 3 and 4 ] and to apply this result to show persistence property of (1.1) in $\mathcal{X}^{s,\theta}$ (see

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definition in (1.15)) for $s \geq 2\theta \geq 2$ and persistence property of (1.2) in $\mathcal{X}^{s,\theta}$ for $s \geq \theta \geq 1$. The notation we took are from [2].

In what follows we introduce the notion of well-posedness that we are going to use throughout this work. We say that (1.1) is locally well-posed in a Banach space $X$, if the following hold.

(1) There exist $T > 0$ and a unique solution $u$ in the time interval $[-T, T]$ (unique existence).
(2) The solution varies continuously depending upon the initial data (continuous dependence); that is, continuity of the application
$$u_0 \to u \quad \text{from } X \text{ to } \mathcal{C}([-T,T];X).$$
In particular if $u_0^n \to u_0$ when $n \to \infty$, then
$$\sup_{t\in[-T,T]} \|u_n(t) - u(t)\|_{H^s} \to 0, \tag{1.4}$$
where $u_n(t)$ is solution of (1.1) with initial data $u_0^n$.
(3) The solution describes a continuous curve in $X$ in the time interval $[-T, T]$ whenever initial data belongs to $X$ (persistence).

Moreover, we say that (1.1) is globally well-posed in $X$ if the same properties hold for all time $T > 0$. If some of the hypotheses in the definition of local well-posedness fail, we say that the IVP is well-posed.

Our main focus in this work will be to show the persistence property. In [2] they proved the persistence property for an equation mixed Korteweg-de Vries - Nonlinear Schrödinger with a weight of low regularity. To accomplish this they used an abstract interpolation lemma ([2, Lemma 2.2]).

The interpolation lemma proved in [2] is quite general and applies to several equations provided they satisfy certain *a priori* estimates. These *a priori* estimates are related to the conserved quantities and are as follows.

$$\|u(t)\|_{L^2} \leq C\|u_0\|_{L^2}. \tag{1.5}$$
$$\|u(t)\|_{\dot{H}^1} \leq C\|u_0\|_{\dot{H}^1} + A_1(\|u_0\|_{L^2}). \tag{1.6}$$
$$\|u(t)\|_{\dot{H}^2} \leq CA_2(\|u_0\|_{\dot{H}^2}, \|u_0\|_{\dot{H}^1}, \|u_0\|_{L^2}). \tag{1.7}$$
$$\|u(t)\|_{L^2(d\dot{\mu}_r)} \leq C\|u_0\|_{L^2(d\dot{\mu}_r)} + A_3(\|D_x^a u_0\|_{L^2}, \|D_x^{a-1} u_0\|_{L^2}, \dots, \|u_0\|_{L^2}), \tag{1.8}$$

where $a = a(r) \geq 1$, $r \in \mathbb{Z}^+$, $A_j$ are continuous functions with $A_1(0) = 0$, $A_2(0,0,0) = 0$ and $A_3(0,\dots,0) = 0$.

It can be inferred that, if one has local well-posedness result for given data in $H^s$ and if the model under consideration satisfies *a priori* estimates (1.5)-(1.8), then with the help of an abstract interpolation lemma, it is easy to prove persistence property in weighted Sobolev spaces.

A typical example of (1.1) that satisfies the properties (1.5)–(1.8) listed above is the IVP associated to the generalized Korteweg-de Vries (gKdV) equation ($a(x) = x^k$ in (1.1))

$$\begin{gathered} \partial_t u + \partial_{xxx} u + u^k \partial_x u = 0, \quad (t,x) \in \mathbb{R}^2,\ k = 1,2,3,\dots \\ u(x,0) = u_0(x). \end{gathered} \tag{1.9}$$

Another typical example is the IVP associated to the Nonlinear Schrödinger (NLS) equation, (1.2) when $F(x) = \mu|x|^{\alpha-1}$.

$$\begin{gathered} i\partial_t u + \Delta u = \mu|u|^{\alpha-1}u, \quad \mu = \pm 1,\ \alpha > 1,\ x \in \mathbb{R}^n,\ t \in \mathbb{R} \\ u(x,0) = u_0(x), \end{gathered} \tag{1.10}$$

the local well-posedness has been studied in [3] for given data in the weighted Sobolev spaces. More precisely, the following result that deals with the persistence property has been proved in [3].

**Theorem 1.1.** *Suppose that $u_0 \in H^s(\mathbb{R}^n) \cap L^2(|x|^{2m}dx)$, $m \in \mathbb{Z}^+$, with $m \le \alpha - 1$ if $\alpha$ is not an odd integer.*

(A) *If $s \ge m$, then there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(x,t)$ of (1.10) with*
$$u \in C([-T,T]; H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T]; L^p_s \cap L^p(|x|^{2m}dx)). \tag{1.11}$$

(B) *If $1 \le s < m$, then (1.11) holds with $[s]$ instead of $m$, and*
$$\Gamma^\beta u = (x_j + 2it\partial_{x_j})^\beta u \in C([-T,T]; L^2) \cap L^q([-T,T]; L^p), \tag{1.12}$$
*for any $\beta \in (\mathbb{Z}^+)^n$ with $|\beta| \le m$.*

The power $m$ of the weight in Theorem 1.1 is assumed to be a positive integer. In the recent work of Nahas and Ponce [10], this restriction in $m$ is relaxed by proving that the persistence property holds for positive real $m$. To be more precise, the result in [10] is the following.

**Theorem 1.2.** *Suppose that $u_0 \in H^s(\mathbb{R}^n) \cap L^2(|x|^{2m}dx)$, $m > 0$, with $m \le \alpha - 1$ if $\alpha$ is not an odd integer.*

(A) *If $s \ge m$, then there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(x,t)$ of (1.10) with*
$$u \in C([-T,T]; H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T]; L^p_s \cap L^p(|x|^{2m}dx)). \tag{1.13}$$

(B) *If $1 \le s < m$, then (1.13) holds with $[s]$ instead of $m$, and*
$$\Gamma^b\Gamma^\beta u \in C([-T,T]; L^2) \cap L^q([-T,T]; L^p), \tag{1.14}$$
*where $\Gamma^b = e^{i|x|^2/4t}2^b t^b D^b(e^{i|x|^2/4t}.)$ with $|\beta| = [m]$ and $b = m - [m]$.*

Kato [5] studied the IVP associated to the gKdV equation for given data in the weighted Sobolev spaces and proved the following result.

**Theorem 1.3** (Kato)**.** *Let $r \in \mathbb{Z}^+$, then the IVP for (1.9) is locally well-posed in weighted Sobolev spaces $\mathcal{X}^{2r,r}$, and globally well-posed in $\mathcal{X}^{2r,r}$ if the initial data satisfies $\|u_0\|_{L^2} < \gamma$.*

In this work we are interested in removing the requirement that the power of the weight in Theorem 1.3 is integer, by proving the similar result for the non integer values of $r \ge 1$ and also we present a proof simples for the persistence in weighted Sobolev spaces for the generalized non-linear Schrödinger equation (1.10) for the non integer values of $r \ge 1$. In [10] they cover all possible values of the parameters $s, \theta$ in the spaces $\mathcal{X}^{s,\theta}$. The main results of this paper are the following.

**Theorem 1.4.** *Problems (1.1) and (1.2) are local well-posed in weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, for $s \ge 2\theta \ge 2$ and $\mathcal{X}^{s,\theta}$, for $s \ge \theta \ge 1$ respectively.*

Without loss of generality in the proof of Theorem 1.4, we will restrict our attention to (1.9) and (1.10). As an application of Theorem 1.4 we have the following result.

**Theorem 1.5.** *Problem* (1.1) *is globally well-posed in* $\mathcal{X}^{s,\theta}$*, for* $s \geq 2\theta \geq 2$*, if the initial data satisfies* $\|u_0\|_{L^2} < \gamma$.

Is not difficult to see that a similar proof as in [2] proves local well-posedness for (1.1), in weighted Sobolev spaces $\mathcal{X}^{s,\theta}$, $s \geq 2$ and $\theta \in [0,1]$.

For other results about persistence, for the problems (1.9) and (1.10) see the work of Nahas and Ponce in [9], see also Nahas, [8] for persistence of the modified Korteweg-de Vries equation (k=2 in (1.10)).

**Notation and Background:** We follow the notation introduced in earlier paper [2]. For the sake of clarity we recall them here. We use $dx$ to denote the Lebesgue measure on $\mathbb{R}$ and, for $\theta \geq 0$, we use

$$d\mu_\theta(x) := (1+|x|^2)^\theta\, dx,$$
$$d\dot{\mu}_\theta(x) := |x|^{2\theta}\, dx$$

to denote the Lebesgue-Stieltjes measures on $\mathbb{R}$. Hence, given a set $X$, a measurable function $f \in L^2(X; d\mu_\theta)$ means that

$$\|f\|^2_{L^2(X;d\mu_\theta)} = \int_X |f(x)|^2\, d\mu_\theta(x) < \infty.$$

When $X = \mathbb{R}$, we write: $L^2(d\mu_\theta) \equiv L^2(\mathbb{R}; d\mu_\theta)$, and for simplicity

$$L^2 \equiv L^2(d\mu_0), \quad L^2(d\mu) \equiv L^2(d\mu_1).$$

Analogously, for the measure $d\dot{\mu}_\theta$. We will use the Lebesgue space-time $L^p_x\mathcal{L}^q_\tau$ endowed with the norm

$$\|f\|_{L^p_x\mathcal{L}^q_\tau} = \big\|\|f\|_{\mathcal{L}^q_\tau}\big\|_{L^p_x} = \Big(\int_{\mathbb{R}}\Big(\int_0^\tau |f(x,t)|^q dt\Big)^{p/q} dx\Big)^{1/p} \quad (1 \leq p,q < \infty).$$

When the integration in the time variable is on the whole real line, we use the notation $\|f\|_{L^p_xL^q_t}$. The notation $\|u\|_{L^p}$ is used when there is no doubt about the variable of integration. Similar notations when $p$ or $q$ are $\infty$.

As usual, $H^s \equiv H^s(\mathbb{R}^n)$, $\dot{H}^s \equiv \dot{H}^s(\mathbb{R}^n)$ are the classic Sobolev spaces in $\mathbb{R}^n$, endowed respectively with the norms

$$\|f\|_{H^s} := \|\widehat{f}\|_{L^2(d\mu_s)}, \quad \|f\|_{\dot{H}^s} := \|\widehat{f}\|_{L^2(d\dot{\mu}_s)}.$$

In this work, we study the solutions of (1.1) in the Sobolev spaces with weight $\mathcal{X}^{s,\theta}$, defined as

$$\mathcal{X}^{s,\theta} := H^s \cap L^2(d\mu_\theta), \tag{1.15}$$

with the norm

$$\|f\|_{\mathcal{X}^{s,\theta}} := \|f\|_{H^s} + \|f\|_{L^2(d\mu_\theta)}.$$

**Remark 2.** We remark that, $\mathcal{X}^{s,1} \subseteq \mathcal{X}^{s,\theta}$, for all $s \in \mathbb{R}$ and $\theta \in [0,1]$.

Indeed, using Hölder's inequality

$$\|f\|_{L^2(d\dot{\mu}_\theta)} \leq \|f\|^{1-\theta}_{L^2}\, \|f\|^\theta_{L^2(d\dot{\mu})}.$$

**Remark 3.** Let $b \in \mathbb{R}$ to denote

$$D^b f(x) = ((2\pi|\xi|)^b \hat{f}\ )^\vee(x).$$

We follow the notation of the classical $\psi$. d.o's in $\mathcal{S}^m_{1,0}$:

$$\mathcal{S}^m_{1,0} := \{a \in \mathcal{C}^\infty(\mathbb{R}^{2n}) : |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \quad \forall \alpha, \beta \in (\mathbb{Z}^+)^n\}.$$

The proof of the following lemmas can be found in [10].

**Lemma 3.1.** *If $a \in \mathcal{S}^0_{1,0}$ and $\langle x\rangle := (1+|x|^2)^{1/2}$, then*

$$a(x,D) : L^2(\mathbb{R}^n; d\mu_b) \to L^2(\mathbb{R}^n; d\mu_b), \quad \forall\, b \ge 0.$$

*is the differential, limited operator.*

**Lemma 3.2.** *Let $a, b > 0$. Suppose that $D^a f \in L^2(\mathbb{R}^n)$ and $\langle x\rangle^b f = (1+|x|^2)^{b/2} f \in L^2(\mathbb{R}^n)$. Then*

$$\|\langle x\rangle^{\theta b} D^{(1-\theta)a} f\|_{L^2} \le C \|\langle x\rangle^b f\|_{L^2}^\theta \|D^a f\|_{L^2}^{1-\theta}. \tag{3.1}$$

## 4. Statement of the well-posedness result

In this section we prove the well-posedness of the Cauchy problem (1.1) in the weighted Sobolev space $\mathcal{X}^{s,\theta}$, for $\theta \ge 1$ and $s \ge 2\theta$.

**Lemma 4.1.** *If $u_0 \in L^2(d\dot\mu_\theta)$, $\theta \in [0,1]$, $\lambda > 0$ and $u_0^\lambda(x) = \mathcal{F}^{-1}(\chi_{\{|\xi|<\lambda\}}\widehat{u_0})(x)$, then*

$$\|u_0^\lambda\|_{L^2(d\dot\mu_\theta)} \le \|u_0\|_{L^2(d\dot\mu_\theta)}. \tag{4.1}$$

If $\theta = 0$, (4.1) is a direct consequence of Plancherel's theorem and definition of $u_0^\lambda$. If $\theta = 1$, using properties of Fourier transform we obtain

$$|\widehat{xu_0^\lambda}(\xi)| = |\partial_\xi \widehat{u_0^\lambda}(\xi)| = |\chi_{\{|\xi|<\lambda\}}\partial_\xi \widehat{u_0}(\xi)| = \chi_{\{|\xi|<\lambda\}}|\widehat{xu_0}(\xi)|.$$

Thus by Plancherel's equality

$$\int_{\mathbb{R}} x^2 |u_0^\lambda(x)|^2 dx = \int_{\mathbb{R}} |\widehat{xu_0^\lambda}(\xi)|^2 d\xi \le \int_{\mathbb{R}} |\widehat{xu_0}(\xi)|^2 d\xi = \int_{\mathbb{R}} |xu_0(x)|^2 dx.$$

When $\theta \in (0,1)$, we obtain (4.1) by interpolation between the cases $\theta = 0$ and $\theta = 1$, see [1].

Lemmas 4.4 and 4.5 tells nothing new; we present a proof for the sake of completeness

### 4.1. **A priori estimates for the nonlinear Schrödinger equation.**

**Lemma 4.2.** *If $u \in \mathbb{S}(\mathbb{R}^n)$, $r \ge 1$. Then*

$$\int_{\mathbb{R}^n} \langle x\rangle^{2r-2} |D_x u|^2 \, dx \le \Big(\int_{\mathbb{R}^n} \langle x\rangle^{2r} |u|^2 \, dx\Big)^{1-\frac{1}{r}} \Big(\int_{\mathbb{R}^n} |D^r u|^2 \, dx\Big)^{1/r}.$$

*Proof.* We apply the Lemma 3.2, taking $a = b = r$ and $\theta = 1 - \dfrac{1}{r}$, then $r \ge 1$ since $0 \le \theta \le 1$. □

**Lemma 4.3.** *If $u \in \mathbb{S}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \langle x\rangle^{2b} |\nabla u(t,x)|^2 dx \le b_{r,n} \int_{\mathbb{R}^n} \langle x\rangle^{2b} |D_x u(t,x)|^2 dx + b_{r,n} \int_{\mathbb{R}^n} \langle x\rangle^{2b} |u(t,x)|^2 dx.$$

*Proof.* Since $\widehat{D_x u}(\xi) = |\xi|\widehat{u}(\xi)$, we consider $a(x,\xi) := \dfrac{\xi_j}{1+|\xi|}$ and using the Lemma 3.1, we can see that the operator $a(x,\xi)$ is bounded and $a \in S^0_{1,0}$. □

**Lemma 4.4.** *If $u$ is a solution of the IVP for the NLS (1.10) with $u_0 \in \mathcal{X}^{s,r}$, $s \geq r \geq 1$. Then*

$$\int_{\mathbb{R}^n} \varphi|u|^2\,dx \leq \big\{C_{r,n} \sup_{t\in[-T,T]} \|u(t)\|^2_{H^r(\mathbb{R}^n)} + \|u(0)\|^2_{L^2(d\mu_r)}\big\} e^{c_{r,n}T}. \tag{4.2}$$

*Proof.* Consider $\varphi(x) := (1+|x|^2)^r = \langle x\rangle^{2r}$ to $x \in \mathbb{R}^n$ Multiplying the term $\varphi\overline{u}$ where $u \in S(\mathbb{R}^n)$ in equation (1.10) and after integrating on $\mathbb{R}^n$, we obtain taking real part

$$2\Re\big\{\int_{\mathbb{R}^n} u_t\varphi\overline{u}\,dx\big\} - 2\Re\big\{i\int_{\mathbb{R}^n} \Delta u\varphi\overline{u}\,dx\big\} = -2\mu\Re\big\{i\int_{\mathbb{R}^n} |u|^\alpha\varphi\,dx\big\} \tag{4.3}$$

observe that $\partial_t u.\overline{u} = 2\Re\{u.\overline{u}_t\}$. Replacing in (4.3), we obtain

$$\partial_t\int_{\mathbb{R}^n} \varphi\,dx|u|^2 = 2\Re\big\{i\int_{\mathbb{R}^n} \Delta u\varphi\overline{u}\,dx\big\}, \tag{4.4}$$

on the other hand

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi\partial^2_{x_i} u\overline{u}\,dx &= -\int_{\mathbb{R}^n} \partial_{x_i}(\varphi\overline{u})\partial_{x_i} u\,dx \\ &= \int_{\mathbb{R}^n} (\varphi\partial^2_{x_i}\overline{u} + 2\partial_{x_i}\varphi\partial_{x_i}\overline{u} + \partial^2_{x_i}\varphi\overline{u})u\,dx, \end{aligned} \tag{4.5}$$

of (4.5), we obtain

$$\int_{\mathbb{R}^n} \varphi\Delta u\overline{u}\,dx = \int_{\mathbb{R}^n} (\varphi\Delta\overline{u} + 2\nabla\varphi.\nabla\overline{u} + \Delta\varphi\overline{u}\ )u\,dx,$$

which leads us to

$$2i\int_{\mathbb{R}^n} \varphi\Im\{\Delta u\overline{u}\}\,dx = \int_{\mathbb{R}} \Delta\varphi|u|^2\,dx + 2\int_{\mathbb{R}^n} \nabla\varphi.\nabla\overline{u}u\,dx, \tag{4.6}$$

of (4.4) and (4.6), we obtain

$$\partial_t\int_{\mathbb{R}^n} \varphi|u|^2\,dx = i\int_{\mathbb{R}} \Delta\varphi|u|^2\,dx + 2i\int_{\mathbb{R}^n} \nabla\varphi.\nabla\overline{u}u\,dx,$$

and taking real part

$$\partial_t\int_{\mathbb{R}^n} \varphi|u|^2\,dx = 2\Re\big\{i\int_{\mathbb{R}^n} \nabla\varphi.\nabla\overline{u}u\,dx\big\}. \tag{4.7}$$

Notice that

$$|\nabla\varphi| \leq 2r\langle x\rangle^{2r-1}, \tag{4.8}$$

so

$$\begin{aligned} \big|\Im\big\{\int_{\mathbb{R}^n} \nabla\varphi.\nabla\overline{u}u\,dx\big\}\big| &\leq \int_{\mathbb{R}^n} |\nabla\varphi||\nabla u||u|\,dx \\ &\leq 2r\int_{\mathbb{R}^n} \langle x\rangle^r|u|\langle x\rangle^{r-1}|\nabla u|\,dx \\ &\leq r\int_{\mathbb{R}^n} \varphi|u|^2\,dx + r\int_{\mathbb{R}^n} \langle x\rangle^{2r-2}|\nabla u|^2\,dx. \end{aligned} \tag{4.9}$$

Applying Lemma 4.3, (4.4) and (4.9), we have

$$\partial_t\int_{\mathbb{R}^n} \varphi|u|^2\,dx \leq c_{r,n}\int_{\mathbb{R}^n} \varphi|u|^2\,dx + c_{r,n}\int_{\mathbb{R}^n} \langle x\rangle^{2r-1}|D_x u|^2\,dx, \tag{4.10}$$

and using Lemma 4.2, we obtain

$$\partial_t \int_{\mathbb{R}^n} \varphi|u|^2\,dx \le c_{r,n}\int_{\mathbb{R}^n} \varphi|u|^2\,dx + c_{r,n}\int_{\mathbb{R}^n} |D_x u|^2\,dx + c_{r,n}\int_{\mathbb{R}^n} |D^r u|^2\,dx.$$

Thus

$$\partial_t \int_{\mathbb{R}^n} \varphi|u|^2\,dx \le c_{r,n}\int_{\mathbb{R}^n} \varphi|u|^2\,dx + c_{r,n}\|u\|^2_{H^r(\mathbb{R}^n)}, \tag{4.11}$$

applying Gronwall, we obtain the result. □

### 4.2. **A priori estimates for the generalized Korteweg-de Vries equation.**

**Lemma 4.5.** *If $u$ is a solution of the IVP for* (1.9) *with $u_0 \in \mathcal{X}^{s,\theta}$, $s \ge 2\theta \ge 2$. Then*

$$\int_{\mathbb{R}} \varphi|u|^2\,dx \le C_{\theta,k}\{\sup_{t\in[-T,T]}\|u\|^2_{H^1(\mathbb{R})} + \sup_{t\in[-T,T]}\|u\|^2_{H^{2\theta}(\mathbb{R})}\}e^{c_{\theta,k}T} + \|u(0)\|^2_{L^2(d\mu_\theta)}\,e^{c_{\theta,k}T}.$$

*Proof.* Let $u \in S(\mathbb{R})$. In (1.9) consider $k \in \mathbb{N}$, $s \ge 2\theta$, $\theta \ge 1$. Now multiply the equation by the term $\varphi u$ and after integrating on $\mathbb{R}$, where $\varphi(x) := (1+|x|^2)^\theta$.

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \varphi|u|^2\,dx &= -2\int_{\mathbb{R}} \varphi u u_{xxx}\,dx - 2\int_{\mathbb{R}} \varphi u^{k+1}u_x\,dx\\ &= -\frac{1}{2}\int_{\mathbb{R}} \varphi_{xxx}u^2\,dx + 3\int_{\mathbb{R}} \varphi_x u_{xx} u\,dx - \frac{2}{k+2}\int_{\mathbb{R}} \varphi\partial_x u^{k+2}\,dx\\ &= \int_{\mathbb{R}} \varphi_{xxx}u^2\,dx - 3\underbrace{\int_{\mathbb{R}} \varphi_x|u_x|^2\,dx}_{I_3} - \frac{2}{k+2}\underbrace{\int_{\mathbb{R}} \varphi\partial_x u^{k+2}\,dx}_{I_4}.\end{aligned} \tag{4.12}$$

Is obvious that

$$\int_{\mathbb{R}} \varphi_{xxx}|u|^2\,dx \le C_\theta \int_{\mathbb{R}} \varphi|u|^2\,dx.$$

Applying interpolation

$$\begin{aligned}|I_3| &\le C_\theta\|u\|^2_{H^1(\mathbb{R})} + \Big(\int_{\mathbb{R}} |x|^{2\theta}|u|^2\,dx\Big)^{1-\frac{1}{2\theta}}\Big(\int_{\mathbb{R}} |D_x^{2\theta}|^2\,dx\Big)^{1/(2\theta)}\\ |I_4| &\le C_\theta \sup_{t\in[-T,T]}\|u(t)\|^k_{H^1(\mathbb{R})}\int_{\mathbb{R}} \varphi|u|^2\,dx.\end{aligned} \tag{4.13}$$

Using Young

$$\begin{aligned}|I_3| &\le C_{\theta,k}\Big(\sup_{t\in[-T,T]}\|u(t)\|_{H^1(\mathbb{R})^2} + \sup_{t\in[-T,T]}\|u(t)\|^2_{H^{2\theta}(\mathbb{R})}\Big)\\ &\quad + C_{\theta,k}\Big(1 + \sup_{t\in[-T,T]}\|u(t)\|^k_{H^1(\mathbb{R})}\Big)\int_{\mathbb{R}} \varphi|u|^2\,dx.\end{aligned} \tag{4.14}$$

Applying similar ideas to the case Nonlinear Schrödinger (NLS) equation and using Gronwall, we complete the proof. □

### 4.3. **Proof of Theorems 1.4 and 1.5.**

*Proof of Theorem 1.4 (case gKdV).* The case NLS follows a similar argument. Let $u_0 \in \mathcal{X}^{s,\theta}$, $s \geq 2\theta \geq 2$, $u_0 \neq 0$, we know that that there exists an function $u \in C([-T,T], H^s)$ such that (1.9) is local well-posed in $H^s$. Is well know that $\mathbf{S}(\mathbb{R})$ is dense in $\mathcal{X}^{s,\theta}$. Then for $u_0 \in \mathcal{X}^{s,\theta}$ there exist a sequence $(u_0^\lambda)$ in $\mathbf{S}(\mathbb{R})$ such that

$$u_0^\lambda \to u_0 \quad \text{in } \mathcal{X}^{s,\theta}. \tag{4.15}$$

By (1.4) (continuous dependence) the sequence of solutions $u^\lambda(t)$ associated to IVP (1.1) with initial data $u_0^\lambda$

$$\begin{gathered} \partial_t u^\lambda + u^\lambda_{xxx} + (u^\lambda)^k u^\lambda_x = 0, \quad (t,x) \in \mathbb{R}^2, \\ u^\lambda(x,0) = u_0^\lambda(x), \end{gathered} \tag{4.16}$$

satisfy

$$\sup_{t\in[-T,T]} \|u^\lambda(t) - u(t)\|_{H^s} \overset{\lambda\to\infty}{\to} 0, \quad s \geq 2\theta \geq 2. \tag{4.17}$$

The solutions $u^\lambda$ of (4.16) satisfy the conditions (1.5)-(1.8) of Section 1. Therefore, Lemma 4.5 gives

$$\begin{aligned} \int_{\mathbb{R}} \varphi|u^\lambda|^2\,dx \leq C_{\theta,k}\{ &\sup_{t\in[-T,T]} \|u^\lambda\|^2_{H^1(\mathbb{R})} + \sup_{t\in[-T,T]} \|u^\lambda\|^2_{H^{2\theta}(\mathbb{R})}\} e^{c_{\theta,k}T} \\ &+ \|u^\lambda(0)\|^2_{L^2(d\mu_\theta)}\, e^{c_{\theta,k}T}, \end{aligned}$$

Taking the limit when $\lambda \to \infty$, (4.17) implies

$$\begin{aligned} \int_{\mathbb{R}} \varphi|u|^2\,dx \leq C_{\theta,k}\{ &\sup_{t\in[-T,T]} \|u\|^2_{H^1(\mathbb{R})} + \sup_{t\in[-T,T]} \|u\|^2_{H^{2\theta}(\mathbb{R})}\} e^{c_{\theta,k}T} \\ &+ \|u(0)\|^2_{L^2(d\mu_\theta)}\, e^{c_{\theta,k}T}. \end{aligned}$$

Thus $u(t) \in \mathcal{X}^{s,\theta}$, $t \in [-T,T]$, which proves the persistence. The local well-posedness theory in $H^s$ implies the uniqueness and continuous dependence upon the initial data in $H^s$, this imply uniqueness in $\mathcal{X}^{s,\theta}$.

Now we will prove continuous dependence in the norm $\|\cdot\|_{L^2(d\dot{\mu}_\theta)}$. Let $u(t)$ and $v(t)$ be two solutions in $\mathcal{X}^{s,\theta}$, of (1.10) with initial dates $u_0$ and $v_0$ respectively, let $u^\lambda(t)$, $v^\lambda(t)$ be the solutions associated with (1.10) with initial dates $u_0^\lambda$ and $v_0^\lambda$ respectively such that $u_0^\lambda, v_0^\lambda \in \mathbf{S}(\mathbb{R})$,

$$u_0^\lambda \to u_0, \quad v_0^\lambda \to v_0 \quad \text{in } \mathcal{X}^{s,\theta} \tag{4.18}$$

and with $\lambda \gg 1$, we have

$$\begin{aligned} \|u(t) - v(t)\|_{L^2(d\dot{\mu}_\theta)} \leq \|u(t) - u^\lambda(t)\|_{L^2(d\dot{\mu}_\theta)} &+ \|u^\lambda(t) - v^\lambda(t)\|_{L^2(d\dot{\mu}_\theta)} \\ &+ \|v^\lambda(t) - v(t)\|_{L^2(d\dot{\mu}_\theta)}. \end{aligned}$$

The convergence

$$\sup_{t\in[-T,T]} \|u^\lambda(t) - u(t)\|_{H^s} \to 0, \qquad \sup_{t\in[-T,T]} \|v^\lambda(t) - v(t)\|_{H^s} \to 0, \tag{4.19}$$

as $\lambda \to \infty$, where $s \geq 2\theta \geq 2$, implies for $\lambda \gg 1$ that

$$|u(x,t) - u^\lambda(x,t)| \leq 2|u(x,t)| \quad \text{and} \quad |v(x,t) - v^\lambda(x,t)| \leq 2|v(x,t)|.$$

The Dominated Convergence Lebesgue's Theorem gives

$$\|u(t) - u^\lambda(t)\|_{L^2(d\dot{\mu}_\theta)} \to 0 \quad \text{and} \quad \|v^\lambda(t) - v(t)\|_{L^2(d\dot{\mu}_\theta)} \to 0.$$

Let $w^\lambda := u^\lambda - v^\lambda$, then $w^\lambda$ satisfies the equation

$$w_t^\lambda + w_{xxx}^\lambda + (u^\lambda)^k w_x^\lambda + v_x^\lambda A(u^\lambda, u^\lambda) w^\lambda = 0,$$

where $A(x, y) = x^{k-1} + x^{k-2}y + \dots + xy^{k-2} + y^{k-1}$.

Then, we multiply the above equation by $\varphi \bar{w}^\lambda$, integrate on $\mathbb{R}$, to obtain by Gronwall's Lemma that

$$\int_{\mathbb{R}} \varphi |w^\lambda(t, x)|^2 \, dx \le \Big\{ \int_{\mathbb{R}} \varphi |w^\lambda(0, x)|^2 \, dx + c_\theta \sup_{t \in [-T,T]} \|w^\lambda(t)\|_{H^{2\theta}}^2 \Big\} e^{k_0 T}, \quad (4.20)$$

where $k_0$ is a constant to $\lambda \gg 1$. Observe that the convergence (4.18) and (4.19) imply

$$\|w^\lambda(0)\|_{L^2(d\mu_\theta)} = \|u_0^\lambda - v_0^\lambda\|_{L^2(d\mu_\theta)} \le 2\|u_0 - v_0\|_{L^2(d\mu_\theta)},$$

and

$$\|w^\lambda(t)\|_{H^{2\theta}} = \|u^\lambda(t) - v^\lambda(t)\|_{H^{2\theta}} \le 2 \sup_{t \in [-T,T]} \|u(t) - v(t)\|_{H^{2\theta}},$$

if $\lambda \gg 1$, which together with (4.20) gives the continuous dependence. □

*Proof of Theorem 1.5.* Is a direct consequence of the proof of Theorem 1.4 and the global theory for the gKDV equation (see [5]). □

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