Electronic Journal of Differential Equations, Vol. 2010(2010), No. 176, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLE POSITIVE SOLUTIONS FOR A NONLINEAR 3N-TH ORDER THREE-POINT BOUNDARY-VALUE PROBLEM

K. L. SARASWATHI DEVI, KAPULA R. PRASAD

Abstract. In this article we establish the existence of at least three positive solutions for 3n-th order three-point boundary value problem by using five functional fixed point theorem. We also establish the existence of at least $2m-1$ positive solutions of the problem for arbitrary positive integer m.

1. Introduction

The general theory of differential equations has emerged as an important area of investigation due to its powerful and versatile applications to almost all areas of science, engineering and technology. Much interest has been developed since last decade regarding the study of existence of positive solutions to the boundary value problems as they are arising in the branches of applied mathematics, physics and technological problems.

In this article, we prove the existence of multiple positive solutions of $3n^{th}$ order ordinary differential equation

$$(-1)^n y^{(3n)} = f(y(t), y^{(3)}(t), y^{(6)}(t), \dots, y^{(3(n-1))}(t)), \quad t \in [t_1, t_3], \tag{1.1}$$

satisfying the general three point boundary conditions

$$\begin{gathered}
\alpha_{3i-2,1} y^{(3i-3)}(t_1) + \alpha_{3i-2,2} y^{(3i-2)}(t_1) + \alpha_{3i-2,3} y^{(3i-1)}(t_1) = 0, \\
\alpha_{3i-1,1} y^{(3i-3)}(t_2) + \alpha_{3i-1,2} y^{(3i-2)}(t_2) + \alpha_{3i-1,3} y^{(3i-1)}(t_2) = 0, \\
\alpha_{3i,1} y^{(3i-3)}(t_3) + \alpha_{3i,2} y^{(3i-2)}(t_3) + \alpha_{3i,3} y^{(3i-1)}(t_3) = 0,
\end{gathered} \tag{1.2}$$

where the coefficients $\alpha_{3i-j,1}, \alpha_{3i-j,2}, \dots, \alpha_{3i-j,3}$ for $j = 0, 1, 2$ and $i = 1, \dots, n-1$, are real constants.

Boundary-value problems of the form (1.1)-(1.2) constitute a natural extension of third order three-point boundary-value problems studied in many papers with simple boundary conditions. Here we refer to Graef, Yang [9], Eloe and Henderson [5], Yang [20], Anderson[2], Anderson and Davis [3], Guo, Sun and Zhao [11] and references there in. Recently Prasad and Murali [17] studied the multiple positive solutions for nonlinear third order general three-point boundary-value problem.

2000 *Mathematics Subject Classification.* 34B18, 34A10.
Key words and phrases. Boundary value problem; multiple positive solutions; fixed point; cone.

Submitted June 23, 2010. Published December 17, 2010.

For convenience we adopt the notation Let

$$\beta_{ij} = \alpha_{3i-3+j,1}t_j + \alpha_{3i-3+j,2}, \quad \gamma_{ij} = \alpha_{3i-3+j,1}t_j^2 + 2\alpha_{3i-3+j,2}t_j + 2\alpha_{3i-3+j,3},$$
$$l_{ij} = \alpha_{3i-3+j,1}s^2 - 2\beta_{ij}s + \gamma_{ij},$$

and define

$$m_{i_{kj}} = \frac{\alpha_{3i-2+k,1}\gamma_{ij} - \alpha_{3i-2+j,1}\gamma_{ik}}{2(\alpha_{3i-2+k,1}\beta_{ij} - \alpha_{3i-2+j,1}\beta_{ik})}, \quad M_{i_{kj}} = \frac{\beta_{3i-2+k,1}\gamma_{ij} - \beta_{ij}\gamma_{ik}}{(\alpha_{3i-2+k,1}\beta_{ij} - \alpha_{3i-2+j,1}\beta_{ik})}$$

for $k = 1, 2, 3$, $j = 1, 2, 3$. Also let $m = \max\{m_{i_{12}}, m_{i_{13}}, m_{i_{23}}\}$,

$$M_i = \min\Big\{m_{i_{23}} + \sqrt{m_{i_{23}}^2 - M_{i_{23}}},\ m_{i_{13}} + \sqrt{m_{i_{13}}^2 - M_{i_{13}}}\Big\}$$

and

$$d_i = [\alpha_{3i-2,1}(\beta_{i2}\gamma_{i3} - \beta_{i3}\gamma_{i2)} - \beta_{i1}(\alpha_{3i-1,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i2}) + \gamma_{i1}(\alpha_{3i-1,1}\beta_{i3} - \alpha_{3i,1}\beta_{i2})].$$

We assume the following conditions throughout this paper:

- (A1) $f : \mathbb{R}^n \to \mathbb{R}^+$ is continuous;
- (A2) $\alpha_{3i-2,1} > 0$, $\alpha_{3i-1,1} > 0$ and $\alpha_{3i,1} > 0$ for $1 \le i \le n$ are real constants, $\frac{\alpha_{3i-2,2}}{\alpha_{3i-2,1}} < \frac{\alpha_{3i-1,2}}{\alpha_{3i-1,1}} < \frac{\alpha_{3i,2}}{\alpha_{3i,1}}$.
- (A3) $m_i \le t_1 \le t_2 \le t_3 \le M_i$, $2\alpha_{3i-1,3}\alpha_{3i-1,1} > \alpha_{3i-1,2}^2$, $2\alpha_{3i-2,3}\alpha_{3i-2,1} < \alpha_{3i-2,2}^2$, $2\alpha_{3i,3}\alpha_{3i,3} > \alpha_{3i,2}^2$.
- (A4) $m_{i_{23}}^2 > M_{i_{23}}$, $m_{i_{12}}^2 < M_{i_{12}}$, $m_{i_{13}}^2 > M_{i_{13}}$ and $d_i > 0$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous boundary value problem corresponding to (1.1)-(1.2) and estimate the bounds for the Green's function. In Section 3, we establish the existence of at least three positive solutions for (1.1)-(1.2), using five functional fixed point theorem. We also establish the existence of at least $2m - 1$ positive solutions of (1.1)-(1.2), for arbitrary positive integer m.

2. The Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous boundary value problem corresponding to (1.1)-(1.2) and estimate the bounds of the Greens function. We prove certain lemmas which are needed to establish our main results.

Let $G_i(t, s)$ be the Green's function for the homogeneous problem

$$-y''' = 0, \quad t \in [t_1, t_3] \tag{2.1}$$

satisfying the general three point boundary conditions (1.2). First we establish results on the related third order homogeneous boundary-value problem (2.1) and (1.2).

Lemma 2.1. *The homogeneous boundary-value problem* (2.1) *and* (1.2) *has only the trivial solution if and only if* $d_i = [\alpha_{3i-2,1}(\beta_2\gamma_3 - \beta_3\gamma_2) - \beta_1(\alpha_{3i-1,1}\gamma_3 - \alpha_{3i,1}\gamma_2) + \gamma_1(\alpha_{3i-1,1}\beta_3 - \alpha_{3i,1}\beta_2)] \ne 0$ *for* $1 \le i \le n$.

Proof. On application of boundary conditions (1.2) to the general solution of (2.1), it can be established. □

Lemma 2.2. *For $1 \le i \le n$, the Green's function for the homogeneous boundary value problem* (2.1) *and* (1.2) *is*

$$G_i(t,s) = \begin{cases} G_{i_1}(t,s), & t_1 < s < t \le t_2 < t_3 \\ G_{i_2}(t,s), & t_1 \le t < s < t_2 < t_3 \\ G_{i_3}(t,s), & t_1 \le t < t_2 < s < t_3 \\ G_{i_4}(t,s), & t_1 < t_2 < s < t \le t_3 \\ G_{i_5}(t,s), & t_1 < t_2 \le t < s < t_3 \\ G_{i_6}(t,s), & t_1 \le s < t_2 < t < t_3 \end{cases} \tag{2.2}$$

where

$$\begin{aligned}
G_{i_1}(t,s) &= \frac{1}{2d_i}[-(\beta_{i2}\gamma_{i3} - \beta_{i3}\gamma_{i2}) + t(\alpha_{3i-1,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i2}) \\
&\quad - t^2(\alpha_{3i-1,1}\beta_{i3} - \alpha_{3i,1}\beta_{i2})] \times l_{i1}, \\
G_{i_2}(t,s) &= \frac{1}{2d_i}[-(\beta_{i1}\gamma_{i3} - \beta_{i3}\gamma_{i1}) + t(\alpha_{3i-2,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i1}) \\
&\quad - t^2(\alpha_{3i-2,1}\beta_{i3} - \alpha_{3i,1}\beta_{i1})] \times l_{i2} \\
&\quad + \frac{1}{2d_i}[(\beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1}) - t(\alpha_{3i-2,1}\gamma_{i2} - \alpha_{3i-1,1}\gamma_{i1}) \\
&\quad + t^2(\alpha_{3i-2,1}\beta_{i2} - \alpha_{3i-1,1}\beta_{i1}] \times l_{i3}, \\
G_{i_3}(t,s) &= \frac{1}{2d_i}[(\beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1}) - t(\alpha_{3i-2,1}\gamma_{i2} - \alpha_{3i-1,1}\gamma_{i1}) \\
&\quad + t^2(\alpha_{3i-2,1}\beta_{i2} - \alpha_{3i-1,1}\beta_{i1})] \times l_{i3}, \\
G_{i_4}(t,s) &= \frac{1}{2d_i}[-(\beta_{i2}\gamma_{i3} - \beta_{i3}\gamma_{i2}) + t(\alpha_{3i-1,2,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i2}) \\
&\quad - t^2(\alpha_{3i-1,1}\beta_{i3} - \alpha_{3i,1}\beta_{i2})] \times l_{i1} \\
&\quad + \frac{1}{2d_i}[(\beta_{i1}\gamma_{i3} - \beta_{i3}\gamma_{i1}) - t(\alpha_{3i-2,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i1}) \\
&\quad + t^2(\alpha_{3i-2,1}\beta_{i3} - \alpha_{3i,1}\beta_{i1}] \times l_{i2}, \\
G_{i_5}(t,s) &= \frac{1}{2d_i}[(\beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1}) - t(\alpha_{3i-2,1}\gamma_{i2} - \alpha_{3i-1,1}\gamma_{i1}) \\
&\quad + t^2(\alpha_{3i-2,1}\beta_{i2} - \alpha_{3i-1,2,1}\beta_{i1})] \times l_{i3}, \\
G_{i_6}(t,s) &= \frac{1}{2d_i}[-(\beta_{i2}\gamma_{i3} - \beta_{i3}\gamma_{i2}) + t(\alpha_{3i-1,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i2}) \\
&\quad - t^2(\alpha_{3i-1,1}\beta_{i3} - \alpha_{3i,1}\beta_{i2})] \times l_{i1}.
\end{aligned}$$

Proof. $G_i(t,s)$ is constructed by using standard methods [18]. □

Lemma 2.3. *Assume the conditions* (A1)–(A4) *are satisfied. Then, for $1 \le i \le n$, the Green's function $G_i(t,s)$ of the boundary-value problem* (2.1) *and* (1.2) *satisfies $G_i(t,s) > 0$, for $(t,s) \in [t_1,t_3] \times [t_1,t_3]$.*

Proof. For $(t,s) \in [t_1,t_3] \times [t_1,t_3]$, $G_i(t,s)$ as stated in (2.2), if we consider sequentially, from (A2)–(A4), we obtain

$$G_i(t,s) > 0, \quad \text{for } (t,s) \in [t_1,t_3] \times [t_1,t_3]. \tag{2.3}$$

□

Lemma 2.4. *Assume the conditions* (A1)–(A4) *are satisfied. Then, for* $1 \le i \le n$, *the Green's function* $G_i(t,s)$ *given by* (2.2) *satisfies*

$$G_i(t,s) \le \max\big\{G_i(t_1,s), G_i(s,s), G_i(t_3,s)\big\}.$$

Proof. This can be proved by proceeding sequentially with the branches of $G_i(t,s)$ in (2.2).

Case 1. For $t_1 < s < t < t_2 < t_3$.

$$\begin{aligned} G_i(t,s) = G_{i_1}(t,s) = \frac{1}{2d_i}&[-(\beta_{i2}\gamma_{i3} - \beta_{i3}\gamma_{i2}) + t(\alpha_{3i-1,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i2}) \\ &- t^2(\alpha_{3i-1,1}\beta_{i3} - \alpha_{3i,1}\beta_{i2}] \times l_i1 \end{aligned}$$

which is decreasing in t, by (A2)-(A4). Therefore, $G_{i_1}(t,s) \le G_{i_1}(s,s) \le G_{i_1}(t_1,s)$. Hence $G_i(t,s) \le G_i(t_1,s)$.

Case 2. For $t_1 \le t < t_2 < s < t_3$.

$$\begin{aligned} G_i(t,s) = G_{i_3}(t,s) = \frac{1}{2d_i}&[(\beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1}) - t(\alpha_{3i-2,1}\gamma_{i2} - \alpha_{3i-1,1}\gamma_{i1}) \\ &+ t^2(\alpha_{3i-2,1}\beta_{i2} - \alpha_{3i-1,1}\beta_{i1})] \times l_3 \end{aligned}$$

which is increasing in t from (A2)-(A4). Therefore, $G_{i_3}(t,s) \le G_{i_3}(s,s) \le G_{i_3}(t_3,s)$. Hence $G_i(t,s) \le G_i(t_3,s)$.

Case 3. For $t_1 \le t < s < t_2 < t_3$.

$$\begin{aligned} G_i(t,s) = G_{i_2}(t,s) = &\frac{1}{2d_i}[-(\beta_{i1}\gamma_{i3} - \beta_{i3}\gamma_{i1}) + t(\alpha_{3i-2,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i1}) \\ &- t^2(\alpha_{3i-2,1}\beta_{i3} - \alpha_{3i,1}\beta_{i1})] \times l_{i2} \\ &+ \frac{1}{2d_i}[(\beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1}) - t(\alpha_{3i-2,1}\gamma_{i2} - \alpha_{3i-1,1}\gamma_{i1}) \\ &+ t^2(\alpha_{3i-2,1}\beta_{i2} - \alpha_{3i-1,1}\beta_{i1}] \times l_{i3} \end{aligned}$$

which is increasing in t by (A2)-(A4) and case 2. Therefore, $G_{i_2}(t,s) \le G_{i_2}(s,s)$. Hence $G_i(t,s) \le G_i(s,s)$.

Case 4. For $t_1 < t < t_2 < s < t < t_3$.

$$\begin{aligned} G_i(t,s) = G_{i_4}(t,s) = &\frac{1}{2d_i}[-(\beta_{i2}\gamma_{i3} - \beta_{i3}\gamma_{i2}) + t(\alpha_{3i-1,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i2}) \\ &- t^2(\alpha_{3i-1,1}\beta_{i3} - \alpha_{3i,1}\beta_{i2})] \times l_{i1} \\ &+ \frac{1}{2d_i}[(\beta_{i1}\gamma_{i3} - \beta_{i3}\gamma_{i1}) - t(\alpha_{3i-2,1}\gamma_{i3} - \alpha_{3i,1}\gamma_{i1}) \\ &+ t^2(\alpha_{3i-2,1}\beta_{i3} - \alpha_{3i,1}\beta_{i1})] \times l_{i2} \end{aligned}$$

which is decreasing in t from case 1 and case 2. Therefore, $G_{i_4}(t,s) \le G_{i_4}(s,s)$. Hence $G_i(t,s) \le G_i(s,s)$.

Similarly we can establish the inequality when the Green's function $G_i(t,s) = G_{i_5}(t,s)$ and $G_i(t,s) = G_{i_6}(t,s)$ as in case 2 and case 1 respectively, where $G_{i_5}(t,s)$, $G_{i_6}(t,s)$ are given as in (2.2). From all above cases

$$G_i(t,s) \le \max\{G_i(t_1,s), G_i(s,s), G_i(t_3,s)\}.$$

□

Lemma 2.5. *Assume that the conditions* (A1)–(A4) *hold. For* $1 \leq i \leq n$, *and fixed* $s \in [t_1, t_3]$, *the Green's function* $G_i(t,s)$ *in* (2.2) *satisfies*

$$\min_{t \in [t_2, t_3]} G_i(t,s) \geq m_i \|G_i(.,s)\|,$$

where

$$m_i = \min\Big\{\frac{G_{i_1}(t_3,s)}{G_{i_1}(t_2,s)}, \frac{G_{i_4}(t_3,s)}{G_{i_4}(t_2,s)}, \frac{G_{i_5}(t_2,s)}{G_{i_5}(t_3,s)}\Big\}$$

and $\|\cdot\|$ *is defined by* $\|x\| = \max\{x(t) : t \in [t_1, t_3]\}$.

Proof. For $s \in [t_1, t_2]$, $G_i(t,s) = G_{i_1}(t,s)$ which is decreasing in t by (A2)-(A4). Therefore,

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i_1}(t,s)}{G_{i_1}(s,s)} \geq \frac{G_{i_1}(t_3,s)}{G_{i_1}(t_2,s)}.$$

For $s \in [t_2, t_3]$ and $t_1 < t_2 \leq t < s < t_3$. $G_i(t,s) = G_{i_5}(t,s)$ which is increasing in t on $[t_1, t_3]$ by (A2)-(A4). Therefore,

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i_5}(t,s)}{G_{i_5}(s,s)} \leq \frac{G_{i_5}(t_2,s)}{G_{i_5}(t_3,s)}.$$

For $s \in [t_2, t_3]$ and $t_1 < t_2 < s < t < t_3$. $G_i(t,s) = G_{i_4}(t,s)$ which is decreasing in t on $[t_1, t_3]$ by $(A2)$-$(A4)$. Therefore

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i_4}(t,s)}{G_{i_4}(s,s)} \geq \frac{G_{i_4}(t,s)}{G_{i_4}(t_2,s)} \geq \frac{G_{i_4}(t_3,s)}{G_{i_4}(t_2,s)}.$$

Therefore, from Lemma 2.4 and by all the above cases we have

$$\min_{t \in [t_2, t_3]} G_i(t,s) \geq m_i \|G(.,s)\|,$$

where

$$m_i = \min\Big\{\frac{G_{i_1}(t_3,s)}{G_{i_1}(t_2,s)}, \frac{G_{i_4}(t_3,s)}{G_{i_4}(t_2,s)}, \frac{G_{i_5}(t_2,s)}{G_{i_5}(t_3,s)}\Big\}.$$

□

Lemma 2.6. *Assume the conditions* (A1)-(A4) *are satisfied and* $G_i(t,s)$ *as in* (2.2). *Let us define* $H_1(t,s) = G_1(t,s)$ *and recursively define*

$$H_j(t,s) = \int_{t_1}^{t_3} H_{j-1}(t,r) G_j(r,s) dr$$

for $2 \leq j \leq n$, *then* $H_n(t,s)$ *is the Green's function for the homogeneous problem corresponding to* (1.1)-(1.2).

Lemma 2.7. *Assume the conditions* (A1)-(A4) *holds. If we define*

$$K = \prod_{j=1}^{n-1} K_j, \quad L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ *in Lemma 2.6 satisfies*

$$0 \leq H_n(t,s) \leq K \|G_n(s,s)\|, \quad (t,s) \in [t_1, t_3] \times [t_1, t_3], \tag{2.4}$$

$$H_n(t,s) \geq m_n L \|G_n(s,s)\|, \quad (t,s) \in [t_2, t_3] \times [t_1, t_3] \tag{2.5}$$

where m_n *is given as in Lemma 2.5,*

$$K_j = \int_{t_1}^{t_3} \|G_j(s,s)\| ds > 0, \quad \text{for } 1 \leq j \leq n,$$

$$L_j = \int_{t_2}^{t_3} \|G_j(s,s)\| ds > 0, \quad for\ 1 \le j \le n.$$

Using Lemma 2.5 and induction on n, we can easily establish the proof of the above lemma.

Let $C = \{v \mid v : [t_1, t_3] \to \mathbb{R}$ is continuous function $\}$. For each $1 \le j \le n-1$, define the operator $T_j : C \to C$ by

$$(T_j v)(t) = \int_{t_1}^{t_3} H_j(t,s)v(s)ds, \quad t \in [t_1, t_3].$$

By the construction of T_j, and the properties of $H_j(t,s)$, it is clear that

$$\begin{gathered}
(-1)^j (T_j v)^{(3j)}(t) = v(t), \quad t \in [t_1, t_3], \\
\alpha_{3i-2,1} T_j v^{(3i-3)}(t_1) + \alpha_{3i-2,2} T_j v^{(3i-2)}(t_1) + \alpha_{3i-2,3} T_j v^{(3i-1)}(t_1) = 0, \\
\alpha_{3i-1,1} T_j v^{(3i-3)}(t_2) + \alpha_{3i-1,2} T_j v^{(3i-2)}(t_2) + \alpha_{3i-1,3} T_j v^{(3i-1)}(t_2) = 0, \\
\alpha_{3i,1} T_j v^{(3i-3)}(t_3) + \alpha_{3i,2} T_j v^{(3i-2)}(t_3) + \alpha_{3i,3} T_j v^{(3i-1)}(t_3) = 0,
\end{gathered}$$

for $i = 1, 2, \dots, j-1$. Hence, we see that (1.1)-(1.2) has a solution if and only if the following boundary-value problem has a solution

$$v^{(3)}(t) + f(T_{n-1}v(t), T_{n-2}v(t), \dots, T_1 v(t), v(t)) = 0, \quad t \in [t_1, t_3], \tag{2.6}$$

$$\begin{gathered}
\alpha_{3i-2,1} v^{(3i-3)}(t_1) + \alpha_{3i-2,2} v^{(3i-2)}(t_1) + \alpha_{3i-2,3} v^{(3i-1)}(t_1) = 0, \\
\alpha_{3i-1,1} v^{(3i-3)}(t_2) + \alpha_{3i-1,2} v^{(3i-2)}(t_2) + \alpha_{3i-1,3} v^{(3i-1)}(t_2) = 0, \\
\alpha_{3i,1} v^{(3i-3)}(t_3) + \alpha_{3i,2} v^{(3i-2)}(t_3) + \alpha_{3i,3} v^{(3i-1)}(t_3) = 0.
\end{gathered} \tag{2.7}$$

for $i = 1, 2, \dots, j-1$. Indeed, if y is a solution of (1.1)-(1.2), then $v(t) = y^{3(n-1)}(t)$ is a solution of (2.6)-(2.7). Conversely, if v is a solution of (2.6)-(2.7), then $y(t) = T_{n-1}v(t)$ is a solution of (1.1)-(1.2). In fact, $y(t)$ is represented as

$$y(t) = \int_{t_1}^{t_3} H_n(t,s)v(s)ds,$$

where

$$v(s) = \int_{t_1}^{t_3} G_1(s,\tau) f(T_{n-1}v(\tau), T_{n-2}v(\tau), \dots, T_1 v(\tau), v(\tau)) d\tau.$$

is a solution of (1.1)-(1.2).

3. Existence of multiple positive solutions

In this section, we establish the existence of multiple positive solutions for (1.1)-(1.2), by using five functional fixed point theorem which is Avery generalization of the Leggett-Williams fixed point theorem. And then, we establish $2m-1$ positive solutions for an arbitrary positive integer m.

Let B be a real Banach space with cone P. A map $\alpha : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P if α is continuous and

$$\alpha(\lambda x + (1-\lambda)y) \ge \lambda \alpha(x) + (1-\lambda)\alpha(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Similarly, we say that a map $\beta : P \to [0, \infty)$ is said to be a nonnegative continuous convex functional on P if β is continuous and

$$\beta(\lambda x + (1-\lambda)y) \le \lambda \beta(x) + (1-\lambda)\beta(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let γ, β, θ be nonnegative continuous convex functional on P and α, ψ be nonnegative continuous concave functionals on P, then for nonnegative numbers h', a', b', d' and c', we define the following convex sets

$$\begin{gathered}
P(\gamma, c') = \{y \in P | \gamma(y) < c'\}, \\
P(\gamma, \alpha, a', c') = \{y \in P | a' \leq \alpha(y), \gamma(y) \leq c'\}, \\
Q(\gamma, \beta, d', c') = \{y \in P | \beta(y) \leq d', \gamma(y) \leq c'\}, \\
P(\gamma, \theta, \alpha, a', b', c') = \{y \in P | a' \leq \alpha(y), \theta(y) \leq b', \gamma(y) \leq c'\}, \\
Q(\gamma, \beta, \psi, h', d', c') = \{y \in P | h' \leq \psi(y), \beta(y) \leq d', \gamma(y) \leq c'\}.
\end{gathered}$$

In obtaining multiple positive solutions of (1.1)-(1.2), the following so called Five Functionals Fixed Point Theorem will be fundamental.

Theorem 3.1. *Let P be a cone in a real Banach space B. Suppose α and ψ are nonnegative continuous concave functionals on P and γ, β θ are nonnegative continuous convex functionals on P such that, for some positive numbers c' and k,*

$$\alpha(y) \leq \beta(y) \quad \text{and} \quad \|y\| \leq k\gamma(y) \quad \text{for all } y \in \overline{P(\gamma, c')}.$$

Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants $h', d', a', b' \geq 0$ with $0 < d' < a'$ such that each of the following is satisfied.

- (B1) *$\{y \in P(\gamma, \theta, \alpha, a', b', c') | \alpha(y) > a'\} \neq \emptyset$ and $\alpha(Ty) > a'$ for $y \in P(\gamma, \theta, \alpha, a', b', c')$;*
- (B2) *$\{y \in Q(\gamma, \beta, \psi, h', d', c') | \beta(y) < d'\} \neq \emptyset$ and $\beta(Ty) < d'$ for $y \in Q(\gamma, \beta, \psi, h', d', c')$;*
- (B3) *$\alpha(Ty) > a'$ provided $y \in P(\gamma, \alpha, a', c')$ with $\theta(Ty) > b'$;*
- (B4) *$\beta(Ty) < d'$ provided $y \in Q(\gamma, \beta, d', c')$ with $\psi(Ty) < h'$.*

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that

$$\beta(y_1) < d', a < \alpha(y_2) andd' < \beta(y_3) \quad \text{with } \alpha(y_3) < a'.$$

Let $B = \{v | v : C[t_1, t_3] \to \mathbb{R}\}$ be the Banach space equipped with the norm

$$\|v\| = \max_{t \in [t_1, t_3]} |v(t)|.$$

Define the cone $P \subset B$ by

$$P = \Big\{v \in B : v(t) \geq 0 \text{ on } [t_1, t_3] \text{ and } \min_{t \in [t_2, t_3]} v(t) \geq M\|v\|\Big\}.$$

where $M = m_j L/K$ and m_j, L, K are defined as in Lemma 2.7. Now, let

$$[t_2', t_3'] \subset [t_2, t_3],$$

and define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P by

$$\begin{gathered}
\gamma(v) = \max_{t \in [t_1, t_3]} |v(t)|, \quad \psi(v) = \min_{t \in [t_2', t_3']} |v(t)|, \quad \beta(v) = \max_{t \in [t_2', t_3']} |v(t)|, \\
\alpha(v) = \min_{t \in [t_2, t_3]} |v(t)|, \quad \theta(v) = \max_{t \in [t_2, t_3]} |v(t)|.
\end{gathered}$$

We observe that for any $v \in P$,

$$\alpha(v) = \min_{t \in [t_2, t_3]} |v(t)| \leq \max_{t \in [t_2', t_3']} |v(t)| = \beta(v), \tag{3.1}$$

$$\|v\| \le \frac{1}{M} \min_{t\in[t_2,t_3]} v(t) \le \frac{1}{M} \max_{t\in[t_1,t_3]} |v(t)| = \frac{1}{M}\gamma(v). \tag{3.2}$$

We are now ready to present the main result of this section. Let

$$\begin{aligned}
\overline{L} &= \min\Big\{ \int_{t_2}^{t_3} G_1(s,s)ds, \int_{t_2}^{t_3} G_2(s,s)ds, \dots, \int_{t_2}^{t_3} G_n(s,s)ds \Big\},\\
\overline{L}' &= \min\Big\{ \int_{t_2'}^{t_3'} G_1(s,s)ds, \int_{t_2'}^{t_3'} G_2(s,s)ds, \dots, \int_{t_2'}^{t_3'} G_n(s,s)ds \Big\},\\
\overline{K} &= \max\Big\{ \int_{t_1}^{t_3} G_1(s,s)ds, \int_{t_1}^{t_3} G_2(s,s)ds, \dots, \int_{t_1}^{t_3} G_n(s,s)ds \Big\}.
\end{aligned}$$

Theorem 3.2. *Suppose there exist $0 < a' < b' < b'/M \le c'$ such that f satisfies the following conditions:*

- (D1) $f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) < a'/\overline{L}'$ *for all* $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ *in* $\prod_{j=n-1}^{1}[\frac{a'LL'm_n}{M}, \frac{c'KK_j}{M}] \times [Ma', a']$;
- (D2) $f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) > b'/M\overline{K}$ *for all* $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ *in* $\prod_{j=n-1}^{1}[\frac{b'm_n\overline{L}'L}{M}, \frac{c'KK_j}{M}] \times [b', \frac{b'}{M}]$;
- (D3) $f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) < c'/\phi$ *for all* $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ *in* $\prod_{j=n-1}^{1}[0, \frac{c'KK_j}{M}] \times [0, c']$.

Then (1.1)-(1.2) *has at least three positive solutions.*

Proof. Define the completely continuous operator $T : P \to B$ by

$$Tv(t) = \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds. \tag{3.3}$$

It is obvious that a fixed point of T is a solution of (2.6)-(2.7). We seek three fixed points $v_1, v_2, v_3 \in P$ of T. First, we show that $T : P \to P$. Let $v \in P$. Clearly, $Tv(t) \ge 0$ for $t \in [t_1, t_3]$. Also, noting that Tv satisfies the boundary conditions (1.2), we have

$$\begin{aligned}
\min_{t\in[t_2,t_3]} Tv(t) &= \min_{t\in[t_2,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds\\
&\ge M \int_{t_1}^{t_3} G_1(s,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds\\
&= M\|Tv\|.
\end{aligned}$$

Thus, $T : P \to P$. Next, for all $v \in P$, by (3.1)-(3.2), we have $\alpha(v) \le \beta(v)$ and $\|v\| \le \frac{c'}{M}$. To show that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$, let $v \in \overline{P(\gamma, c')}$. This implies $\|v\| \le \frac{c'}{M}$. For $1 \le j \le n-1$ and $t \in [a, b]$,

$$\begin{aligned}
T_jv(t) &= \int_{t_1}^{t_3} H_j(t,s)v(s)ds\\
&\le \frac{c'}{M} \int_{t_1}^{t_3} H_j(t,s)ds\\
&\le \frac{c'}{M} K \int_{t_1}^{t_3} G_j(s,s)ds
\end{aligned}$$

$$= \frac{c'}{M} K K_j.$$

We may now use condition $(D3)$ to obtain

$$\begin{aligned}
\gamma(Tv) &= \max_{t\in[t_1,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1 v(s), v(s)) ds \\
&\le \frac{c'}{\overline{\overline{K}}} \int_{t_1}^{t_3} G_1(s,s) ds \\
&\le c'.
\end{aligned}$$

Therefore, $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$.

We first verify that conditions (B1), (B2) of Theorem 3.1 are satisfied. It is obvious that

$$\{v \in P(\gamma, \theta, \alpha, b', \frac{b'}{M}, c') : \alpha(v) > b'\} \neq \emptyset,$$
$$\{v \in Q(\gamma, \beta, \psi, Ma', a', c') : \beta(v) < a'\} \neq \emptyset.$$

Next, let $v \in P(\gamma, \theta, \alpha, b', \frac{b'}{M}, c')$ or $v \in Q(\gamma, \beta, \psi, Ma', a', c')$. Then, for $1 \le j \le n-1$,

$$\begin{aligned}
T_j v(t) &= \int_{t_1}^{t_3} H_j(t,\tau) v(\tau) d\tau \\
&\le \frac{c'}{M} \int_{t_1}^{t_3} H_j(t,\tau) d\tau, \\
&\le \frac{c'}{M} K \int_{t_1}^{t_3} \|G_j(\tau,\tau)\| d\tau, \\
&\le \frac{c' K K_j}{M}
\end{aligned}$$

and for $v \in P(\gamma, \theta, \alpha, b', \frac{b'}{M}, c')$,

$$\begin{aligned}
T_j v(t) &= \int_{t_1}^{t_3} H_j(t,\tau) v(\tau) d\tau \\
&\ge \min_{t\in[t_2,t_3]} \int_{t_2}^{t_3} H_j(t,\tau) v(\tau) d\tau \\
&\ge \frac{m_n L b'}{M} \int_{t_2}^{t_3} \|G_j(\tau,\tau)\| d\tau \\
&\ge \frac{m_n L b' \overline{L}}{M}.
\end{aligned}$$

Also for $v \in Q(\gamma, \beta, \psi, Ma', a', c')$,

$$\begin{aligned}
T_j v(t) &= \int_{t_1}^{t_3} H_j(t,\tau) v(\tau) d\tau \\
&\ge \max_{t\in[t'_2,t'_3]} \int_{t'_2}^{t'_3} H_j(t,\tau) v(\tau) d\tau \\
&\ge \frac{m_n L a'}{M} \int_{t'_2}^{t'_3} \|G_j(\tau,\tau)\| d\tau
\end{aligned}$$

$$\geq \frac{m_n L a' \overline{L'}}{M}.$$

Now, we may apply condition (D2) to obtain

$$\begin{aligned}
\alpha(Tv) &= \min_{t\in[t_2,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\geq M \int_{t_1}^{t_3} G_1(s,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \geq b'.
\end{aligned}$$

Clearly, by condition (D1), we obtain

$$\begin{aligned}
\beta(Tv) &= \max_{t\in[t_2',t_3']} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\leq \frac{a'}{\overline{L'}} \max_{t\in[t_2',t_3']} \int_{t_1}^{t_3} G_1(s,s)ds \\
&\leq \frac{a'}{\overline{L'}} \overline{L'} = a'.
\end{aligned}$$

To see that (B3) is satisfied, let $v \in P(\gamma, \alpha, b', c')$ with $\theta(Tv) > b'/M$, we obtain

$$\begin{aligned}
\alpha(Tv) &= \min_{t\in[t_2,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\geq M \int_{t_1}^{t_3} G_1(s,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\geq M \max_{t\in[t_1,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\geq M \max_{t\in[t_2,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&= M\theta(Tv) > b'.
\end{aligned}$$

Finally, we show that (B4) holds. Let $v \in Q(\gamma, \beta, a', c')$ with $\psi(Tv) < Ma'$, we have

$$\begin{aligned}
\beta(Tv) &= \max_{t\in[t_2',t_3']} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\leq \max_{t\in[t_1,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\leq \int_{t_1}^{t_3} G_1(s,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&= \frac{1}{M} \int_{t_1}^{t_3} G_1(s,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\leq \frac{1}{M} \min_{t\in[t_2,t_3]} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\leq \frac{1}{M} \min_{t\in[t_2',t_3']} \int_{t_1}^{t_3} G_1(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1v(s), v(s))ds \\
&\leq \frac{1}{M}\psi(Tv) < a'.
\end{aligned}$$

We have proved that all the conditions of Theorem 3.1 are satisfied and so there exist at least three positive solutions $v_1, v_2, v_3 \in \overline{P(\gamma, c')}$ for (2.6)-(2.7). Therefore, (1.1)-(1.2) has at least three positive solutions y_1, y_2, y_3 of the form

$$y_i(t) = T_{n-1}v_i(t) = \int_a^b H_{n-1}(t,s)v_i(s)ds, \quad i = 1, 2, 3.$$

This completes the proof. $\square$

Now we prove the existence of $2m-1$ positive solutions for (1.1)-(1.2) by using induction on m.

Theorem 3.3. *Let m be an arbitrary positive integer. Assume that there exist numbers $a_i (1 \le i \le m)$ and $b_j (1 \le j \le m-1)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \dots < a_{m-1} < b_{m-1} < \frac{b_{m-1}}{M} < a_m$ such that*

$$f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) < \frac{a_i}{\overline{L'}} \tag{3.4}$$

for all $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ in $\prod_{j=n-1}^{1}[\frac{a_i L L' m_n}{M}, \frac{a_m K K_j}{M}] \times [Ma_i, a_i]$, $1 \le i \le m$, and

$$f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) > \frac{b_l}{K\overline{M}} \tag{3.5}$$

for all $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ in $\prod_{j=n-1}^{1}[\frac{b_l m_n \overline{L}' L}{M}, \frac{b_{m-1} K K_j}{M}] \times [b_l, \frac{b_l}{M}]$, $1 \le l \le m-1$. Then (1.1)-(1.2) has at least $2m-1$ positive solutions in $\overline{P}_{a_m}$.

Proof. We use induction on m. First, for $m = 1$, we know from (3.4) that $T : \overline{P}_{a_1} \to P_{a_1}$, then, it follows from Schauder fixed point theorem that (1.1)-(1.2) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $m = k$. In order to prove that this conclusion holds for $m = k+1$, we suppose that there exist numbers $a_i (1 \le i \le k+1)$ and $b_j (1 \le j \le k)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \dots < a_k < b_k < \frac{b_k}{M} < a_{k+1}$ such that

$$f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) < \frac{a_i}{\overline{L'}} \tag{3.6}$$

for all $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ in $\prod_{j=n-1}^{1}[\frac{a_i L L' m_n}{M}, \frac{a_m K K_j}{M}] \times [Ma_i, a_i]$, $1 \le i \le k+1$;

$$f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) > \frac{b_l}{K\overline{M}} \tag{3.7}$$

for all $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ in $\prod_{j=n-1}^{1}[\frac{b_l m_n \overline{L}' L}{M}, \frac{b_{m-1} K K_j}{M}] \times [b_l, \frac{b_l}{M}]$, $1 \le l \le k$.

By assumption, Problem (1.1)-(1.2) has at least $2k-1$ positive solutions u_i $(i = 1, 2, \dots, 2k-1)$ in $\overline{P}_{a_k}$. At the same time, it follows from Theorem 3.2, and (3.6) and (3.7) that (1.1)-(1.2) has at least three positive solutions u, v and w in $\overline{P}_{a_{k+1}}$ such that, $\|u\| < a_k, b_k < \min_{t\in[t_2,t_3]} v(t), \|w\| > a_k, \min_{t\in[t_2,t_3]} w(t) < b_k$. Obviously, v and w are different from u_i $(i = 1, 2, \dots, 2k-1)$. Therefore, (1.1)-(1.2) has at least $2k+1$ positive solutions in $\overline{P}_{a_{k+1}}$ which shows that this conclusion also holds for $m = k+1$. $\square$

Acknowledgements. K. L. Saraswathi Devi is grateful to the U. G. C. (India) and to the management of Ch. S. D. St. Theresa's College for Women, Eluru, for being accepted in the Faculty Development Programme. The authors thank the referees for their valuable suggestions.

References

[1] R. P. Agarwal, D. O' Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Nethelands, 1999.

[2] D. R. Anderson; Multiple positive solutions for a three-point boundary value problem, *Math. Comp. Modelling.*, **27**(1998), 49-57.

[3] D. R. Anderson and J. M. Davis; Multiple solutions and eigenvalues for third order right focal boundary value problem, *J. Math. Anal. Appl.*, **267**(2002), 135-157.

[4] P. W. Eloe and J. Henderson; Positive solutions for (n-1,1) conjugate boundary value problems, *Nonlinear. Anal.*, **28**(1997), 1669-1680.

[5] P. W. Eloe and J. Henderson; Positive solutions and nonlinear multipoint conjugate eigenvalue problems, *Electron. J. Diff. Eqns.*, **1997**(1997), No.3, 1-11.

[6] P. W. Eloe and J. Henderson; Positive solutions for (k,n-k) conjugate eigenvalue problems, *Diff. Eqns. Dyn. Sys.*, **6**(1998), 309-317.

[7] L. H. Erbe, S. Hu and H. Wang; Multiple positive solutions of some boundary value problems, *J. Math. Anal. Appl.*, **184**(1994), 640-648.

[8] L. H. Erbe and H. Wang; On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, **120**(1994), 743-748.

[9] J. R. Greaf and B. Yang; Positive solutions to a multi-point higher order boundary value problem, *J. Math. Anal. Appl.*, **316**(2006), 409-421.

[10] J. R. Greaf and B. Yang; Positive solutions of nonlinear third order eigenvalue problem, *Dyn. Sys. Appl.*, **15**(2006), 97-110.

[11] L. J. Guo, J. P. Sun, Y. H. Zhao; Multiple positive solutions for nonlinear third-order three-point boundary value problems, *Electron. J. Diff. Eqns.*, **2007**(2007), No. 112, 1-7.

[12] J. Henderson and E. R. Kaufmann; Multiple positive solutions for focal boundary value problems, *Comm. Appl. Anal.*, **1**(1997), 53-60.

[13] E. R. Kaufmann; Multiple positive solutions for higher order boundary value problems, *Rocky Mtn. J. Math.*, **28**(1998), No.3, 1017-1028.

[14] M. A. Krasnosel'skii; *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[15] S. Li; Positive solutions of nonlinear singular third-order two-point boundary value problem, *J. Math. Anal. Appl.*, **323**(2006), 413-425.

[16] W. C. Lian, F. H. Wang and C. C. Yeh; On the existence of positive solution of nonlinear second order differential equations, *Proc. Amer. Math. Soc.*, **124**(1996), 1111-1126.

[17] K. R. Prasad and P. Murali; Multiple positive solutions for nonlinear third order general three point boundary value problems, *Diff. Eqns. Dyn. Sys.*, **16**(2008), 63-75.

[18] D. R. K. S. Rao, K. N. Murthy and M. S. N. Murthy; On three-point boundary value problems containing parameters, *Bull. Inst. Math. Academia Sinica.*, **10**(1982), No. 3, 265-275.

[19] M. E. Shahed; Positive solutions of boundary value problems for n-th order ordinary differential equtations, *Electron. J. Qual. Theory. Diff. Eqns.*, **2008**(2008), No. 1, 1-9.

[20] B. Yang; Positive solutions of a third-order three-point boundary value problem, *Electron. J. Diff. Eqns.*, **2008**(2008), No. 99, 1-10.

K. L. Saraswathi Devi

Department of Applied Mathematics, Andhra University, Visakhapatnam, 530 003, India
Department of Mathematics, Ch. S. D. St. Theresa's Degree College for Women, Eluru, 534 003, India

E-mail address: saraswathikatneni@gmail.com

Kapula Rajendra Prasad

Department of Applied Mathematics, Andhra University, Visakhapatnam, 530 003, India

E-mail address: rajendra92@rediffmail.com