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REGULARITY OF SOLUTIONS TO 3-D NEMATIC LIQUID CRYSTAL FLOWS

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Abstract. In this note we consider the regularity of solutions to 3-D nematic liquid crystal flows, we prove that if either $u \in L^q(0,T;L^p(\mathbb{R}^3))$, $\frac{2}{q}+\frac{3}{p}\leq 1$, $3 < p \leq \infty$; or $u \in L^\alpha(0,T;L^\beta(\mathbb{R}^3))$, $\frac{2}{\alpha}+\frac{3}{\beta}\leq 2$, $\frac{3}{2} < \beta \leq \infty$, then the solution (u,d) is regular on $(0,T]$.

1. Introduction

In this note we study the following hydrodynamical systems modelling the flow of nematic liquid crystal material [4, 5]:

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla P = -\lambda\nabla\cdot(\nabla d\odot\nabla d), \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.1}$$

$$d_t + (u\cdot\nabla)d = \gamma(\Delta d - f(d)), \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.2}$$

$$\operatorname{div} u = 0, \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.3}$$

$$(u,d)|_{t=0} = (u_0,d_0) \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Here $u = u(x,t) = (u_1(x,t),u_2(x,t),u_3(x,t))$ is the velocity field of the flow, $d = d(x,t) = (d_1(x,t),d_2(x,t),d_3(x,t))$ is the (averaged) macroscopic/continuum molecule direction, $P = P(x,t)$ is a scalar function representing the pressure, ν,λ,γ are positive constants, and $f(d) = \frac{1}{\epsilon^2}(|d|^2-1)d$. The term $\nabla d\odot\nabla d$ denotes the 3×3 matrix whose (i,j)-th entry is equal to $\partial_i d\cdot\partial_j d$ (for $1\leq i,j\leq 3$). For simplicity, we assume that $\nu=\lambda=\gamma=\epsilon=1$ throughout this paper.

The above system is a simplified version of the Ericksen-Leslie model (see [4]) which retains many essential features of the hydrodynamic equations for nematic liquid crystal. The existence of global-in-time weak solutions and local-in-time classical solutions for this system have been established by Lin and Liu [4]. Later, in [5], they also proved that the one dimensional spacetime Hausdorff measure of the singular set of the "suitable" weak solutions is zero. Recently, Zhou and Fan in [8] proved a regularity criterion for another system of partial differential equations modelling nematic liquid crystal flows, which is considered by Sun and Liu [7] and

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is similar to (1.1)–(1.4); their result says that if the local solution (u, b) satisfies

$$\int_0^T \frac{\|\nabla u\|_p^r}{1 + ln(e + \|\nabla u\|_p)}\, dt < \infty \quad \text{with} \quad \frac{2}{r} + \frac{3}{p} = 2,\ 2 \le p \le 3,$$

then (u, d) is regular on $(0, T]$.

We notice that if $d \equiv 0$, then the system (1.1)–(1.4) becomes to the Navier-Stokes equations. There have been a lot of works on regularity criteria of the solution to the 3-D Navier-Stokes equations. The following results in this direction are well-known: If one of the following two conditions holds

(1) $u \in L^q(0, T; L^p)$ for $\frac{2}{q} + \frac{3}{p} \le 1$ and $3 < p \le \infty$;
(2) $\nabla u \in L^\alpha(0, T; L^\beta)$ for $\frac{2}{\alpha} + \frac{3}{\beta} \le 2$ and $\frac{3}{2} < \beta \le \infty$,

then the solution to the 3-D Navier-Stokes equations is regular [1, 2, 3, 6]. In this note we want to show that the above regularity criteria still hold for the nematic liquid crystal flow (1.1)–(1.4). More precisely, we have the following results:

Theorem 1.1. *Let* $(u_0, d_0) \in H^1(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$. *Suppose that* (u, d) *is a local smooth solution of the liquid crystal flow* (1.1)–(1.4) *on the time interval* $[0, T)$ *associate with the initial value* (u_0, d_0). *Assume that one of the following two conditions holds*

(a) $u \in L^q(0, T; L^p(\mathbb{R}^3))$, *for* $\frac{2}{q} + \frac{3}{p} \le 1$ *with* $3 < p \le \infty$;
(b) $\nabla u \in L^\alpha(0, T; L^\beta(\mathbb{R}^3))$, *for* $\frac{2}{\alpha} + \frac{3}{\beta} \le 2$ *with* $\frac{3}{2} < \beta \le \infty$.

Then (u, d) *can be extended beyond* T.

We shall give the proof of this result in the following section. As usual, we use the notation C to denote a "generic" constant which may change from line to line, and use $\|\cdot\|_p$ to denote the norm of the Lebesgue space L^p.

2. Proof of Theorem 1.1

Assume that $[0, T_{max})$ is the maximal interval of the existence of local smooth solution. To conclude our proof, we only need to show that $T < T_{max}$. Arguing by contradiction, we assume that $T_{max} \le T$, and either (a) or (b) holds. If we can establish the estimate

$$\lim_{t \to T^-} (\|\nabla u\|_2 + \|\Delta d\|_2) < \infty, \tag{2.1}$$

then $[0, T)$ is not a maximal interval of the existence of solution, which leads to an contradiction.

We multiply (1.1) by u and integrate over $\mathbb{R}^3$, and multiply (1.2) by $-\Delta d + f(d)$ and integrate over $\mathbb{R}^3$. By adding the two results above, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} (|u|^2 + |\nabla d|^2 + 2F(d))\, dx + \int_{\mathbb{R}^3} (|\nabla u|^2 + |\Delta d - f(d)|^2)\, dx = 0, \tag{2.2}$$

where $F(d)$ is the primitive function of $f(d)$; i.e., $F(d) = \frac{|d|^4}{4} - \frac{|d|^2}{2}$. Here we used the condition $\operatorname{div} u = 0$ and the fact that

$$((u \cdot \nabla)u, u) = (u, \nabla P) = ((u \cdot \nabla)d, f(d)) = (u, \nabla \frac{|\nabla d|^2}{2}) = 0.$$

Hence

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \le C. \tag{2.3}$$

Multiply (1.2) by $|d|^2 d$ and integrate by parts yields

$$\frac{1}{4}\frac{d}{dt}\int_{\mathbb{R}^3}|d|^4(t,x)\,dx+\int_{\mathbb{R}^3}(3d^2|\nabla d|^2+|d|^6)(t,x)\,dx=\int_{\mathbb{R}^3}|d|^4(t,x)\,dx,$$

which implies

$$\|d(t,\cdot)\|_{L^\infty(0,T;L^4)}+\int_0^t\int_{\mathbb{R}^3}(3d^2|\nabla d|^2+|d|^6)(\tau,x)\,dx\,d\tau\le C. \tag{2.4}$$

Multiply (1.2) by $f(d)$ and integrate by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}F(d)(t,x)\,dx=\int_{\mathbb{R}^3}(\Delta df(d)-|f(d)|^2)(t,x)\,dx. \tag{2.5}$$

By (2.2)–(2.5), the Gronwall's inequality and the fact $f(d)=(|d|^2-1)d$, we obtain

$$\|d\|_{L^\infty(0,T;H^1)}+\|d\|_{L^2(0,T;H^2)}\le C. \tag{2.6}$$

Noticing that the i-th (i=1,2,3) component of u satisfies

$$\partial_t u_i+(u\cdot\nabla)u_i-\Delta u_i+\partial_i P=-\sum_{j=1}^3\partial_j\Big(\sum_{k=1}^3\partial_i d_k\partial_j d_k\Big). \tag{2.7}$$

Multiplying (2.7) by $-\Delta u_i$, summing over i, using integration by parts, and noting that $\operatorname{div}u=0$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla u|^2\,dx+\int_{\mathbb{R}^3}|\Delta u|^2\,dx\\&=\sum_{i=1}^3\int_{\mathbb{R}^3}(u\cdot\nabla)u_i\Delta u_i\,dx-\sum_{i,k=1}^3\int_{\mathbb{R}^3}\partial_i d_k\Delta d_k\Delta u_i\,dx.\end{aligned} \tag{2.8}$$

Applying Δ to both sides of (1.2), multiplying them with Δd, and using (1.3), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\Delta d|^2\,dx+\int_{\mathbb{R}^3}\big(|\nabla\Delta d|^2+\Delta f(d)\Delta d\big)\,dx\\&=\sum_{i,k}^3\int_{\mathbb{R}^3}\Delta u_i\partial_i d_k\Delta d_k\,dx-2\sum_{i,k=1}^3\int_{\mathbb{R}^3}\nabla u_i\partial_i\nabla d_k\Delta d_k\,dx,\end{aligned} \tag{2.9}$$

where we used the condition $\operatorname{div}u=0$. Putting (2.8) and (2.9) together, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\big(|\nabla u|^2+|\Delta d|^2\big)\,dx+\int_{\mathbb{R}^3}(|\Delta u|^2+|\nabla\Delta d|^2)\,dx\\&=\sum_{i=1}^3\int_{\mathbb{R}^3}(u\cdot\nabla)u_i\Delta u_i\,dx-2\sum_{i,k=1}^3\int_{\mathbb{R}^3}\nabla u_i\partial_i\nabla d_k\Delta d_k\,dx-\int_{\mathbb{R}^3}\Delta f(d)\Delta d\,dx\\&=:I_1+I_2+I_3.\end{aligned} \tag{2.10}$$

Now, we first consider the case that the smooth solution (u,d) satisfies the condition (a). For I_1, we can do estimates for it as

$$\begin{aligned}I_1&\le C\|u\|_p\|\nabla u\|_{\frac{2p}{p-2}}\|\Delta u\|_2\quad\text{(Hölder's inequality)}\\&\le C\|u\|_p\|\nabla u\|_2^{\frac{p-3}{p}}\|\Delta u\|_2^{1+\frac{3}{p}}\quad\text{(Gagliardo-Nirenberg inequality)}\\&\le\frac{1}{2}\|\Delta u\|_2^2+C\|u\|_p^{\frac{2p}{p-3}}\|\nabla u\|_2^2\quad\text{(Young inequality).}\end{aligned} \tag{2.11}$$

Similarly, we can estimate I_2 and I_3 as

$$\begin{aligned} I_2 &= 2\int_{\mathbb{R}^3} u_i\partial_i\nabla d_k\nabla\Delta d_k\,dx \\ &\le C\|u\|_p\|\nabla^2 d\|_{\frac{2p}{p-2}}\|\Delta d\|_2 \quad \text{(Hölder's Inequality)} \\ &\le C\|u\|_p\|\nabla^2 d\|_2^{\frac{p-3}{p}}\|\nabla\Delta d\|_2^{1+\frac{3}{p}} \quad \text{(Gagliardo-Nirenberg inequality)} \\ &\le \frac{1}{4}\|\nabla\Delta d\|_2^2 + C\|u\|_p^{\frac{2p}{p-3}}\|\Delta d\|_2^2 \quad \text{(Young inequality)}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} I_3 &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i[(|d|^2-1)d]\partial_i\Delta d\,dx \\ &= \sum_{i=1}^{3} 3\int_{\mathbb{R}^3}\partial_i d\partial_i\Delta d|d|^2\,dx - \sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i d\partial_i\Delta d\,dx \\ &\le C\|\nabla d\|_6\|\nabla\Delta d\|_2\|d\|_6^2 + C\|\nabla d\|_2\|\nabla\Delta d\|_2 \quad \text{(Hölder's inequality)} \\ &\le C\|\Delta d\|_2\|\nabla\Delta d\|_2\|\nabla d\|_2^2 + C\|\nabla d\|_2\|\nabla\Delta d\|_2 \quad \text{(Sobolev embedding)} \\ &\le \frac{1}{4}\|\nabla\Delta d\|_2^2 + C(\|\nabla d\|_2^2 + \|\nabla d\|_2^2\|\Delta d\|_2^2) \quad \text{(Young inequality)} \\ &\le \frac{1}{4}\|\nabla\Delta d\|_2^2 + C\|\Delta d\|_2^2 + C. \end{aligned} \tag{2.13}$$

Substituting the above estimates (2.11)–(2.13) into (2.10), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |\Delta d|^2\right)\,dx + \int_{\mathbb{R}^3}(|\Delta u|^2 + |\nabla\Delta d|^2)\,dx \\ &\le C\|u\|_p^{\frac{2p}{p-3}}(\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + C\|\Delta d\|_2^2 + C \\ &\le C(1+\|u\|_p^{\frac{2p}{p-3}})(\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + C. \end{aligned} \tag{2.14}$$

Hence, the Gronwall's inequality yields

$$\sup_{0\le t\le T}\{\|\nabla u\|_2^2 + \|\Delta d\|_2^2\} \le Ce^{CT}e^{\int_0^T\|u\|_p^{2p/(p-3)}\,dt} < \infty. \tag{2.15}$$

Next we consider the case that the smooth solution (u,d) satisfies the condition (b). We estimate I_1 as follows:

$$\begin{aligned} I_1 &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(\partial_i u\cdot\nabla)u_i\partial_j u_i\,dx \\ &\le C\|\nabla u\|_\beta\|\nabla u\|_{\frac{2\beta}{\beta-1}}^2 \quad \text{(Hölder's inequality)} \\ &\le C\|\nabla u\|_\beta\|\nabla u\|_2^{\frac{2\beta-3}{\beta}}\|\Delta u\|_2^{\frac{3}{\beta}} \quad \text{(Gagliardo-Nirenberg inequality)} \\ &\le \frac{1}{2}\|\Delta u\|_2^2 + C\|\nabla u\|_\beta^{\frac{2\beta}{2\beta-3}}\|\nabla u\|_2^2 \quad \text{(Young inequality)}. \end{aligned} \tag{2.16}$$

Similarly, we can do estimates for I_2 as

$$\begin{aligned} I_2 &\le C\|\nabla u\|_\beta \|\Delta d\|^2_{\frac{2\beta}{\beta-1}} \quad \text{(Hölder's inequality)} \\ &\le C\|\nabla u\|_\beta \|\Delta d\|_2^{\frac{2\beta-3}{\beta}} \|\nabla\Delta d\|_2^{\frac{3}{\beta}} \quad \text{(Gagliardo-Nirenberg inequality)} \\ &\le \frac{1}{4}\|\nabla\Delta d\|_2^2 + C\|\nabla u\|_\beta^{\frac{2\beta}{2\beta-3}} \|\Delta d\|_2^2 \quad \text{(Young inequality)}, \end{aligned} \tag{2.17}$$

and for I_3 as

$$I_3 \le \frac{1}{4}\|\nabla\Delta d\|_2^2 + C\|\Delta d\|_2^2 + C. \tag{2.18}$$

Putting the above estimates for (2.15)–(2.18) into (2.10), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |\Delta d|^2\right)\,dx + \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla\Delta d|^2)\,dx \\ &\le C(1 + \|\nabla u\|_\beta^{\frac{2\beta}{2\beta-3}})(\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + C. \end{aligned}$$

Hence, the Gronwall's inequality yields

$$\sup_{0\le t\le T}\{\|\nabla u\|_2^2 + \|\Delta d\|_2^2\} \le Ce^{CT} e^{\int_0^T \|\nabla u\|_\beta^{\frac{2\beta}{2\beta-3}}\,dt} < \infty. \tag{2.19}$$

By (2.15) and (2.19), we see that (2.1) follows. This proves Theorem 1.1.

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