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LIFE SPAN OF BLOW-UP SOLUTIONS FOR HIGHER-ORDER SEMILINEAR PARABOLIC EQUATIONS

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Abstract. In this article, we study the higher-order semilinear parabolic equation

$$u_t + (-\Delta)^m u = |u|^p, \quad (t,x) \in \mathbb{R}^1_+ \times \mathbb{R}^N,$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N.$$

Using the test function method, we derive the blow-up critical exponent. And then based on integral inequalities, we estimate the life span of blow-up solutions.

1. Introduction

This article concerns the cauchy problem for the higher-order semilinear parabolic equation

$$\begin{gathered} u_t + (-\Delta)^m u = |u|^p, \quad (t,x) \in \mathbb{R}^1_+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{gathered} \tag{1.1}$$

where $m, p > 1$. Higher-order semilinear and quasilinear heat equations appear in numerous applications such as thin film theory, flame propagation, bi-stable phase transition and higher-order diffusion. For examples of these mathematical models, we refer the reader to the monograph [9]. For studies of higher-order heat equations we refer also to [1, 2, 4, 5, 6, 10] and the references therein.

In [6], under the assumption that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0 \not\equiv 0$ and

$$\int_{\mathbb{R}^N} u_0(x) \mathrm{d}x \ge 0, \tag{1.2}$$

Galaktionov and Pohozaev studied the Fujita critical exponent of problem (1.1) and showed that $p_F = 1 + 2m/N$. The critical exponents p_F is calculated from both sides:

(i) blow-up of any solutions with (1.2) for $1 < p \le p_F$ and
(ii) global existence of small solutions for $p > p_F$.

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Egorov et al [5] studied the asymptotic behavior of global solutions with suitable initial data in the supercritical Fujita range $p > p_F$ by constructing self-similar solutions of higher-order parabolic operators and through a stability analysis of the autonomous dynamical system. For other studies of the problem, we refer to [4] where global non-existence was proved for $p \in (1, p_F]$ by using the test function approach, and [1] where a general situation was discussed with nonlinear function $h(u)$ in place of $|u|^p$.

In a recent paper [10], we discussed the system

$$\begin{gathered} u_t + (-\Delta)^m u = |v|^p, \quad (t,x) \in \mathbb{R}^1_+ \times \mathbb{R}^N, \\ v_t + (-\Delta)^m v = |u|^q, \quad (t,x) \in \mathbb{R}^1_+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}^N. \end{gathered} \tag{1.3}$$

It is proved that if $N/(2m > \max\left\{\frac{1+p}{pq-1}, \frac{1+q}{pq-1}\right\}$ then solutions of (1.3) with small initial data exist globally in time. Moreover the decay estimates $\|u(t)\|_\infty \le C(1+t)^{-\sigma_1}$ and $\|v(t)\|_\infty \le C(1+t)^{-\sigma_2}$ with $\sigma_1 > 0$ and $\sigma_2 > 0$ are also satisfied. On the other hand, under the assumption that

$$\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x > 0, \quad \int_{\mathbb{R}^N} v_0(x)\mathrm{d}x > 0,$$

if $N/(2m) \le \max\left\{\frac{1+p}{pq-1}, \frac{1+q}{pq-1}\right\}$ then every solution of (1.3) blows up in finite time.

In our present work, exploiting the test function method, we shall give the life span of blow-up solution for some special initial data. The main idea comes from [7] for discussing cauchy problem of the second order equation

$$\begin{gathered} \rho(x)u_t - \Delta u^m = h(x,t)u^{1+p}, \quad (t,x) \in \mathbb{R}^1_+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \end{gathered} \tag{1.4}$$

Using the test function method, the author gave the blow-up type critical exponent and the estimates for life span $[0, T)$ like that in [8]. For the construction of a test function, the author mainly based on the eigenfunction Φ corresponding to the principle eigenvalue λ_1 of the Dirichlet problem on unit ball B_1,

$$\begin{gathered} -\Delta w(x) = \lambda_1 w(x), \quad x \in B_1, \\ w(x) = 0, \quad x \in \partial B_1. \end{gathered}$$

However, for the operator $(-\Delta)^m$, the eigenfunction Φ corresponding to the principal eigenvalue λ_1 of the Dirichlet problem may change sign (see [3]). We will use a non-negative smooth function Φ constructed in [1] and [6]. The organization of this paper is as follows. In section 2, by the test function method, we derive some integral inequalities and reacquire the Fujita critical exponent p_F obtained in the paper [6]. Section 3 is for the estimate of life span of blow-up solution.

2. Fujita critical exponent

In this section, we shall use the test function method to derive the Fujita critical exponent and some useful inequalities. From the reference [6], we know that if $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then the solution $u(t,\cdot) \in C^1([0,T]; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ for some $T > 0$. Therefore, without loss of generality, we may consider $u_0(x)$ concentrated around the origin and bounded below by a positive constant in some neighborhood of origin. Further, $u_0(x) \to 0$ as $|x| \to \infty$. With these choices, the solution u and its spatial derivatives vanish as $|x| \to \infty$ for $t > 0$.

First we construct a test function. For this aim, we shall use a non-negative smooth function Φ which was constructed in the papers [1] and [6].

Let

$$\Phi(x) = \Phi(|x|) > 0, \quad \Phi(0) = 1; \quad 0 < \Phi(r) \le 1 \quad \text{for } r > 0,$$

where $\Phi(r)$ is decreasing and $\Phi(r) \to 0$ as $r \to \infty$ sufficiently fast. Moreover, there exists a constant $\lambda_1 > 0$ such that

$$|\Delta^m \Phi| \le \lambda_1 \Phi, \quad x \in \mathbb{R}^N, \tag{2.1}$$

and such that

$$\|\Phi\|_1 = \int_{\mathbb{R}^N} \Phi(x) \mathrm{d}x = 1.$$

This can be done by letting $\Phi(r) = e^{-r^\nu}$ for $r \gg 1$ with $\nu \in (0, 1]$, and then extending Φ to $[0, \infty)$ by a smooth approximation. Take $\theta > p/(p-1)$, and define

$$\phi(t) = \begin{cases} 0, & t > T, \\ (1 - (t-S)/(T-S))^\theta, & 0 \le t \le T, \\ 1, & t < S, \end{cases}$$

where $0 \le S < T$. Now set

$$\xi(t, x) = \phi(t/R^{2m}) \Phi(x/R), \quad R > 0.$$

Suppose that u exists in $[0, t_*) \times \mathbb{R}^N$. For $TR^{2m} < t_*$, multiply both sides of equation (1.1) by ξ and integrate over $[0, TR^{2m}) \times \mathbb{R}^N$ by parts to obtain

$$\int_0^{TR^{2m}} \int_{\mathbb{R}^N} |u|^p \xi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\xi(0, x)\mathrm{d}x \le \int_0^{TR^{2m}} \int_{\mathbb{R}^N} |u|\{|\xi_t| + |\Delta^m \xi|\}\mathrm{d}x\mathrm{d}t. \tag{2.2}$$

Denote

$$I(S, T) = \int_{SR^{2m}}^{TR^{2m}} \int_{\mathbb{R}^N} |u|^p \phi(t/R^{2m})\Phi(x/R)\mathrm{d}x\mathrm{d}t, \quad J = \int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x.$$

We now estimate $I(0, T) + J$. Using the Hölder inequality, since $\phi'(t) = 0$ except on (S, T), we obtain

$$\begin{aligned} \int_0^{TR^{2m}} \int_{\mathbb{R}^N} |u||\xi_t|\mathrm{d}x\mathrm{d}t &= \int_{SR^{2m}}^{TR^{2m}} \int_{\mathbb{R}^N} |u|\phi(t/R^{2m})^{1/p}|\phi'(t/R^{2m})| \\ &\quad \times \phi(t/R^{2m})^{-1/p}\ \Phi(x/R)R^{-2m}\mathrm{d}x\mathrm{d}t \\ &\le I(S, T)^{1/p} R^{-2m} \Big(\int_{SR^{2m}}^{TR^{2m}} \int_{\mathbb{R}^N} |\phi'(t/R^{2m})|^{p/(p-1)} \\ &\quad \times \phi(t/R^{2m})^{-1/(p-1)}\Phi(x/R)\mathrm{d}x\mathrm{d}t \Big)^{(p-1)/p}. \end{aligned} \tag{2.3}$$

Since $\Delta_x^m \Phi(x/R) = R^{-2m}\Delta_y^m \Phi(y)$ for $y = x/R$, using the Hölder inequality and (2.1) we have

$$\begin{aligned}
&\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u||\Delta^m \xi| \mathrm{d}x\mathrm{d}t \\
&= \int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|\phi(t/R^{2m})|\Delta^m \Phi(x/R)| \mathrm{d}x\mathrm{d}t \\
&= R^{-2m}\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|\phi(t/R^{2m})|\Delta_{x/R}^m \Phi(x/R)| \mathrm{d}x\mathrm{d}t \\
&\le \lambda_1 R^{-2m}\int_0^{TR^{2m}}\int_{\mathbb{R}^N} |u|\phi(t/R^{2m})\Phi(x/R) \mathrm{d}x\mathrm{d}t \\
&\le I(0,T)^{1/p}\lambda_1 R^{-2m}\Big(\int_0^{TR^{2m}}\int_{\mathbb{R}^N} \phi(t/R^{2m})\Phi(x/R) \mathrm{d}x\mathrm{d}t\Big)^{(p-1)/p}.
\end{aligned} \tag{2.4}$$

Making the change of variables $\tau = t/R^{2m}$ and $\eta = x/R$, from (2.2), (2.3) and (2.4) we deduce that

$$\begin{aligned}
&I(0,T) + J \\
&\le I(S,T)^{1/p}R^s\Big(\int_{SR^{2m}}^{TR^{2m}}\int_{\mathbb{R}^N} |\phi'(\tau)|^{p/(p-1)}\phi(\tau)^{-1/(p-1)}\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p} \\
&\quad + I(0,T)^{1/p}\lambda_1 R^s\Big(\int_0^T\int_{\mathbb{R}^N} \phi(\tau)\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p},
\end{aligned} \tag{2.5}$$

where $s = -2m + (2m+N)(p-1)/p$. Set

$$A(S,T) = \Big(\int_S^T\int_{\mathbb{R}^N} |\phi'(\tau)|^{p/(p-1)}\phi(\tau)^{-1/(p-1)}\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p},$$

$$B(T) = \Big(\int_0^T\int_{\mathbb{R}^N} \phi(\tau)\Phi(\eta)\mathrm{d}\eta\mathrm{d}\tau\Big)^{(p-1)/p}.$$

Thus (2.5) can be simply written as

$$I(0,T) + J \le R^s[I(S,T)^{1/p}A(S,T) + \lambda_1 I(0,T)^{1/p}B(T)]. \tag{2.6}$$

We have the following result:

Theorem 2.1 (Fujita critical exponent)**.** *If*

$$\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x \ge 0,\ \ u_0(x) \not\equiv 0$$

and $s \le 0$, that is to say $p \le p_c = 1 + 2m/N$, then (1.1) *has no global solution.*

Proof. By slightly shifting the origin in time, we may assume

$$\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x > 0. \tag{2.7}$$

Let u be a global solution with u_0 satisfying (2.7), then

$$\int_0^\infty\int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t > 0.$$

Suppose $s < 0$. Letting R tend to infinity in (2.6) to obtain

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x = 0.$$

Hence $u \equiv 0$, a contradiction.

Suppose $s = 0$. We first show $J \geq 0$ for all $R > 0$. In fact, from the assumptions on initial datum, there exists $\varepsilon_0 > 0$ such that $u_0(x) \geq \delta > 0$ for $|x| \leq \varepsilon_0$. Set

$$\begin{aligned} J &= \int_{|x|\leq\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x + \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\ &> \delta \int_{|x|\leq\varepsilon_0} \Phi(x/R)\mathrm{d}x + \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\ &= \delta R^N \int_{|\eta|\leq\varepsilon_0/R} \Phi(\eta)\mathrm{d}\eta + \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\ &\geq \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x. \end{aligned}$$

By the choice of Φ, we have

$$\lim_{R\to 0} \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x = 0.$$

And so there exists $R_0 > 0$ such that $J \geq 0$ for all $0 < R < R_0$. On the other hand, there exists $M > 0$ such that

$$\int_{|x|\leq R_0 M} u_0(x)\mathrm{d}x > \int_{|x|>R_0 M} |u_0(x)|\mathrm{d}x.$$

In addition, by a slight modification of Φ, we may set $\Phi(x) \equiv 1$ in $\{x : \ |x| \leq M\}$. Note that since $0 \leq \Phi \leq 1$ we have, for $R \geq R_0$,

$$\begin{aligned} J &= \int_{|x|\leq R_0 M} u_0(x)\Phi(x/R)\mathrm{d}x + \int_{|x|>R_0 M} u_0(x)\Phi(x/R)\mathrm{d}x \\ &\geq \int_{|x|\leq R_0 M} u_0(x)\mathrm{d}x - \int_{|x|>R_0 M} |u_0(x)|\Phi(x/R)\mathrm{d}x \\ &\geq \int_{|x|\leq R_0 M} u_0(x)\mathrm{d}x - \int_{|x|>R_0 M} |u_0(x)|\mathrm{d}x > 0. \end{aligned}$$

Now we are in the position to complete the proof of case $s = 0$. Since

$$A(S,T) = \frac{\theta(T-S)^{-1/p}}{[\theta - 1/(p-1)]^{(p-1)/p}}, \quad B(T) = \Big[S + \frac{T-S}{\theta+1}\Big]^{(p-1)/p},$$

we may choose S small and θ large, $T - S$ bounded, such that

$$B(T) \leq \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x / \Big[2\lambda_1 \Big(\int_0^\infty \int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t\Big)^{1/p}\Big]. \tag{2.8}$$

Moreover, note that $J \geq 0$, from (2.6) we get that $I(0,T)$ is uniformly bounded for all $R > 0$. Then, keeping $T - S$ bounded,

$$\lim_{R\to\infty} I(S,T)^{1/p} A(S,T) = 0. \tag{2.9}$$

Letting $R \to \infty$, (2.6)–(2.9) give

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}^N} u_0(x)\mathrm{d}x = 0,$$

which also implies $u \equiv 0$. □

Let σ be an arbitrary positive number. For $x \in [0, \infty)$ and $0 < \omega < 1$, define

$$\Psi(\omega;\sigma) := \max_x(\sigma x^\omega - x).$$

It is easy to check that $\Psi(\omega;\sigma) = (1-\omega)\omega^{\frac{\omega}{1-\omega}}\sigma^{\frac{1}{1-\omega}}$. Set

$$A(T) = A(0,T), \quad S(T) = A(T) + \lambda_1 B(T).$$

We have the following result.

Theorem 2.2. *If u is a solution of* (1.1) *defined on $[0,t_*) \times \mathbb{R}^N$. Then, for $R > 0$ and $0 \le \tau \le t_* R^{-2m}$, we have*

$$\int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \le \Psi\Big(\frac{1}{p};\ S(T)R^s\Big). \tag{2.10}$$

Moreover, if u is a global solution of (1.1)*, then*

$$\lim_{R\to\infty}\sup R^{-\hat{s}}\int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \le \lambda_1^{1/(p-1)}, \tag{2.11}$$

where $\hat{s} = sp/(p-1)$.

Proof. Denote $I(T) = I(0,T)$. Firstly, by the definition of Ψ, from (2.6) we know that

$$J \le I(T)^{1/p}S(T)R^s - I(T) \le \Psi\Big(\frac{1}{p};\ S(T)R^s\Big).$$

This is exactly (2.10). By means of (2.10), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x &\le \Psi\Big(\frac{1}{p};\ S(T)R^s\Big)\\ &= (1-1/p)(1/p)^{\frac{1/p}{1-1/p}}[S(T)R^s]^{\frac{1}{1-1/p}}\\ &= (p-1)p^{p/(1-p)}R^{sp/(p-1)}S(T)^{\frac{p}{p-1}},\end{aligned} \tag{2.12}$$

which leads to

$$\lim_{R\to\infty}\sup R^{-\hat{s}}\int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \le (p-1)p^{p/(1-p)}[\inf_T S(T)]^{\frac{p}{p-1}}. \tag{2.13}$$

To estimate $S(T)$, we need estimate $A(T)$ and $B(T)$ respectively. Denote

$$a_p = \frac{\theta}{[\theta - 1/(p-1)]^{(p-1)/p}}, \quad b_p = \frac{\lambda_1}{(\theta+1)^{(p-1)/p}}.$$

We obtain

$$S(T) = a_p T^{-1/p} + b_p T^{(p-1)/p}.$$

Since

$$\begin{aligned}\min_T S(T) &= p[a_p/(p-1)]^{(p-1)/p}b_p^{1/p}\\ &= \frac{p(p-1)^{-(p-1)/p}\lambda_1^{1/p}\theta^{(p-1)/p}}{[\theta-1/(p-1)]^{(p-1)^2/p^2}(1+\theta)^{(p-1)/p^2}},\end{aligned}$$

we have

$$\lim_{\theta\to\infty}\min_T S_p(T) = p(p-1)^{-(p-1)/p}\lambda_1^{1/p}. \tag{2.14}$$

Combining (2.13) and (2.14), we obtain (2.11). The proof is complete. □

3. Life span of blow-up solutions

In this section, we shall estimate the life span of the blow-up solution with some special initial datum. To this aim, we assume that u_0 satisfies

(H) There exist positive constants C_0, L such that

$$u_0(x) \geq \begin{cases} \delta, & |x| \leq \varepsilon_0, \\ C_0|x|^{-\kappa}, & |x| > \varepsilon_0, \end{cases}$$

where δ and ε_0 are as in the proof of Theorem 2.1, and $N < \kappa < 2m/(p-1)$ if $p < 1 + 2m/N$; $0 < \kappa < N$ if $p = 1 + 2m/N$.

Now we state the main result.

Theorem 3.1. *Let* (H) *be fulfilled and u_ε be the solution of* (1.1) *with initial data $u_\varepsilon(0, x) = \varepsilon u_0(x)$, where $\varepsilon > 0$. Denote $[0, T_\varepsilon)$ be the life span of u_ε. Then there exists a positive constant C such that $T_\varepsilon \leq C\varepsilon^{1/\hat{\beta}}$, where*

$$\hat{\beta} = \frac{\kappa}{2m} - \frac{1}{p-1} < 0.$$

Remark 3.2. When $p = 1 + 2m/N$, note that $\hat{\beta} = (\kappa - N)/(2m)$.

Proof. Choose R such that $R \geq R^0 > 0$. By the definition of J and the assumptions of initial data, we have

$$\begin{aligned} J &= \varepsilon \int_{\mathbb{R}^N} u_0(x)\Phi(x/R)\mathrm{d}x \\ &\geq \varepsilon \int_{|x|>\varepsilon_0} u_0(x)\Phi(x/R)\mathrm{d}x \\ &= \varepsilon R^N \int_{|\eta|>\varepsilon_0/R} u_0(R\eta)\Phi(\eta)\mathrm{d}\eta \\ &\geq \varepsilon C_0 R^{N-\kappa} \int_{|\eta|>\varepsilon_0/R} |\eta|^{-\kappa}\Phi(\eta)\mathrm{d}\eta \\ &\geq \varepsilon C_0 R^{N-\kappa} \int_{|\eta|>\varepsilon_0/R^0} |\eta|^{-\kappa}\Phi(\eta)\mathrm{d}\eta \\ &= \widetilde{C} R^{N-\kappa}. \end{aligned} \tag{3.1}$$

Using (2.12), we know from (3.1) that, for $0 < \tau < T_\varepsilon$,

$$\begin{aligned} \varepsilon &\leq R^{\kappa-N}\widetilde{C}^{-1}(p-1)p^{p/(1-p)}[R^s S(T)]^{p/(p-1)} \\ &= \widetilde{C}^{-1}(p-1)p^{p/(1-p)}H(\tau, R), \end{aligned} \tag{3.2}$$

where $H(\tau, R) = R^{\kappa-N}[S(\tau R^{-2m})R^s]^{p/(p-1)}$. We write

$$H(\tau, R) = [a_p\tau^{-1/p}R^{\alpha_1} + b_p\tau^{(p-1)/p}R^{-\alpha_2}]^{p/(p-1)},$$

where $\alpha_1 = (p-1)\kappa/p$, $\alpha_2 = 2m - (p-1)\kappa/p$. The choice of κ implies $\alpha_1, \alpha_2 > 0$.

Now we derive some estimates on $H(\tau, R)$. If we can find a function $G(\tau)$ such that

$$H(\tau, R) \geq G(\tau), \quad \forall\ \tau > 0,$$

and for each value of $R \geq R^0$ there exists a value of τ_R such that $H(\tau_R, R) = G(\tau_R)$, then (3.2) holds for all $R \geq R^0$ if and only if

$$\varepsilon \leq \widetilde{C}^{-1}(p-1)p^{p/(1-p)}G(\tau). \tag{3.3}$$

Set

$$y = R^{\alpha_1+\alpha_2} = R^{2m}, \quad \beta_1 = \alpha_2/(\alpha_1+\alpha_2) = \alpha_2/(2m).$$

Then

$$H(\tau, R) = \tau^{-1/(p-1)}h(\tau, y)^{p/(p-1)}$$

with $h(\tau, y) = a_p y^{1-\beta_1} + b_p y^{-\beta_1}\tau$. Denote

$$\sigma = a_p b_p^{-1}(1-\beta_1)\beta_1^{-1}y, \quad G(\tau) = \tau^{-1/(p-1)}g(\tau)^{p/(p-1)},$$

where

$$g(\tau) = [a_p y^{1-\beta_1}\sigma^{\beta_1-1} + b_p y^{-\beta_1}\sigma^{\beta_1}]\tau^{1-\beta_1}.$$

It is easy to check that $0 < \beta_1 < 1$. Then, $\zeta = g(\tau)$ ia a concave curve. Furthermore, $\zeta = h(\tau, y)$ is a tangent line of $\zeta = g(\tau)$ at the point of $(\sigma, g(\sigma))$. Therefore, we get that $h(\tau, y) \geq g(\tau)$, for all $\tau > 0$. Hence $H(\tau, R) \geq G(\tau)$, for all $\tau > 0$. Moreover, $H(\tau, R_\tau) = G(\tau)$ with

$$\tau_R = a_p b_p^{-1}(1-\beta_1)\beta_1^{-1}R^{2m}.$$

By computations,

$$G(\tau) = \tau^{-1/(p-1)}g(\tau)^{p/(p-1)} = C_1\tau^{\hat\beta}. \tag{3.4}$$

for some positive constant C, where

$$\hat\beta = \frac{\kappa}{2m} - \frac{1}{p-1}.$$

The choice of κ implies that $\hat\beta < 0$. Combining (3.3) and (3.4), we find that

$$\varepsilon \leq K\tau^{\hat\beta} \tag{3.5}$$

for some $K > 0$. From (3.5), it follows that

$$\tau \leq C\varepsilon^{1/\hat\beta}$$

for some $C > 0$. The proof is complete. □

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