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NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION A THIRD-ORDER DIFFERENTIAL EQUATION

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Abstract. We show that under certain restrictions the following three conditions are equivalent: The equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = f(t)$$

is oscillatory. The equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0$$

is oscillatory. The second-order Riccati equation

$$z'' + 3zz' + a(t)z' = z^3 + a(t)z^2 + b(t)z + c(t)$$

does not admit a non-oscillatory solution that is eventually positive.

Furthermore, we obtain sufficient conditions for the above statements to hold, in terms of the coefficients. These conditions are sharp in the sense that they are both necessary and sufficient when the coefficients $a(t), b(t), c(t)$ are constant.

1. Introduction

Consider the third-order non-homogenous differential equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = f(t), \tag{1.1}$$

the associated homogenous equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0, \tag{1.2}$$

and the second-order Riccati equation

$$z'' + 3zz' + a(t)z' = z^3 + a(t)z^2 + b(t)z + c(t), \tag{1.3}$$

where $t \ge t_0$ for some constant t_0,

(i) $a \in C^2([t_0,\infty),(0,\infty))$, $b \in C^1([t_0,\infty),(-\infty,0))$, $c \in C([t_0,\infty),(0,\infty))$, and $f \in C([t_0,\infty),(0,\infty))$;
(ii) $a'(t) \ge 0$, $f'(t) \le 0$ almost everywhere in $[t_0,\infty)$.

We prove that under these conditions, the following three statements are equivalent:

(A) Equation (1.1) is oscillatory;
(B) Equation (1.2) is oscillatory;

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(C) Equation (1.3) does not admit non-oscillatory solutions which are eventually positive.

Further (C) holds, if

(D)

$$\int_{t_0}^{\infty}\Big\{\frac{2a^3(t)}{27}-\frac{a(t)b(t)}{3}+c(t)-\frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3}-b(t)\Big)^{3/2}\Big\}dt=\infty$$

Condition (D) is sharp in the sense that it is a necessary and sufficient condition for (C) to hold, when $a(t), b(t), c(t)$ are constant.

As usual, a function $y \in C(\mathbb{R},\mathbb{R})$ is said to be non-oscillatory, if there exist a point t_0 such that $y(t) > 0$ for all $t \ge t_0$, or $y(t) < 0$ for all $t \ge t_0$. Otherwise, y is said to be oscillatory.

Equations (1.1) and (1.2) is said to be non-oscillatory if all of their solutions are non-oscillatory. Otherwise, they is said to be oscillatory. Such a classification of definitions has been made, because, there are third order differential equations which admit both oscillatory and non oscillatory solutions. For example $y_1(t) = e^t$ and $y_2(t) = e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)$ are such solutions of $y'''(t) - y(t) = 0$.

The results of this paper are motivated from the properties of solutions to

$$y''' + ay'' + by' + cy = f \tag{1.4}$$

and the corresponding homogenous differential equation

$$x''' + ax'' + bx' + cx = 0, \tag{1.5}$$

where a, b, c, f are constants and $a > 0$, $b < 0$, $c > 0$ and $f > 0$.

It is established in [5, 15] that (1.4) and (1.5) admit oscillatory solutions if and only if

$$\frac{2a^3}{27}-\frac{ab}{3}+c-\frac{2}{3\sqrt{3}}\Big(\frac{a^2}{3}-b\Big)^{3/2}>0$$

In this article, we establish similar results for equations with variable coefficients. The following definition of Hanan [11] is used in the sequel.

Equation (1.2) is said to be of *Class I* (C_I for short) if its solution $x(t)$ with $x(a) = x'(a) = 0$, $x''(a) > 0$ $(t_0 < a < \infty)$ satisfies $x(t) > 0$ in (t_0, a).

2. Proof of $(A) \Leftrightarrow (B)$

The following lemma plays a vital role in the entire paper.

Lemma 2.1. *Suppose that the following conditions hold:*

- (H1) $u, \vartheta \in C^1(\mathbb{R},\mathbb{R})$;
- (H2) $\alpha, \beta \in \mathbb{R}$ *are consecutive zeros of* $\vartheta(t)$, $\vartheta(\alpha) = \vartheta(\beta) = 0$;
- (H3) $u(t)$ *is of one sign in* $[\alpha, \beta]$.

Then there exist a constant $\lambda \ne 0$ *and a point* $t_0 \in (\alpha, \beta)$ *such that the function*

$$w(t) = \lambda u(t) - \vartheta(t)$$

has a double zero at t_0 *(i.e.,* $w(t_0) = w'(t_0) = 0$*) and* $w(t)$ *is of one sign in* (t_0, β)*).*

Proof. Without loss of generality, assume that $u(t) > 0$ and $\vartheta(t) > 0$ in (α, β). consider the set

$$\bar{A} = \{\mu \mid \mu u(t) - \vartheta(t) > 0, \quad t \in (\alpha, \beta)\}$$

Setting

$$k = \max_{\alpha \le t \le \beta} \frac{\vartheta(t)}{u(t)}$$

it follows that k is positive and finite. Any number greater than k is obviously an element of $\bar{A}$. Clearly, the set $\bar{A}$ is nonempty and bounded below.

Let λ be the greatest lower bound of $\bar{A}$. We claim that $w(t) = \lambda u(t) - \vartheta(t)$ has a double zero at some point t_0 in (α, β) and of one sign in (t_0, β).

By definition of λ, there exists a sequence of real numbers $\langle \lambda_n \rangle$ in $\bar{A}$ which converges to λ. The sequence of continuous bounded functions $\langle \lambda_n u(t) - \vartheta(t) \rangle >$ converges uniformly to $\lambda u(t) - \vartheta(t)$ in (α, β). This gives

$$\lambda u(t) - \vartheta(t) \ge 0, \quad t \in (\alpha, \beta) \tag{2.1}$$

Now we claim that $\lambda u(t) - \vartheta(t)$ vanishes at least once in (α, β). If possible suppose that $\lambda u(t) - \vartheta(t) > 0$ in (α, β). Then there exist $\epsilon > 0$ such that

$$\lambda u(t) - \vartheta(t) > \epsilon > 0, \quad t \in (\alpha, \beta), \tag{2.2}$$

and $K > 0$ such that $u(t) < K$, $t \in (\alpha, \beta)$. From (2.2) it follows that

$$(\lambda - \epsilon_1(t))u(t) - \vartheta(t) > 0, \quad t \in (\alpha, \beta) \tag{2.3}$$

where

$$\epsilon_1(t) = \frac{\epsilon}{u(t)} > \frac{\epsilon}{K}$$

From (2.3) and the inequality given above we get

$$\Big(\lambda - \frac{\epsilon}{K}\Big) \in \bar{A}$$

This contradicts to the assumption on λ. Thus $\lambda u(t) - \vartheta(t) \ge 0$ in (α, β) and vanishes at least once in it. This shows that all zeros of $\lambda u(t) - \vartheta(t)$ are double zeros in (α, β). The greatest of all such double zeros will serve the purpose of t_0 with required properties. This completes the proof. $\square$

Remark. In the above lemma $\lambda > 0$ if and only if u, ϑ have same sign and $\lambda > 0$ otherwise.

Lemma 2.2 ([11])**.** *Equation* (1.2) *is of Class I.*

Lemma 2.3. *If* (2.1) *is oscillatory, then any solution of* (2.1) *which vanishes at least once is oscillatory.*

Proof. From Lemma 2.2, it follows that (2.1) is of class I. Now the proof follows from [11, Theorem 3.4]. $\square$

Lemma 2.4. *If* u_1 , u_2 *and* u_3 *are solutions of* (1.2) *satisfying the initial conditions*

$$\begin{gathered} u_1(t_0) = 1, \quad {u_1}'(t_0) = 0, \quad {u_1}''(t_0) = 0, \\ u_2(t_0) = 0, \quad {u_2}'(t_0) = 1, \quad {u_2}''(t_0) = 0, \\ u_3(t_0) = 0, \quad {u_3}'(t_0) = 0, \quad {u_3}''(t_0) = 1 \end{gathered}$$

then

$$y_p(t) = \int_{t_0}^{t} \frac{1}{W(s)} \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1(s) & u_2(s) & u_3(s) \\ {u_1}'(s) & {u_2}'(s) & {u_3}'(s) \end{vmatrix} f(s) ds$$

is a solution of (1.1)*. where* $W(s)$ *is the Wronskian*

$$W(s) = \begin{vmatrix} u_1(s) & u_2(s) & u_3(s) \\ u_1{}'(s) & u_2{}'(s) & u_3{}'(s) \\ u_1{}''(s) & u_2{}''(s) & u_3{}''(s) \end{vmatrix}.$$

Further $y_p(t)$ *satisfies* $y_p(t_0) = 0$, $y_p{}'(t_0) = 0$, $y_p{}''(t_0) = 0$ *and* $y_p{}'''(t_0) = f(t_0)$

Proof. Expanding

$$\begin{aligned} y_p(t) = u_1(t)\int_{t_0}^{t} \frac{1}{W(s)} \begin{vmatrix} u_2(s) & u_3(s) \\ u_2{}'(s) & u_3{}'(s) \end{vmatrix} f(s)ds \\ - u_2(t)\int_{t_0}^{t} \frac{1}{W(s)} \begin{vmatrix} u_1(s) & u_3(s) \\ u_1{}'(s) & u_3{}'(s) \end{vmatrix} f(s)ds \\ + u_3(t)\int_{t_0}^{t} \frac{1}{W(s)} \begin{vmatrix} u_1(s) & u_2(s) \\ u_1{}'(s) & u_3{}'(s) \end{vmatrix} f(s)ds\,. \end{aligned}$$

Differentiating $y_p(t)$ we obtain

$$\begin{aligned} y_p'(t) = \int_{t_0}^{t} \frac{1}{W(s)} \begin{vmatrix} u_1{}'(t) & u_2{}'(t) & u_3{}'(t) \\ u_1(s) & u_2(s) & u_3(s) \\ u_1{}'(s) & u_2{}'(s) & u_3{}'(s) \end{vmatrix} f(s)ds \\ + \frac{1}{W(t)} \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1(t) & u_2(t) & u_3(t) \\ u_1{}'(t) & u_2{}'(t) & u_3{}'(t) \end{vmatrix} f(t)\,. \end{aligned}$$

The second term of the above being equal to zero and proceeding similarly,

$$y_p''(t) = \int_{t_0}^{t} \frac{1}{W(s)} \begin{vmatrix} u_1{}''(t) & u_2{}''(t) & u_3{}''(t) \\ u_1(s) & u_2(s) & u_3(s) \\ u_1{}'(s) & u_2{}'(s) & u_3{}'(s) \end{vmatrix} f(s)ds$$

and

$$y_p'''(t) = \int_{t_0}^{t} \frac{1}{W(s)} \begin{vmatrix} u_1{}'''(t) & u_2{}'''(t) & u_3{}'''(t) \\ u_1(s) & u_2(s) & u_3(s) \\ u_1{}'(s) & u_2{}'(s) & u_3{}'(s) \end{vmatrix} f(s)ds + f(t)\,.$$

Since $u_i(t)$, $i = 1, 2, 3$ are solutions of (1.2), replacing

$$u_i{}'''(t) = -a(t)u_i{}''(t) - b(t)u_i{}'(t) - c(t)u_i(t)$$

for $i = 1, 2, 3$ in $y_p{}'''(t)$ we obtain

$$y_p{}'''(t) = -a(t)y_p{}''(t) - b(t)y_p{}'(t) - c(t)y_p(t) + f(t)$$

and $y_p(t)$ satisfies $y_p(t_0) = y_p'(t_0) = y_p''(t_0) = 0$ and $y_p'''(t_0) = f(t_0)$. This completes the proof . □

Lemma 2.5. *Suppose that* $2b(t) - a'(t) \leq 0$, $2b(t) - a'(t) - c'(t) < 0$ *and* $f'(t) \leq 0$ *for* $t \geq t_0$*. Then any solutions of* (1.1) *do not admit two consecutive double zeros in* $[t_0, \infty)$*.*

Proof. If possible let $y(t)$ have consecutive double zeros at α and β. Then either $y(t) > 0$ for $\alpha < t < \beta$, or $y(t) < 0$ for $\alpha < t < \beta$. In the former case, multiplying (1.1) by $y'(t)$ and integrating the resultant from α to β we obtain

$$0 > -\int_{\alpha}^{\beta} \Big(y''(t)\Big)^2 dt + \int_{\alpha}^{\beta} \Big(2b(t) - a'(t) - c'(t)\Big) \frac{\big(y'(t)\big)^2}{2} dt = -\int_{\alpha}^{\beta} f'(t)y(t)dt > 0$$

which is a contradiction. In the latter case, $y(\alpha) = y(\beta) = 0$ implies that there exists $t_0 \in (\alpha, \beta)$ such that $y'(t_0) = 0$ and $y'(t) > 0$ in (t_0, β). Similarly, there exists point $t_2 \in (t_0, \beta)$ such that $y''(t_2) = 0$ and $y''(t) < 0$ in (t_2, β). Integrating (1.1) from t_2 to β and using the fact that $y(t) < 0$, $y'(t) > 0$ and $y''(t) < 0$ in (t_2, β) we get

$$0 = y''(\beta) - y''(t_2) = \int_{t_2}^{\beta} \big[f(t) - a(t)y''(t) - b(t)y'(t) - c(t)\big]dt > 0$$

which is again a contradiction. This completes the proof. ∎

Lemma 2.6. *If the hypotheses of lemma 2.5 hold, then* (1.1) *does not admit a solution with a double zero followed by two single zeros.*

Proof. If possible, let $y(t)$ be a solution of (1.1) having a double zero at α_3 followed by single zeros at α_1 and α_2. That is,

$$y(\alpha_1) = y(\alpha_2) = 0, \quad y(\alpha_3) = y'(\alpha_3) = 0,$$

where $t_0 < \alpha_1 < \alpha_2 < \alpha_3$. Consider the solution $x(t)$ of (1.2) with initial conditions

$$x(\alpha_3) = x'(\alpha_3) = 0, \quad x''(\alpha_3) = 1$$

By lemma 2.2, $x(t) > 0$ for $t_0 < t < \alpha_3$. From Lemma 2.1, there exists a constant $\lambda \neq 0$ such that

$$\lambda x(t) - y(t)$$

has a double zero at some point $t_1 \in (\alpha_1, \alpha_2)$ and is of constant sign in (t_1, α_2). Thus, $y(t) - \lambda x(t) = -(\lambda x(t) - y(t))$ is a solution of (1.1) with consecutive double zeros at t_1 and α_3 . This is a contradiction to lemma 2.5. This completes the proof. ∎

Theorem 2.7. *Suppose the hypothesis of lemma 2.5 hold. If* (1.2) *is oscillatory then* (1.1) *is oscillatory.*

Proof. If possible, suppose that (1.1) is non oscillatory. Let $x(t)$ be an oscillatory solution of (1.2) and $y(t)$ be a non oscillatory solution of (1.1). So, there exists $t_0 > 0$ such that either

$$y(t) > 0 \quad \text{for } t \geq t_0 \tag{2.4}$$

or

$$y(t) < 0 \quad \text{for } t \geq t_0 \tag{2.5}$$

Suppose that (2.4) holds (the proof for (2.5) follows similarly). Let α, β, $(t_0 < \alpha < \beta)$ be the consecutive zeros of $x(t)$ such that $x(t) > 0$ for $t \in (\alpha, \beta)$. From lemma 2.1, it follows that there exist a $\mu > 0$ such that

$$\mu y(t) - x(t) \tag{2.6}$$

has a double zero at some point $t_1 \in (\alpha, \beta)$. Putting $t = t_1$ in (2.6) we obtain

$$\mu = \frac{x(t_1)}{y(t_1)} > 0$$

Setting $\lambda_1 = 1/\mu$, it follows that

$$w(t) = y(t) - \lambda_1 x(t) \tag{2.7}$$

has a double zero at t_1. By assumption all solutions of (1.1) are non oscillatory and so $w(t)$ is non oscillatory. Let $t_2 \geq t_1$ such that $w(t) > 0$, $t \geq t_2$ because otherwise, $w(t) < 0$, $t \geq t_2$ gives

$$0 < y(t) < \lambda_1 x(t)$$

a contradiction to the fact that $x(t)$ is oscillatory. Now, let α_1 and β_1 ($t_2 < \alpha_1 < \beta_1$) be two consecutive zeros of $x(t)$ and $x(t) > 0$ for $t \in (\alpha_1, \beta_1)$. By lemma 2.1 and proceeding in the lines of (2.6) to (2.7), there exists $\lambda_2 > 0$ such that $y(t) - \lambda_2 x(t)$ has a double zero at some point $t_3 \in (\alpha_1, \beta_1)$. That is

$$\begin{aligned} y(t_3) - \lambda_2 x(t_3) = 0, \\ y'(t_3) - \lambda_2 x'(t_3) = 0 \end{aligned} \tag{2.8}$$

Again $w(t) > 0$ for $t \geq t_2$ implies that

$$w(t_3) = y(t_3) - \lambda_1 x(t_3) > 0 \tag{2.9}$$

From (2.8) and (2.9) it follows that $\lambda_2 > \lambda_1$. This in turn implies that

$$y(t_1) - \lambda_2 x(t_1) < 0 \tag{2.10}$$

Since $y(t) - \lambda_2 x(t)$ is continuous in (α, β) and positive for $t = \alpha$ and $t = \beta$, from (2.10) it follows that $y(t) - \lambda_2 x(t)$ has at least two zeros in (α, β). This contradicts Lemma 2.6 that (1.1) admits a solution $y(t) - \lambda_2 x(t)$ having a double zero at t_3 followed by two single zeros. This completes the proof. $\square$

The theorem stated below follows from [16].

Theorem 2.8. *If* (1.2) *is non oscillatory, then* (1.1) *is non oscillatory.*

3. Proof of $(B) \Leftrightarrow (C)$

We state below a result from Erbe [6], for its use in the sequel.

Lemma 3.1 ([6, Lemma 2.2]). *If $b(t) \leq 0$, $c(t) > 0$ and $x(t)$ is a non oscillatory solution of* (1.2) *with $x(t) \geq 0$ or $x(t) \leq 0$ eventually, then there exists $d > 0$ such that either*

$$x(t)x'(t) < 0 \quad \text{for } t \geq d \tag{3.1}$$

or

$$x(t)x'(t) > 0 \quad \text{and} \quad x(t) > 0, \quad \text{for } t \geq d \tag{3.2}$$

Furthermore, if (3.1) *holds, then for $t \geq d$,*

$$x(t)x'(t)x''(t) \neq 0, \quad \operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t) \tag{3.3}$$

and

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0, \quad \lim_{t\to\infty} x(t) = k \neq \pm\infty.$$

Lemma 3.2 ([6, Lemma 2.3]). *Let $b(t) \leq 0$, $c(t) \geq 0$. A necessary and sufficient condition for* (1.2) *to be oscillatory is that for any nontrivial non oscillatory solution* (3.1) *and* (3.3) *hold.*

Theorem 3.3. *Equation* (1.2) *is oscillatory if and only if all non oscillatory solutions of*

$$z'' + 3zz' + a(t)z' + a(t)z^3 + b(t)z^2 + c(t)z = 0 \tag{3.4}$$

are eventually negative.

Proof. Suppose that (1.2) admits an oscillatory solution. If possible, let $z(t)$ be a non oscillatory solution of (3.4) which is eventually positive. Set

$$\vartheta(t) = \exp\Big\{\int_{t_0}^{t} z(s)ds\Big\}. \tag{3.5}$$

Obviously $\vartheta(t) > 0$, $t > t_0$ and $\vartheta(t)$ is monotonically increasing in (t_0, ∞). Further

$$\vartheta'(t) = z(t)\vartheta(t), \tag{3.6}$$

$$\vartheta''(t) = z(t)\vartheta'(t) + z'(t)v(t) = z^2(t)\vartheta(t) + z'(t)\vartheta(t), \tag{3.7}$$

$$\begin{aligned}\vartheta'''(t) &= z^2(t)\vartheta'(t) + 2z(t)z'(t)\vartheta(t) + z''(t)\vartheta(t) + z'(t)\vartheta'(t)\\ &= z^3(t)\vartheta'(t) + 2z(t)z'(t)\vartheta(t) + z''(t)\vartheta(t) + z(t)z'(t)\vartheta(t).\end{aligned} \tag{3.8}$$

Using (3.5)–(3.8) we obtain

$$\begin{aligned}&\vartheta'''(t) + a(t)\vartheta''(t) + b(t)\vartheta'(t) + c(t)\vartheta(t)\\ &= \vartheta(t)\{z'' + 3zz' + a(t)z' + z^3 + a(t)z^2 + b(t)z + c(t)\} = 0\end{aligned}$$

This shows that $\vartheta(t)$ is a solution of (1.2) which do not satisfy (3.1), a contradiction to Lemma 3.2. This proves the sufficient part of the theorem.

Conversely, suppose that all non oscillatory solutions of (3.4) are eventually negative. If possible, let all solutions of (1.2) be non oscillatory. By Lemma 3.2 there exists at least one solution $x(t)$ of (1.2) which satisfy (3.2). Without loss of generality assume that $x(t) > 0$ and $x'(t) > 0$ for $t \geq d > 0$. Setting

$$z(t) = \frac{x'(t)}{x(t)}, \quad t \geq d$$

it is easy to verify that $z(t)$ is a solution of (3.4) which is eventually positive, a contradiction to our assumption. This completes the proof. □

In the following, sufficient conditions are established in terms of the coefficients a, b and c ensuring oscillation of (1.2).

Theorem 3.4. *Suppose that $a(t) \geq 0$, $b(t) \leq 0$, $c(t) \geq 0$ and $a'(t) \leq 0$. If*

$$\int_{t_0}^{\infty} \Big\{\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3} - b(t)\Big)^{3/2}\Big\}dt = \infty \tag{3.9}$$

then (1.2) *admits oscillatory solutions.*

The proof of the above theorem follows from [17].

Theorem 3.5. *Suppose that $a(t) \geq 0$, $a(t) + b(t) + 1 \leq 0$, $c(t) \geq 0$. Further, if*

$$\begin{aligned}&\frac{2\big(a(t)+3\big)^3}{27} - \frac{\big(a(t)+3\big)\Big(a(t)+b(t)+1\Big)}{3}\\ &+ c(t) - \frac{2}{3\sqrt{3}}\Big[\frac{\big(a(t)+3\big)^2}{3} - \Big(a(t)+b(t)+1\Big)\Big]^{3/2} > 0\end{aligned} \tag{3.10}$$

for $t \geq t_0$, $t_0 > 0$ then (1.2) *is oscillatory.*

Proof. For the sake of contradiction, suppose that all solutions of (1.2) are non oscillatory. By Lemma 3.2, there exists a solution $x(t)$ of (1.2) satisfying (3.2). Set

$$e^z = \frac{x'(t)}{x(t)} \tag{3.11}$$

Differentiating successively, it may be shown that

$$\frac{x''(t)}{x(t)} = e^z + e^{2z} \tag{3.12}$$

and

$$\frac{x'''(t)}{x(t)} = e^{3z} + 3e^{2z} + e^z \tag{3.13}$$

Dividing(1.2) throughout by $x(t)$ and using (3.11), (3.12) and (3.13) in the resulting equation we obtain

$$F(e^z,t) = e^{3z} + \Big(3 + a(t)\Big)e^{2z} + \Big(a(t) + b(t) + 1\Big)e^z + c(t) = 0 \tag{3.14}$$

It may be shown that the minimum of $F(e^z,t)$ attains at

$$e^z = \frac{1}{3}\Big[-a(t) - 3 + \sqrt{\big(a(t)+3\big)^2 - 3\Big(1 + a(t) + b(t)\Big)}\Big]$$

and its minimum value is

$$\min_z F(e^z,t) = \frac{2A^3(t)}{27} - \frac{A(t)B(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\Big(\frac{A^2(t)}{3} - B(t)\Big)^{3/2} \tag{3.15}$$

where $A(t) = a(t) + 3$ and $B(t) = a(t) + b(t) + 1$. Combining (3.14) and (3.15) we have the inequality

$$\frac{2A^3(t)}{27} - \frac{A(t)B(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\Big(\frac{A^2(t)}{3} - B(t)\Big)^{3/2} \le 0.$$

This contradicts assumption (3.10). This completes the proof. □

Theorem 3.6. *Suppose that $A(t) \ge 0$, $B(t) \le 0$, and $C(t) \ge 0$ with*

$$\int_{t_0}^{\infty} \frac{1}{t^{2n}}\Big\{\frac{2A^3(t)}{27} - \frac{A(t)B(t)}{3} + C(t) - \frac{2}{3\sqrt{3}}\Big(\frac{A^2(t)}{3} - B(t)\Big)^{3/2}\Big\}dt = \infty,$$

where

$$\begin{gathered} A(t) = t^n a(t) - \frac{3n}{2}t^{n-1}, \\ B(t) = (n^2 + 3n)t^{2n-2} + t^{2n}\Big(a'(t) + b(t)\Big) - na(t)t^{2n-1}, \\ C(t) = t^{3n}c(t) \end{gathered}$$

for $t \ge t_0$, $t_0 > 0$ then (1.2) is oscillatory.

Proof. If possible, suppose that all solutions of (1.2) are non oscillatory. By Lemma 3.2, there exists a solution $x(t)$ of (1.2) satisfying (3.2) for $t \ge t_0$. Now, set

$$z = t^n \frac{x'(t)}{x(t)}.$$

Clearly, $z(t) > 0$ and satisfies

$$\begin{aligned} &z'' + \Big(a(t) - \frac{2n}{t}\Big)z' + \frac{3}{t^n}zz' \\ &= -\frac{1}{t^{2n}}\Big[z^3 + \Big(t^n a(t) - 3nt^{n-1}\Big)z^2 \\ &\quad + \Big(n(n+1)t^{2n-2} + t^{2n}b(t) - na(t)t^{2n-1}\Big)z + t^{3n}c(t)\Big] \end{aligned} \tag{3.16}$$

Integrating (3.16) from t_0 to t and rearranging terms we have

$$\begin{aligned} &z'(t) - z'(t_0) + \Big(a(t) - \frac{2n}{t}\Big)z(t) - \Big(a(t_0) - \frac{2n}{t_0}\Big)z(t_0) - \int_{t_0}^{t}\Big(a'(s) + \frac{2n}{s^2}\Big)z\,ds \\ &+ \frac{3}{t^n}\frac{z^2(t)}{2} - \frac{3z^2(t_0)}{2t_0^n} - \int_{t_0}^{t}\Big(-\frac{3nz^2}{2s^{n+1}}\Big)ds \\ &= \int_{t_0}^{t} -\frac{1}{s^{2n}}\Big[z^3 + \Big(s^n a(s) - 3ns^{n-1}\Big)z^2 \\ &\quad + \Big(n(n+1)s^{2n-2} + s^{2n}b(s) - na(s)s^{2n-1}\Big)z + s^{3n}c(s)\Big]ds \end{aligned}$$

Simplifying it further, we obtain

$$\begin{aligned} &z'(t) - z'(t_0) + \Big(a(t) - \frac{2n}{t}\Big)z(t) - \Big(a(t_0) - \frac{2n}{t_0}\Big)z(t_0) + \frac{3}{t^n}\Big[\frac{z^2(t)}{2}\Big] - \Big[\frac{3z^2(t_0)}{2t_0^n}\Big] \\ &= -\int_{t_0}^{t}\frac{1}{t^{2n}}\Big[z^3 + A(t)z^2 + B(t)z + C(t)\Big]dt, \end{aligned} \tag{3.17}$$

where

$$\begin{gathered} A(t) = t^n a(t) - \frac{3nt^{n-1}}{2}, \\ B(t) = (n^2 + 3n)t^{2n-2} + t^{2n}\Big(a'(t) + b(t)\Big) - na(t)t^{2n-1}, \\ C(t) = t^{3n}c(t). \end{gathered}$$

Moreover, the minimum of

$$F(z,t) = z^3 + A(t)z^2 + B(t)z + C(t)$$

for $z > 0$ is attained at

$$z(t) = \frac{1}{3}\Big(-A(t) + \sqrt{A^2 - 3B}\Big)$$

and the minimum is given by

$$\min F(z,t) = \frac{2A^3}{27} - \frac{AB}{3} + C - \frac{2}{3\sqrt{3}}\Big(\frac{A^2}{3} - B\Big)^{3/2}. \tag{3.18}$$

Substituting (3.18) in (3.17), then taking limit as $t \to \infty$ we see that $z'(t) \to -\infty$ as $t \to \infty$. This further implies that $z(t) \to -\infty$ as $t \to \infty$, a contradiction to our assumption. This completes the Proof. $\square$

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