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# MONOTONE POSITIVE SOLUTIONS FOR $p$-LAPLACIAN EQUATIONS WITH SIGN CHANGING COEFFICIENTS AND MULTI-POINT BOUNDARY CONDITIONS

JIANYE XIA, YUJI LIU

Abstract. We prove the existence of three monotone positive solutions for the second-order multi-point boundary value problem, with sign changing coefficients,

$$
\begin{gathered}
[p(t)\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x'(0) = -\sum_{i=1}^{l} a_i x'(\xi_i) + \sum_{i=l+1}^{m} a_i x'(\xi_i), \\
x(1) + \beta x'(1) = \sum_{i=1}^{k} b_i x(\xi_i) - \sum_{i=k+1}^{m} b_i x(\xi_i) - \sum_{i=1}^{m} c_i x'(\xi_i).
\end{gathered}
$$

To obtain these results, we use a fixed point theorem for cones in Banach spaces. Also we present an example that illustrates the main results.

## 1. Introduction

As is well known, a differential equation defined on the interval $a \le t \le b$ having the form

$$
\begin{gathered}
[p(t)x'(t)]' + (q(t) + \lambda r(t))x(t) = 0, \\
a_1 x(a) + a_2 x'(a) = a_3 x(b) + a_4 x'(b) = 0,
\end{gathered}
$$

is called a Sturm-Liouville boundary-value problem or Sturm-Liouville system. Here $p(t) > 0$, $q(t)$, the weighting function $r(t)$, the constants $a_1, a_2, a_3, a_4$ are given, and the eigenvalue $\lambda$ is an unspecified parameter.

Sturm-Liouville boundary-value problems for nonlinear second-order $p$-Laplacian differential equations have been studied extensively; see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16]. The study of existence of positive solutions for such problems is complicated since there is no Green's function for the p-Laplacian ($p \neq 2$).

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Some authors have extend Sturm-Liouville boundary conditions to nonlinear cases. For example, He, Ge and Peng [7], by means of the Leggett-Williams fixed-point theorem, established criteria for the existence of at least three positive solutions to the one-dimensional p-Laplacian boundary-value problem

$$\begin{gathered}(\phi(y'))' + g(t)f(t,y) = 0, \quad t \in (0,1),\\ y(0) - B_0(y'(0)) = 0,\\ y(1) + B_1(y'(1)) = 0,\end{gathered} \tag{1.1}$$

where $\phi(v) = |v|^{p-2}v$ with $v > 1$, under the assumptions $xB_0(x) \geq 0$, $xB_1(x) \geq 0$ and that there exist constants $M_i > 0$ such that

$$|B_i(x)| \leq M_i|x|, \quad x \in \mathbb{R}. \tag{1.2}$$

In [9, 10, 11, 7], the authors extended (1.1) to a more general case. They established some existence results of at least one positive solution of Sturm-Liouville boundary value problems of higher order differential equations. Liu [12] studied the boundary-value problem

$$\begin{gathered}[\phi(x^{(n-1)}(t))]' = f(t, x(t), x'(t), \dots, x^{(n-2)}(t)), \quad 0 < t < 1,\\ x^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \dots, n-3,\\ x^{(n-2)}(0) - B_0(x^{(n-1)}(0)) = 0,\\ B_1(x^{(n-2)}(1)) + x^{(n-1)}(1) = 0.\end{gathered} \tag{1.3}$$

which is a general case of (1.1). Liu [12] established existence of at least one positive solution of (1.3) without assumption (1.2).

The Sturm-Liouville boundary conditions have been extended to multi-point cases. For example, Ma [15, 16] studied the problem

$$\begin{gathered}[p(t)x'(t)]' - q(t)x(t) + f(t, x(t)) = 0, \quad t \in (0,1),\\ \alpha x(0) - \beta p(0)x'(0) = \sum_{i=1}^{m} a_i x(\xi_i),\\ \gamma x(1) + \delta p(1)x'(1) = \sum_{i=1}^{m} b_i x(\xi_i),\end{gathered} \tag{1.4}$$

where $0 < \xi_1 < \dots < \xi_m < 1$, $\alpha, \beta, \gamma, \delta \geq 0$, $a_i, b_i \geq 0$ with $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$. By using Green's functions (which is complicate for studying (1.1)) and Guo-Krasnoselskii fixed point theorem [4, 6], the existence and multiplicity of positive solutions for (1.4) were given. There is no paper discussing the existence of multiple positive solutions of (1.4) by using Leggett-Williams fixed point theorem. Liu in [10, 11] also studied some Sturm-Liouville type multi-point boundary value problems.

In recent papers [8, 17], the authors studied the four-point boundary-value problem

$$\begin{gathered}(\phi(x'))' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1),\\ \alpha x(0) - \beta x'(\xi) = 0,\\ \gamma x(1) + \delta x'(\eta) = 0,\end{gathered} \tag{1.5}$$

where $\phi(x) = |x|^{p-2}x$, $p > 1$, $\phi^{-1}(x) = |x|^{q-2}x$ with $1/p + 1/q = 1$, $\alpha > 0$, $\beta \geq 0$, $\gamma > 0$, $\delta \geq 0$, $0 < \xi < \eta < 1$, $f$ is continuous and nonnegative. When $\xi \to 0$ and

$\eta \to 1$, (1.5) converges to a Sturm-Liouville boundary-value problem. So (1.5) can be seen as a generalized Sturm-Liouville boundary-value problem.

Xu [18] proved the existence of at least one or two positive solutions of the differential equation

$$\begin{gathered}
[\phi(x'(t))]' + a(t)f(x(t)) = 0, \quad t \in (0,1), \\
x'(0) = \sum_{i=1}^{m} a_i x'(\xi_i), \\
x(1) = \sum_{i=1}^{k} b_i x(\xi_i) - \sum_{i=k+1}^{s} b_i x(\xi_i) - \sum_{i=s+1}^{m} c_i x'(\xi_i),
\end{gathered} \tag{1.6}$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, $a_i, b_i, c_i \geq 0$, $a$ and $f$ are continuous functions, $\phi(x) = |x|^{p-2}x$ with $p > 1$.

Motivated by above mentioned papers, we investigate the boundary-value problem

$$\begin{gathered}
[p(t)\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\
x'(0) = -\sum_{i=1}^{l} a_i x'(\xi_i) + \sum_{i=l+1}^{m} a_i x'(\xi_i), \\
x(1) + \beta x'(1) = \sum_{i=1}^{k} b_i x(\xi_i) - \sum_{i=k+1}^{m} b_i x(\xi_i) - \sum_{i=1}^{m} c_i x'(\xi_i),
\end{gathered} \tag{1.7}$$

where

- $0 < \xi_1 < \cdots < \xi_m < 1$, $\beta \geq 0$, $1 \leq k, l \leq m$ and $a_i \geq 0, b_i \geq 0, c_i \geq 0$ for all $i = 1, \ldots, m$;
- $f$ is defined on $[0,1] \times \mathbb{R} \times \mathbb{R}$, continuous, nonnegative with $f(t,0,0) \not\equiv 0$ on each subinterval of $[0,1]$;
- $p$ is defined on $[0,1]$, continuous and positive;
- $\phi$ is called $p$-Laplacian, $\phi(x) = |x|^{p-2}x$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x)$ with $\phi^{-1}(x) = |x|^{q-2}x$ with $1/p + 1/q = 1$.

Sufficient conditions for the existence of at least three monotone positive solutions of (1.7) are established by using a fixed point theorem for cones in Banach spaces.

We remark that the fixed point theorem used here is different form the one in [18]. Our results improve the the results in [18] since: Three positive solutions are obtained, while one or two positive solutions were obtained in [18]; the nonlinearity in (1.7) depends on $t, x, x'$, while the nonlinearity in [18] depends only on $t, x$. The main result for this article is presented in Section 2, and an example is given in Section 3.

## 2. Main Results

In this section, we first present some background definitions and state an important fixed point theorem. Then the main results are given and proved.

**Definition 2.1.** Let $X$ be a semi-ordered real Banach space. The nonempty convex closed subset $P$ of $X$ is called a cone in $X$ if $x + y \in P$ and $ax \in P$ for all $x, y \in P$ and $a \geq 0$, and $x \in X$ and $-x \in X$ imply $x = 0$.

**Definition 2.2.** Let $X$ be a semi-ordered real Banach space and $P$ a cone in $X$. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave (or convex) functional map provided $\psi$ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq \text{ (or } \leq) \ t\psi(x) + (1-t)\psi(y) \quad \text{for all } x, y \in P, t \in [0,1].$$

**Definition 2.3.** Let $X$ be a semi-ordered real Banach space. An operator $T; X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let $a_1, a_2, a_3, a_4 > 0$ be positive constants, $\psi$ be a nonnegative continuous functional on the cone $P$. Define the sets as follows:

$$\begin{gathered}
P(\beta_1; a_4) = \{x \in P : \beta_1(x) < a_4\}, \\
P(\beta_1, \alpha_1; a_2, a_4) = \{x \in P : \alpha_1(x) \geq a_2,\ \beta_1(x) \leq a_4\}, \\
P(\beta_1, \beta_2, \alpha_1; a_2, a_3, a_4) = \{x \in P : \alpha_1(x) \geq a_2,\ \beta_2(x) \leq a_3,\ \beta_1(x) \leq a_4\}
\end{gathered}$$

and a closed set

$$R(\beta_1, \psi; a_1, a_4) = \{x \in P : \psi(x) \geq a_1, \beta_1(x) \leq a_4\}.$$

**Theorem 2.4** ([3]). *Let $P$ be a cone in a real Banach space $X$ with the norm $\|\cdot\|$. Suppose that*

1. *$T : P \to P$ is completely continuous;*
2. *$\beta_1$ and $\beta_2$ be nonnegative continuous convex functionals on $P$, $\alpha_1$ be a nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$ satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for all $x \in P$ and $\lambda \in [0,1]$, and $\alpha_1(x) \leq \psi(x)$ and there exists a constant $M > 0$ such that $\|x\| \leq M\beta_1(x)$ for all $x \in P$;*
3. *there exist positive numbers $a_1 < a_2, a_3$ and $a_4$ such that*
   - (E1) *$T(\overline{P(\beta_1; a_4)}) \subseteq \overline{P(\beta_1; a_4)}$;*
   - (E2) *$\alpha_1(Tx) > a_2$ for all $x \in P(\beta_1, \alpha_1; a_2, a_4)$ with $\beta_2(Tx) > a_3$;*
   - (E3) *$\{x \in P(\beta_1, \beta_2, \alpha_1; a_2, a_3, a_4) : \alpha_1(x) > a_2\} \neq \emptyset$ and $\alpha_1(Tx) > b$ for all $x \in P(\beta_1, \beta_2, \alpha_1; a_2, a_3, a_4)$;*
   - (E4) *$0 \notin R(\beta_1, \psi; a_1, a_4)$ and $\psi(Tx) < a_1$ for all $x \in R(\beta_1, \psi; a_1, a_4)$ with $\psi(x) = a_1$;*

*Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\beta_1; a_4)}$ such that*

$$\begin{gathered}
\beta_1(x_i) \leq a_4, \quad i = 1, 2, 3; \\
\alpha_1(x_1) > a_2, \quad \psi(x_2) > a_1, \quad \alpha_1(x_2) < a_2, \quad \psi(x_3) < a_1.
\end{gathered}$$

Choose $X = C^1[0,1]$. We call $x \leq y$ for $x, y \in X$ if $x(t) \leq y(t)$ for all $t \in [0,1]$, for $x \in X$, define its norm by

$$\|x\| = \max\big\{ \max_{t \in [0,1]} |x(t)|,\ \max_{t \in [0,1]} |x'(t)| \big\}.$$

It is easy to see that $X$ is a semi-ordered real Banach space. Let

$$\begin{aligned}
P = \Big\{ & y \in X : y(t) \geq 0 \text{ for all } t \in [0,1],\ y'(t) \leq 0 \text{ is decreasing on } [0,1], \\
& y(t) \geq (1-t)y(0) \text{ for all } t \in [0,1], \\
& x(1) + \beta p(1)x'(1) = \sum_{i=1}^{k} b_i x(\xi_i) - \sum_{i=k+1}^{m} b_i x(\xi_i) - \sum_{i=1}^{m} b_i x'(\xi_i),
\end{aligned}$$

$$x'(0) = -\sum_{i=1}^{l} a_i x'(\xi_i) + \sum_{i=l+1}^{m} a_i x'(\xi_i)\Big\}.$$

Then $P$ is a nonempty cone in $X$. For $\sigma \in (0, 1/2)$. Define functionals from $P$ to $\mathbb{R}$ by

$$\beta_1(y) = \max_{t\in[0,1]} |y'(t)|, \quad \psi(x) = \max_{t\in[0,1]} |y(t)|,$$
$$\beta_2(y) = \max_{t\in[\sigma,1-\sigma]} |y(t)|, \quad \alpha_1(y) = \min_{t\in[\sigma,1-\sigma]} |y(t)|, \quad y \in P.$$

Let us list some conditions to be used in this article.

(A1) $p : [0, 1] \to (0, +\infty)$ is continuous;

(A2) $a_i \geq 0$, $b_i \geq 0, c_i \geq 0$ for all $i = 1, \dots, m$ satisfying

$$\sum_{i=l+1}^{m} a_i\Big(\frac{p(0)}{p(\xi_i)}\Big) - \sum_{i=1}^{l} a_i \frac{p(0)}{p(\xi_i)} < 1, \quad \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i < 1,$$
$$\sum_{i=l+1}^{m} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)}\Big) \geq \sum_{i=1}^{l} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)}\Big), \quad \sum_{i=1}^{k} b_i \geq \sum_{i=k+1}^{m} b_i;$$

(A2') $a_i \geq 0$, $b_i \geq 0$ for all $i = 1, \dots, m$ satisfying

$$\sum_{i=l+1}^{m} a_i\Big(\frac{p(0)}{p(\xi_i)}\Big) < 1, \quad \sum_{i=1}^{k} b_i < 1,$$
$$\sum_{i=l+1}^{m} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)}\Big) \geq \sum_{i=1}^{l} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)}\Big), \quad \sum_{i=1}^{k} b_i \geq \sum_{i=k+1}^{m} b_i;$$

(A3) $\beta \geq 0$;

(A4) $f : [0, 1] \times [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous with $f(t, 0, 0) \not\equiv 0$ on each sub-interval of $[0, 1]$;

(A5) $\theta : [0, 1] \to [0, +\infty)$ is a continuous function and $\theta(t) \not\equiv 0$ on each subinterval of $[0, 1]$.

It is easy to see that (A2) holds if (A2') holds.

**Lemma 2.5.** *Suppose that $p : [0, 1] \to (0, +\infty)$ with $p \in C^1[0, 1]$, $x \in X$, $x(t) \geq 0$ for all $t \in [0, 1]$ and $[p(t)\phi(x'(t))]' \leq 0$ on $[0, 1]$. Then $x$ is concave and*

$$x(t) \geq \min\{t, 1 - t\} \max_{t\in[0,1]} x(t), \quad t \in [0, 1]. \tag{2.1}$$

*Proof.* Suppose $x(t_0) = \max_{t\in[0,1]} x(t)$. If $t_0 < 1$, for $t \in (t_0, 1)$, since $x'(t_0) \leq 0$, we have $p(t_0)\phi(x'(t_0)) \leq 0$. Then $p(t)\phi(x'(t)) \leq 0$ for all $t \in (t_0, 1]$. It follows that $x'(t) \leq 0$ for all $t \in [0, 1]$. Thus $x(t_0) \geq x(t) \geq x(1) \geq 0$. Let

$$\tau(t) = \frac{\int_{t_0}^{t} \phi^{-1}\big(\frac{1}{p(s)}\big)ds}{\int_{t_0}^{1} \phi^{-1}\big(\frac{1}{p(s)}\big)ds}.$$

Then $\tau \in C([t_0, 1], [0, 1])$ and is increasing on $[t_0, 1]$ since

$$\frac{d\tau}{dt} = \frac{\phi^{-1}\big(\frac{1}{p(t)}\big)}{\int_{t_0}^{1} \phi^{-1}\big(\frac{1}{p(s)}\big)ds} > 0,$$

$\tau(t_0)=0$ and $\tau(1)=1$. Thus

$$\frac{dx}{dt}=\frac{dx}{d\tau}\frac{d\tau}{dt}=\frac{dx}{d\tau}\frac{\phi^{-1}\big(\frac{1}{p(t)}\big)}{\int_{t_0}^1\phi^{-1}\big(\frac{1}{p(s)}\big)ds}$$

which implies

$$p(t)\phi(x'(t))=\phi\Big(\frac{dx}{d\tau}\Big)\frac{1}{\phi\Big(\int_{t_0}^1\phi^{-1}\big(\frac{1}{p(s)}\big)ds\Big)}.$$

Hence

$$\phi'\Big(\frac{dx}{d\tau}\Big)\frac{\phi^{-1}\big(\frac{1}{p(t)}\big)}{\int_{t_0}^1\phi^{-1}\big(\frac{1}{p(s)}\big)ds}\frac{d^2x}{d\tau^2}=\phi\Big(\int_{t_0}^1\phi^{-1}\big(\frac{1}{p(s)}\big)ds\Big)\big[\,p(t)\phi(x'(t))\big]'\le 0$$

for all $t\in[t_0,1]$. Note that $\phi'(x)\ge 0$. It follows that $\frac{d^2x}{d\tau^2}\le 0$. Together with $x''(\tau)\le 0(\tau\in[0,1])$. Then $x'$ is decreasing on $[0,1]$. We get that there exist $t_0\le\eta\le t\le\xi\le 1$ such that

$$\begin{aligned}\frac{x(t_0)-x(1)}{t_0-1}-\frac{x(t)-x(1)}{t-1}&=-\frac{(t-1)[x(t_0)-x(t)]+(t_0-t)[x(1)-x(t)]}{(t-1)(t_0-1)}\\&=-\frac{(t-1)(t_0-t)x'(\eta)+(t_0-t)(1-t)tx'(\xi)}{(t-1)(t_0-1)}\\&\le-\frac{(t-1)(t_0-t)x'(\xi)+(t_0-t)(1-t)tx'(\xi)}{(t-1)(t_0-1)}=0.\end{aligned}$$

It follows that for $t\in(t_0,1)$,

$$x(t)\ge x(1)+(t-1)\frac{x(t_0)-x(1)}{t_0-1}=x(1)\Big(1-\frac{1-t}{1-t_0}\Big)+\frac{1-t}{1-t_0}x(t_0)\ge(1-t)x(t_0).$$

If $t_0>0$, for $t\in(0,t_0)$, similarly to above discussion, we have

$$x(t)\ge tx(t_0),\ t\in(0,t_0).$$

Then one gets that $x(t)\ge\min\{t,1-t\}\max_{t\in[0,1]}x(t)$ for all $t\in[0,1]$. The proof is complete. □

Consider the boundary-value problem

$$\begin{gathered}[p(t)\phi(x'(t))]'+\theta(t)=0,\quad t\in(0,1),\\ x'(0)=-\sum_{i=1}^{l}a_ix'(\xi_i)+\sum_{i=l+1}^{m}a_ix'(\xi_i),\\ x(1)+\beta x'(1)=\sum_{i=1}^{k}b_ix(\xi_i)-\sum_{i=k+1}^{s}b_ix(\xi_i)-\sum_{i=s+1}^{m}c_ix'(\xi_i).\end{gathered}\tag{2.2}$$

**Lemma 2.6.** *Suppose that* (A1)–(A5) *hold. If $x$ is a solution of* (2.2)*, then $x$ is concave, decreasing and positive on* $(0,1)$.

*Proof.* Suppose $x$ satisfies (2.2). It follows from the assumptions that $px'$ is decreasing on $[0,1]$. Lemma 2.5 implies that $x'$ is decreasing on $[0,1]$. Then $x$ is concave on $[0,1]$.

First, we prove that $x'(0)\le 0$. One sees from (A2) and the deceasing property of $p(t)\phi(x'(t))$ that

$$
\begin{aligned}
x'(1) &= -\sum_{i=1}^{l} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big)\phi^{-1}(p(\xi_i)\phi(x'(\xi_i))) + \sum_{i=l+1}^{m} a_i\big(\frac{1}{p(\xi_i)}\big)\phi^{-1}(p(\xi_i)\phi(x'(\xi_i)))\\
&\le -\sum_{i=1}^{l} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big)\phi^{-1}(p(\xi_l)\phi(x'(\xi_l)))\\
&\quad + \sum_{i=l+1}^{m} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big)\phi^{-1}(p(\xi_{l+1})\phi(x'(\xi_{l+1})))\\
&= \Big(\sum_{i=l+1}^{m} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big) - \sum_{i=1}^{l} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big)\Big)\phi^{-1}(p(\xi_{l+1})\phi(x'(\xi_{l+1})))\\
&\quad + \sum_{i=1}^{l} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big)[\phi^{-1}(p(\xi_{l+1})\phi(x'(\xi_{l+1}))) - \phi^{-1}(p(\xi_l)\phi(x'(\xi_l)))]\\
&\le \Big(\sum_{i=l+1}^{m} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big) - \sum_{i=1}^{l} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big)\Big)\phi^{-1}(p(\xi_{l+1}\phi(x'(\xi_{l+1})))\\
&\le \Big(\sum_{i=l+1}^{m} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big) - \sum_{i=1}^{l} a_i\phi^{-1}\big(\frac{1}{p(\xi_i)}\big)\Big)\phi^{-1}(p(0)\phi(x'(0))).
\end{aligned}
$$

It follows that

$$
\Big(1-\sum_{i=l+1}^{m} a_i\Big(\frac{p(0)}{p(\xi_i)}\Big) + \sum_{i=1}^{l} a_i\frac{p(0)}{p(\xi_i)}\Big)x'(0)\le 0.
$$

It follows that $x'(0)\le 0$. Then $x'(t)\le 0$ for all $t\in[0,1]$.

Second, we prove that $x(1)\ge 0$. It follows from the boundary conditions in (2.2) that

$$
\begin{aligned}
x(1)+\beta x'(1) &= \sum_{i=1}^{k} b_i x(\xi_i) - \sum_{i=k+1}^{s} b_i x(\xi_i) - \sum_{i=s+1}^{m} c_i x'(\xi_i)\\
&\ge \sum_{i=1}^{k} b_i x(\xi_i) - \sum_{i=k+1}^{s} b_i x(\xi_i)\\
&\ge \sum_{i=1}^{k} b_i x(\xi_k) - \sum_{i=k+1}^{s} b_i x(\xi_{k+1})\\
&= \Big(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\Big)x(\xi_k) + \sum_{i=k+1}^{s} b_i[x(\xi_k)-x(\xi_{k+1})]\\
&\ge \Big(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\Big)x(\xi_k)\\
&\ge \Big(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\Big)x(1).
\end{aligned}
$$

Then

$$\Big(1-\sum_{i=1}^{k} b_i+\sum_{i=k+1}^{s} b_i\Big)x(1)+\beta x'(1)\geq 0.$$

We get $x(1)\geq 0$ since $x'(1)\leq 0$ and $\beta\geq 0$. Then $x(t)>x(1)\geq 0$ for all $t\in[0,1)$. The proof is complete. ☐

**Lemma 2.7.** *Suppose that* (A1),(A2'), (A3)–(A5) *hold. Let*

$$\mu=\phi\Big(\frac{1}{\sum_{i=l+1}^{m} a_i\phi^{-1}\big(\frac{p(0)}{p(\xi_i)}\big)}\Big)-1.$$

*If $y$ is a solution of* (2.2), *then*

$$y(t)=B_\theta-\int_t^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_\theta)-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds,\quad t\in[0,1],$$

*where*

$$A_\theta\in\Big[-\phi^{-1}\Big(\frac{\int_0^1\theta(u)du}{\mu p(0)}\Big),0\Big],$$

$$\begin{aligned}A_\theta=&-\sum_{i=1}^{l}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(A_\theta)-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)\\&+\sum_{i=l+1}^{m}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(A_\theta)-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)\end{aligned}$$

*and*

$$\begin{aligned}B_\theta=&\frac{1}{1-\sum_{i=1}^{k}b_i+\sum_{i=k+1}^{m}b_i}\Big[-\beta\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(A_\theta)-\frac{1}{p(1)}\int_0^1\theta(u)du\Big)\\&-\sum_{i=1}^{k}b_i\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_\theta)-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds\\&+\sum_{i=k+1}^{m}b_i\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_\theta)-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds\\&-\sum_{i=1}^{m}c_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(A_\theta)-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)ds\Big].\end{aligned}$$

*Proof.* Since $y$ is solution of (2.2),

$$\begin{gathered}y'(t)=\phi^{-1}\Big(\frac{1}{p(t)}p(0)\phi(y'(0))-\frac{1}{p(t)}\int_0^t\theta(u)du\Big),\\y(t)=y(1)-\int_t^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds.\end{gathered}$$

The boundary conditions in (2.2) imply

$$\begin{aligned}y'(0)=&-\sum_{i=1}^{l}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(y'(0))-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)\\&+\sum_{i=l+1}^{m}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(y'(0))-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)\end{aligned}$$

and

$$\begin{aligned}
&y(1)+\beta\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(y'(0))-\frac{1}{p(1)}\int_0^1\theta(u)du\Big)\\
&=\sum_{i=1}^{k}b_i\Big(y(1)-\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds\Big)\\
&\quad-\sum_{i=k+1}^{m}b_i\Big(y(1)-\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds\Big)\\
&\quad-\sum_{i=1}^{m}c_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(y'(0))-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
y(1)&=\frac{1}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\Big[-\beta\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(y'(0))-\frac{1}{p(1)}\int_0^1\theta(u)du\Big)\\
&\quad-\sum_{i=1}^{k}b_i\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds\\
&\quad+\sum_{i=k+1}^{m}b_i\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\theta(u)du\Big)ds\\
&\quad-\sum_{i=1}^{m}c_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(y'(0))-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)ds\Big].
\end{aligned}$$

Lemma 2.6 implies that $y'(0)\le 0$. One finds that

$$\begin{aligned}
y'(0)&=-\sum_{i=1}^{l}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(y'(0))-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)\\
&\quad+\sum_{i=l+1}^{m}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(y'(0))-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)\\
&\ge\sum_{i=l+1}^{m}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(y'(0))-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big).
\end{aligned}$$

If $y'(0)\neq 0$, one gets

$$1-\sum_{i=l+1}^{m}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)-\frac{1}{\phi(y'(0))}\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big)\le 0. \tag{2.3}$$

Let

$$G(c)=c-\sum_{i=l+1}^{m}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(c)-\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big).$$

Then

$$\frac{G(c)}{c}=1-\sum_{i=l+1}^{m}a_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)-\frac{1}{\phi(c)}\frac{1}{p(\xi_i)}\int_0^{\xi_i}\theta(u)du\Big).$$

Note that

$$\mu=\phi\Big(\frac{1}{\sum_{i=l+1}^m a_i\phi^{-1}\big(\frac{p(0)}{p(\xi_i)}\big)}\Big)-1.$$

It is easy to see that $\frac{G(c)}{c}$ is decreasing on $(-\infty, 0)$ and on $(0, +\infty)$. First, since $\lim_{t\to 0^+} G(c)/c = +\infty$ and

$$\lim_{c\to+\infty} \frac{G(c)}{c} = 1 - \sum_{i=l+1}^{m} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0)\Big) > 0,$$

we get that $\frac{G(c)}{c} > 0$ for all $c > 0$. Second, we have $\lim_{c\to 0^-} G(c)/c = -\infty$ and

$$\begin{aligned}
\frac{G\Big(-\phi^{-1}\Big(\frac{\int_0^1 \theta(u)du}{\mu p(0)}\Big)\Big)}{-\phi^{-1}\Big(\frac{\int_0^1 \theta(u)du}{\mu p(0)}\Big)} &= 1 - \sum_{i=l+1}^{m} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0) + \frac{p(0)\mu}{\int_0^1 \theta(u)du} \frac{1}{p(\xi_i)} \int_0^{\xi_i} \theta(u)du\Big) \\
&= 1 - \sum_{i=t+1}^{m} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0) + \mu \frac{\int_0^{\xi_i} \theta(u)du}{\int_0^1 \theta(u)du} \frac{1}{p(\xi_i)} p(0)\Big) \\
&\ge 1 - \phi^{-1}(1+\mu) \sum_{i=l+1}^{m} a_i \phi^{-1}\Big(\frac{p(0)}{p(\xi_i)}\Big) = 0.
\end{aligned}$$

It follows from (2.3) that $\frac{G(y'(0))}{y'(0)} \le 0$. We get

$$0 \ge y'(0) \ge -\phi^{-1}\Big(\frac{\int_0^1 \theta(u)du}{\mu p(0)}\Big).$$

The proof is complete. □

Define the nonlinear operator $T : X \to X$ by

$$(Tx)(t) = B_x - \int_t^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^s f(u, x(u), x'(u))du\Big) ds,$$

for $t \in [0, 1]$, where

$$A_\sigma \in \Big[-\phi^{-1}\Big(\frac{\int_0^1 f(u, x(u), x'(u))du}{\Big(\phi\Big(\frac{1}{\sum_{i=t+1}^m a_i \phi^{-1}\big(\frac{p(0)}{p(\xi_i)}\big)}\Big) - 1\Big)p(0)}\Big), 0\Big],$$

$$\begin{aligned}
A_x = &-\sum_{i=1}^{l} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0)\phi(A_x) - \frac{1}{p(\xi_i)} \int_0^{\xi_i} f(u, x(u), x'(u))du\Big) \\
&+ \sum_{i=l+1}^{m} a_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0)\phi(A_x) - \frac{1}{p(\xi_i)} \int_0^{\xi_i} f(u, x(u), x'(u))du\Big),
\end{aligned}$$

$$\begin{aligned}
B_x = &\frac{1}{1 - \sum_{i=1}^k b_i + \sum_{i=k+1}^m b_i} \\
&\times \Big[-\beta\phi^{-1}\Big(\frac{1}{p(1)} p(0)\phi(A_x) - \frac{1}{p(1)} \int_0^1 f(u, x(u), x'(u))du\Big) \\
&- \sum_{i=1}^{k} b_i \int_{\xi_i}^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^s f(u, x(u), x'(u))du\Big) ds \\
&+ \sum_{i=k+1}^{m} b_i \int_{\xi_i}^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^s f(u, x(u), x'(u))du\Big) ds
\end{aligned}$$

$$-\sum_{i=1}^{m} c_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(A_x)-\frac{1}{p(\xi_i)}\int_0^{\xi_i} f(u,x(u),x'(u))du\Big)ds\Big].$$

**Lemma 2.8.** *Suppose that* (A1),(A2'), (A3), (A4) *hold. Then*

(i) *the following equalities hold:*

$$[p(t)\phi((Ty)'(t))]'+f(t,y(t),y'(t))=0,\quad t\in(0,1),$$
$$(Ty)'(0)=-\sum_{i=1}^{l} a_i(Ty)'(\xi_i)+\sum_{i=l+1}^{m} a_i(Ty)'(\xi_i),$$
$$(Ty)(1)+\beta(Ty)'(1)=\sum_{i=1}^{k} b_i(Ty)(\xi_i)-\sum_{i=k+1}^{m} b_i(Ty)(\xi_i)-\sum_{i=1}^{m} c_i(Ty)'(\xi_i);$$

(ii) *$Ty\in P$ for each $y\in P$;*

(iii) *$x$ is a positive solution of* (1.7) *if and only if $x$ is a solution of the operator equation $y=Ty$ in $P$;*

(iv) *$T:P\to P$ is completely continuous.*

*Proof.* The proofs of (i), (ii) and (iii) are simple. To prove (iv), it suffices to prove that $T$ is continuous on $P$ and $T$ is relative compact. We divide the proof into two steps:

**Step 1.** For each bounded subset $D\subset P$, and each $x_0\in D$, since $f(t,u,v)$ is continuous, we can prove that $T$ is continuous at $y(t)$.

**Step 2.** For each bounded subset $D\subset P$, prove that $T$ is relative compact on $D$. It is similar to that of the proof of Lemmas in [13] and are omitted. □

**Lemma 2.9.** *Suppose that* (A1), (A2'), (A3), (A4) *hold. Then there exists a constant $M>0$ such that*

$$\max_{t\in[0,1]}(Tx)(t)\le M\max_{t\in[0,1]}|(Tx)'(t)|\quad \textit{for each } x\in P.$$

*Proof.* For $x\in P$, Lemma 2.6 implies that $(Tx)(t)\ge 0$ and $(Tx)'(t)\le 0$ for all $t\in[0,1]$. Lemma 2.8 implies

$$(Tx)(1)+\beta(Tx)'(1)=\sum_{i=1}^{k} b_i(Tx)(\xi_i)-\sum_{i=k+1}^{m} b_i(Tx)(\xi_i)-\sum_{i=1}^{m} c_i(Tx)'(\xi_i).$$

Then there exist numbers $\eta_i\in[\xi_i,1]$ such that

$$\begin{aligned}(Tx)(1)&=\frac{(Tx)(1)-\sum_{i=1}^k b_i(Tx)(1)+\sum_{i=k+1}^m b_i(Tx)(1)}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\\&=\frac{-\beta(Tx)'(1)-\sum_{i=1}^m c_i(Tx)'(\xi_i)}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\\&\quad+\frac{\sum_{i=1}^k b_i[(Tx)(\xi_i)-(Tx)(1)]-\sum_{i=k+1}^m b_i[(Tx)(\xi_i)-(Tx)(1)]}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\\&=\frac{-\beta(Tx)'(1)-\sum_{i=1}^m c_i(Tx)'(\xi_i)}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\end{aligned}$$

$$+\frac{\sum_{i=1}^k b_i(\xi_i-1)(Tx)'(\eta_i)-\sum_{i=k+1}^m b_i(\xi_i-1)(Tx)'(\eta_i)}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}.$$

It follows that

$$\begin{aligned}
&|(Tx)(t)|\\
&=|(Tx)(t)-(Tx)(1)+(Tx)(1)|\\
&\le |(Tx)(1)|+(1-t)|(Tx)'(\xi)| \quad \text{where } \xi\in[t,1]\\
&\le \Big(1+\frac{\beta p(1)+\sum_{i=1}^k b_i(1-\xi_i)+\sum_{i=k+1}^m b_i(1-\xi_i)+\sum_{i=1}^m c_i}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\Big)\max_{t\in[0,1]}|(Tx)'(t)|.
\end{aligned}$$

Then

$$\begin{aligned}
&\max_{t\in[0,1]}|(Tx)(t)|\\
&\le \Big(1+\frac{\beta p(1)+\sum_{i=1}^k b_i(1-\xi_i)+\sum_{i=k+1}^m b_i(1-\xi_i)+\sum_{i=1}^m c_i}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\Big)\max_{t\in[0,1]}|(Tx)'(t)|.
\end{aligned}$$

It follows that there exists a constant $M>0$ such that for all $x\in P$,

$$\max_{t\in[0,1]}(Tx)(t)\le M\beta_1((Tx))\,.$$

□

Denote

$$\mu=\phi\Big(\frac{1}{\sum_{i=t+1}^m a_i\phi^{-1}\big(\frac{p(0)}{p(\xi_i)}\big)}\Big)-1,$$

$$M=1+\frac{\beta p(1)+\sum_{i=1}^k b_i(1-\xi_i)+\sum_{i=k+1}^m b_i(1-\xi_i)+\sum_{i=1}^m c_i}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i},$$

$$\begin{aligned}
L_1=\beta\phi^{-1}\Big(\frac{1+\mu}{\mu p(1)}\Big)&+\sum_{i=1}^k b_i\int_{\xi_i}^1\phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds+\sum_{i=k+1}^m b_i\int_0^{\xi_i}\phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds\\
&+\sum_{i=1}^m c_i\phi^{-1}\left(\frac{1+\mu\xi_i}{\mu p(\xi_i)}\right)ds+\Big(1-\sum_{i=1}^k b_i\Big)\int_0^1\phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds,\\
L_2=\beta\phi^{-1}\Big(\frac{1}{p(1)}\Big)&+\sum_{i=1}^k b_i\int_{\xi_i}^1\phi^{-1}\big(\frac{s}{p(s)}\big)ds+\sum_{i=k+1}^m b_i\int_0^{\xi_i}\phi^{-1}\big(\frac{s}{p(s)}\big)ds\\
&+\sum_{i=1}^m c_i\phi^{-1}\Big(\frac{\xi_i}{p(\xi_i)}\Big)ds+\Big(1-\sum_{i=1}^k b_i\Big)\int_0^1\phi^{-1}\big(\frac{s}{p(s)}\big)ds.
\end{aligned}$$

**Theorem 2.10.** *Choose $k\in(0,1)$. Suppose that* (A1), (A2'), (A3), (A4) *hold. Let $e_1,e_2,c$ be positive numbers and*

$$Q=\min\Big\{\phi\Big(\frac{c\big(1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i\big)}{L_1}\Big),\quad \frac{\mu p(1)\phi(c)}{1+\mu}\Big\};$$

$$W=\phi\Big(\frac{e_2\big(1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i\big)}{L_2}\Big);$$

$$E = \phi\Big(\frac{e_1\big(1 - \sum_{i=1}^k b_i + \sum_{i=k+1}^m b_i\big)}{L_1}\Big).$$

*If* $Mc > e_2 > \frac{e_1}{k} > e_1 > 0$ *and*

- (A6) $f(t,u,v) \le Q$ *for all* $t \in [0,1], u \in [0, Mc], v \in [-c, c]$*;*
- (A7) $f(t,u,v) \ge W$ *for all* $t \in [0,k], u \in [e_2, e_2/k], v \in [-c, c]$*;*
- (A8) $f(t,u,v) \le E$ *for all* $t \in [0,1], u \in [0, e_1], v \in [-c, c]$*;*

*then* (1.7) *has at least three solutions* $x_1, x_2, x_3$ *such that*

$$x_1(0) < e_1, \quad x_2(k) > e_2, \quad x_3(0) > e_1, \quad x_3(k) < e_2.$$

*Proof.* To apply Theorem 2.4, we check that all its conditions are satisfied. By the definitions, it is easy to see that $\alpha_1$ is a nonnegative continuous concave functional on the cone $P$, $\beta_1, \beta_2$ are three nonnegative continuous convex functionals on the cone $P$, $\psi$ a nonnegative continuous functional on the cone $P$. Lemma 2.8 implies that $x = x(t)$ is a positive solution of (1.7) if and only if $x$ is a solution of the operator equation $y = Ty$ and $T$ is completely continuous.

For $x \in P$, one sees that $\psi(\lambda x) \le \lambda\psi(x)$ for all $x \in P$ and $\lambda \in [0,1]$, and $\alpha_1(x) \le \psi(x)$ for all $x \in P$. There exists a constant $M > 0$ such that $\|x\| \le M\beta_1(x)$ for all $x \in P$. Then (1) and (2) of Theorem 2.4 hold.

Corresponding to Theorem 2.4, we have $a_4 = c$, $a_3 = \frac{e_2}{k}$, $a_2 = e_2$, $a_1 = e_1$. Now, we check that (1.3) of Theorem 2.4 holds. One sees that $0 < a_1 < a_2$, $a_3 > 0$, $a_4 > 0$. The rest is divided into four steps.

**Step 1.** Prove that $T(\overline{P_1(\beta_1; a_4)}) \subseteq \overline{P_1(\beta_1; a_4)}$; For $x \in \overline{P_1(\beta_1; a_4)}$, we have $\beta_1(x) \le a_4 = c$. Then

$$\begin{gathered} 0 \le x(t) \le \max_{t\in[0,1]} x(t) \le Mc \quad \text{for } t \in [0,1], \\ -c \le x'(t) \le c \quad \text{for all } t \in [0,1]. \end{gathered}$$

So (A6) implies that $f(t, x(t), x'(t)) < Q$ for $t \in [0,1]$. Then Lemma 2.7 implies

$$0 \ge A_x \ge -\phi^{-1}\Big(\frac{\int_0^1 f(u, x(u), x'(u))du}{\mu p(0)}\Big) \ge -\phi^{-1}\Big(\frac{Q}{\mu p(0)}\Big).$$

Since

$$c \ge \max\big\{\frac{e_2}{\sigma_0},\ La,\ \phi^{-1}\big(\frac{2}{p(0)}\big)a,\ \phi^{-1}\big(\frac{1}{p(1)}\big)a\big\},$$

we obtain

$$\phi(a) \le \min\big\{\phi\big(\frac{c}{L}\big),\ \frac{\phi(c)p(0)}{2},\ \phi(c)p(1)\big\} = Q.$$

Then $\max_{t\in[0,1]}(Ty)(t) = (Ty)(0)$ implies

$$\begin{aligned} &(Tx)(0) \\ &= B_x - \int_0^1 \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x) - \frac{1}{p(s)}\int_0^s f(u, x(u), x'(u))du\Big)ds \\ &= \frac{1}{1 - \sum_{i=1}^k b_i + \sum_{i=k+1}^m b_i} \\ &\quad \times \Big[ -\beta\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(A_x) - \frac{1}{p(1)}\int_0^1 f(u, x(u), x'(u))du\Big) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x) - \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds \\
&+ \sum_{i=k+1}^{m} b_i \int_{\xi_i}^{1} \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x) - \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds \\
&- \sum_{i=1}^{m} c_i \phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(A_x) - \frac{1}{p(\xi_i)}\int_0^{\xi_i} f(u,x(u),x'(u))du\Big)ds\Big] \\
&- \int_0^1 \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x) - \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds \\
=\;& \frac{1}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^m b_i} \\
&\times \Big[ -\beta\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(A_x) - \frac{1}{p(1)}\int_0^1 f(u,x(u),x'(u))du\Big) \\
&- \sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x) - \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds \\
&- \sum_{i=k+1}^{m} b_i \int_0^{\xi_i} \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x) - \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds \\
&- \sum_{i=1}^{m} c_i \phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(A_x) - \frac{1}{p(\xi_i)}\int_0^{\xi_i} f(u,x(u),x'(u))du\Big)ds \\
&- \Big(1-\sum_{i=1}^k b_i\Big)\int_0^1 \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x) - \frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds\Big] \\
\le\;& \frac{1}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^m b_i}\Big[\beta\phi^{-1}\Big(\frac{Q}{\mu p(1)} + \frac{Q}{p(1)}\Big) \\
&+ \sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \phi^{-1}(\frac{Q}{\mu p(s)} + \frac{Qs}{p(s)})ds + \sum_{i=k+1}^{m} b_i \int_0^{\xi_i} \phi^{-1}(\frac{Q}{\mu p(s)} + \frac{Qs}{p(s)})ds \\
&+ \sum_{i=1}^{m} c_i \phi^{-1}(\frac{Q}{\mu p(\xi_i)} + \frac{Q\xi_i}{p(\xi_i)})ds + \Big(1-\sum_{i=1}^k b_i\Big)\int_0^1 \phi^{-1}(\frac{Q}{\mu p(s)} + \frac{Qs}{p(s)})ds\Big] \\
=\;& \frac{\phi^{-1}(Q)}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^m b_i} \\
&\times \Big[\beta\phi^{-1}\Big(\frac{1+\mu}{\mu p(1)}\Big) + \sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds \\
&+ \sum_{i=k+1}^{m} b_i \int_0^{\xi_i} \phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds + \sum_{i=1}^{m} c_i \phi^{-1}\Big(\frac{1+\mu\xi_i}{\mu p(\xi_i)}\Big)ds \\
&+ \Big(1-\sum_{i=1}^k b_i\Big)\int_0^1 \phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds\Big] \le c.
\end{aligned}
$$

On the other hand, similarly to above discussion, since $(Ty)'(t)$ is decreasing and $(Ty)'(0)\le 0$, from Lemma 2.7 we have

$$(Tx)'(1)=\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(A_x)-\frac{1}{p(1)}\int_0^1 f(u,x(u),x'(u))du\Big),$$

$$\begin{aligned}\max_{t\in[0,1]}|(Tx)'(t)|&=-\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(A_x)-\frac{1}{p(1)}\int_0^1 f(u,x(u),x'(u))du\Big)\\&\le \phi^{-1}\Big(\frac{Q}{\mu p(1)}+\frac{Q}{p(1)}\Big)\le c.\end{aligned}$$

It follows that $\|Tx\|=\max\big\{\max_{t\in[0,1]}|(Tx)(t)|,\max_{t\in[0,1]}|(Tx)'(t)|\big\}\le c$. Then $T(\overline{P(\beta_1;a_4)})\subseteq\overline{P(\beta_1;a_4)}$. This completes the proof of (E1) in Theorem 2.4.

**Step 2.** Prove that $\alpha_1(Tx)>a_2$ for all $x\in P(\beta_1,\alpha_1;a_2,a_4)$ with $\beta_2(Tx)>a_3$; For $y\in P(\beta_1,\alpha_1;a_2,a_4)=P(\beta_1,\alpha_1;e_2,c)$ with $\beta_2(Ty)>a_3=\frac{e_2}{k}$, we have

$$\alpha_1(y)=\min_{t\in[0,k]}y(t)\ge e_2,\quad \beta_1(y)=\max_{t\in[0,1]}|y'(t)|\le c,\quad \max_{t\in[0,k]}(Ty)(t)>\frac{e_2}{k}.$$

Then

$$\alpha_1(Ty)=\min_{t\in[0,k]}(Ty)(t)\ge k\beta_2(Ty)>k\frac{e_2}{k}=e_2=a_2.$$

This completes the proof of (E2) in Theorem 2.4.

**Step 3.** Prove that $\{x\in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4):\alpha_1(x)>a_2\}\ne\emptyset$ and $\alpha_1(Tx)>b$ for all $x\in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4)$; It is easy to check that $\{x\in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4):\alpha_1(x)>a_2\}\ne\emptyset$. For $y\in P(\beta_1,\beta_2,\alpha_1;a_2,a_3,a_4)$, one has that

$$\alpha_1(y)=\min_{t\in[0,k]}y(t)\ge e_2,\quad \beta_2(y)=\max_{t\in[0,k]}y(t)\le\frac{e_2}{k},\quad \beta_1(y)=\max_{t\in[0,1]}|y'(t)|\le c.$$

Then

$$e_2\le y(t)\le\frac{e_2}{k},\quad t\in[0,k],\ |y'(t)|\le c.$$

Thus (A7) implies

$$f(t,y(t),y'(t))\ge W,\quad t\in[0,k].$$

Since

$$\alpha_1(Ty)=\min_{t\in[0,k]}(Ty)(t)\ge k\max_{t\in[0,1]}(Ty)(t),$$

we obtain $\alpha_1(Ty)\ge k\max_{t\in[0,1]}(Ty)(t)$. Then

$$\alpha_1(Ty)\ge k\max_{t\in[0,1]}(Ty)(t)\ge k(Ty)(0).$$

From $A_x\le 0$, we obtain

$$\begin{aligned}\alpha_1(T_1y)\ge k\Big[&\frac{1}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^m b_i}\\&\times\Big(-\beta\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(A_x)-\frac{1}{p(1)}\int_0^1 f(u,x(u),x'(u))du\Big)\\&-\sum_{i=1}^k b_i\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x)-\frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds\\&+\sum_{i=k+1}^m b_i\int_{\xi_i}^1\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x)-\frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m} c_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0)\phi(A_x) - \frac{1}{p(\xi_i)} \int_0^{\xi_i} f(u,x(u),x'(u))du\Big) ds\Big) \\
&- \int_0^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^{s} f(u,x(u),x'(u))du\Big) ds\Big] \\
= &\frac{k}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{m} b_i} \\
&\times \Big[ -\beta \phi^{-1}\Big(\frac{1}{p(1)} p(0)\phi(A_x) - \frac{1}{p(1)} \int_0^{1} f(u,x(u),x'(u))du\Big) \\
&- \sum_{i=1}^{k} b_i \int_{\xi_i}^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^{s} f(u,x(u),x'(u))du\Big) ds \\
&+ \sum_{i=k+1}^{m} b_i \int_{\xi_i}^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^{s} f(u,x(u),x'(u))du\Big) ds \\
&- \sum_{i=1}^{m} c_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0)\phi(A_x) - \frac{1}{p(\xi_i)} \int_0^{\xi_i} f(u,x(u),x'(u))du\Big) ds \\
&- \Big(1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{m} b_i\Big) \int_0^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) \\
&- \frac{1}{p(s)} \int_0^{s} f(u,x(u),x'(u))du\Big) ds\Big] \\
= &\frac{k}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{m} b_i} \\
&\times \Big[ -\beta \phi^{-1}\Big(\frac{1}{p(1)} p(0)\phi(A_x) - \frac{1}{p(1)} \int_0^{1} f(u,x(u),x'(u))du\Big) \\
&- \sum_{i=1}^{k} b_i \int_{\xi_i}^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^{s} f(u,x(u),x'(u))du\Big) ds \\
&- \sum_{i=k+1}^{m} b_i \int_0^{\xi_i} \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) - \frac{1}{p(s)} \int_0^{s} f(u,x(u),x'(u))du\Big) ds \\
&- \sum_{i=1}^{m} c_i \phi^{-1}\Big(\frac{1}{p(\xi_i)} p(0)\phi(A_x) - \frac{1}{p(\xi_i)} \int_0^{\xi_i} f(u,x(u),x'(u))du\Big) ds \\
&- \Big(1-\sum_{i=1}^{k} b_i\Big) \int_0^1 \phi^{-1}\Big(\frac{1}{p(s)} p(0)\phi(A_x) \\
&- \frac{1}{p(s)} \int_0^{s} f(u,x(u),x'(u))du\Big) ds\Big] \\
\geq &\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{m} b_i} \\
&\times \Big[\beta \phi^{-1}(\frac{W}{p(1)}) + \sum_{i=1}^{k} b_i \int_{\xi_i}^1 \phi^{-1}(\frac{Ws}{p(s)}) ds + \sum_{i=k+1}^{m} b_i \int_0^{\xi_i} \phi^{-1}(\frac{Ws}{p(s)}) ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m} c_i\phi^{-1}\big(\frac{W\xi_i}{p(\xi_i)}\big)ds+\Big(1-\sum_{i=1}^{k} b_i\Big)\int_0^1 \phi^{-1}\big(\frac{Ws}{p(s)}\big)ds\Big]\\
&=\frac{\phi^{-1}(W)}{1-\sum_{i=1}^{k} b_i+\sum_{i=k+1}^{m} b_i}\\
&\quad\times\Big[\beta\phi^{-1}\big(\frac{1}{p(1)}\big)+\sum_{i=1}^{k} b_i\int_{\xi_i}^1 \phi^{-1}\big(\frac{s}{p(s)}\big)ds+\sum_{i=k+1}^{m} b_i\int_0^{\xi_i}\phi^{-1}\big(\frac{s}{p(s)}\big)ds\\
&\quad+\sum_{i=1}^{m} c_i\phi^{-1}\big(\frac{\xi_i}{p(\xi_i)}\big)ds+\Big(1-\sum_{i=1}^{k} b_i\Big)\int_0^1 \phi^{-1}\big(\frac{s}{p(s)}\big)ds\Big]=e_2.
\end{aligned}
$$

This completes the proof of (E3) in Theorem 2.4.

**Step 4.** Prove that $0\notin R(\beta_1,\psi;a_1,a_4)$ and that $\psi(Tx)<a_1$ for all $x$ in $R(\beta_1,\psi;a_1,a_4)$ with $\psi(x)=a_1$; It is easy to see that $0\notin R(\beta_1,\psi;a_1,a_4)$. For $y\in R(\beta_1,\psi;a_1,a_4)$ with $\psi(x)=a_1$, one has that

$$
\psi(y)=\max_{t\in[0,1]} y(t)=a_1,\quad \beta_1(y)=\max_{t\in[0,1]}|y'(t)|\le a_4.
$$

Hence

$$
0\le y(t)\le a_1,\quad t\in[0,1];\quad -c\le y'(t)\le c,\quad t\in[0,1].
$$

Then (A8) implies $f(t,y(t),y'(t))\le E$, $t\in[0,1]$. So

$$
\begin{aligned}
&\psi(Ty)=\max_{t\in[0,1]}(Ty)(t)=(Ty)(0)\\
&=\frac{1}{1-\sum_{i=1}^{k} b_i+\sum_{i=k+1}^{m} b_i}\\
&\quad\times\Big[-\beta\phi^{-1}\Big(\frac{1}{p(1)}p(0)\phi(A_x)-\frac{1}{p(1)}\int_0^1 f(u,x(u),x'(u))du\Big)\\
&\quad-\sum_{i=1}^{k} b_i\int_{\xi_i}^1 \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x)-\frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds\\
&\quad-\sum_{i=k+1}^{m} b_i\int_0^{\xi_i}\phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x)-\frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds\\
&\quad-\sum_{i=1}^{m} c_i\phi^{-1}\Big(\frac{1}{p(\xi_i)}p(0)\phi(A_x)-\frac{1}{p(\xi_i)}\int_0^{\xi_i} f(u,x(u),x'(u))du\Big)ds\\
&\quad-\Big(1-\sum_{i=1}^{k} b_i\Big)\int_0^1 \phi^{-1}\Big(\frac{1}{p(s)}p(0)\phi(A_x)-\frac{1}{p(s)}\int_0^s f(u,x(u),x'(u))du\Big)ds\Big]\\
&\le\frac{1}{1-\sum_{i=1}^{k} b_i+\sum_{i=k+1}^{m} b_i}\Big[\beta\phi^{-1}\Big(\frac{E}{\mu p(1)}+\frac{E}{p(1)}\Big)\\
&\quad+\sum_{i=1}^{k} b_i\int_{\xi_i}^1 \phi^{-1}\Big(\frac{E}{\mu p(s)}+\frac{Es}{p(s)}\Big)ds+\sum_{i=k+1}^{m} b_i\int_0^{\xi_i}\phi^{-1}\Big(\frac{E}{\mu p(s)}+\frac{Es}{p(s)}\Big)ds\\
&\quad+\sum_{i=1}^{m} c_i\phi^{-1}\Big(\frac{E}{\mu p(\xi_i)}+\frac{E\xi_i}{p(\xi_i)}\Big)ds+\Big(1-\sum_{i=1}^{k} b_i\Big)\int_0^1 \phi^{-1}\Big(\frac{E}{\mu p(s)}+\frac{Es}{p(s)}\Big)ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\phi^{-1}(E)}{1-\sum_{i=1}^{k} b_i+\sum_{i=k+1}^{m} b_i}\Big[\beta\phi^{-1}\Big(\frac{1+\mu}{\mu p(1)}\Big)+\sum_{i=1}^{k} b_i\int_{\xi_i}^{1}\phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds\\
&\quad+\sum_{i=k+1}^{m} b_i\int_0^{\xi_i}\phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds+\sum_{i=1}^{m} c_i\phi^{-1}\Big(\frac{1+\mu\xi_i}{\mu p(\xi_i)}\Big)ds\\
&\quad+\Big(1-\sum_{i=1}^{k} b_i\Big)\int_0^1\phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds\Big]\le a_1.
\end{aligned}
$$

This completes the proof of (E4) in Theorem 2.4.

Then Theorem 2.4 implies that $T$ has at least three fixed points $x_1$, $x_2$ and $x_3$ such that

$$
\beta(x_1)<e_1,\quad \alpha(x_2)>e_2,\quad \beta(x_3)>e_1,\quad \alpha(x_3)<e_2.
$$

Hence (1.7) has three decreasing positive solutions $x_1, x_2$ and $x_3$ as needed in the statement of the theorem; therefore, the proof is complete. □

## 3. Examples

Now, we present an example that illustrates our main results, and that can not be solved by results in [3, 9, 10, 15, 16].

**Example 3.1.** Consider the boundary-value problem

$$
\begin{gathered}
x''(t)+f(t,x(t),x'(t))=0,\quad t\in(0,1),\\
x'(0)=-\frac{1}{4}x'(1/4)+\frac{1}{2}x'(1/2),\\
x(1)+2x'(1)=\frac{1}{4}x(1/4)-\frac{1}{4}x(1/2)-\frac{1}{2}x'(1/4)-\frac{1}{4}x'(1/2),
\end{gathered}
\tag{3.1}
$$

where $f(t,u,v)=f_0(u)+\frac{t|v|}{25500000}$ and

$$
f_0(u)=\begin{cases}
\frac{2}{51}u & u\in[0,4],\\
\frac{8}{51} & u\in[4,12],\\
564000+\frac{564000-\frac{8}{51}}{1004-12}(u-1004) & u\in[12,1004],\\
564000 & u\in[1004,4004],\\
564000 & u\in[4004,2000004],\\
564000e^{u-2000004} & u\ge 2000004.
\end{cases}
$$

Corresponding to (1.7), one sees that $\phi^{-1}(x)=x$, $\xi_1=1/4, \xi_2=1/2$, $a_1=1/4, a_2=1/2$, $b_1=1/4, b_2=1/4$, $c_1=1/2, c_2=1/4$. It is easy to see that (A1)–(A4) hold. Choose constants $k=\frac{1}{2}$, $e_1=2, e_2=100$ and $c=20000$. One obtains

$$
\begin{gathered}
M=1+\frac{\beta p(1)+\sum_{i=1}^{k} b_i(1-\xi_i)+\sum_{i=k+1}^{m} b_i(1-\xi_i)+\sum_{i=1}^{m} c_i}{1-\sum_{i=1}^{k} b_i+\sum_{i=k+1}^{m} b_i}=\frac{51}{4},\\
\mu=\phi\Big(\frac{1}{\sum_{i=t+1}^{m} a_i\phi^{-1}\Big(\frac{p(0)}{p(\xi_i)}\Big)}\Big)-1=1,
\end{gathered}
$$

$$
\begin{aligned}
L_1 = \beta\phi^{-1}\Big(\frac{1+\mu}{\mu p(1)}\Big) &+ \sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds \\
&+ \sum_{i=k+1}^{m} b_i \int_0^{\xi_i} \phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds + \sum_{i=1}^{m} c_i\phi^{-1}\Big(\frac{1+\mu\xi_i}{\mu p(\xi_i)}\Big)ds \\
&+ \Big(1-\sum_{i=1}^{k} b_i\Big)\int_0^1 \phi^{-1}\Big(\frac{1+\mu s}{\mu p(s)}\Big)ds, \\
L_2 = \beta\phi^{-1}\big(\frac{1}{p(1)}\big) &+ \sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \phi^{-1}\big(\frac{s}{p(s)}\big)ds \\
&+ \sum_{i=k+1}^{m} b_i \int_0^{\xi_i} \phi^{-1}\big(\frac{s}{p(s)}\big)ds + \sum_{i=1}^{m} c_i\phi^{-1}\big(\frac{\xi_i}{p(\xi_i)}\big)ds \\
&+ \Big(1-\sum_{i=1}^{k} b_i\Big)\int_0^1 \phi^{-1}\big(\frac{s}{p(s)}\big)ds, \\
Q = \min\big\{\phi\Big(\frac{c\Big(1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{m} b_i\Big)}{L_1}\Big)&,\ \frac{\mu p(1)\phi(c)}{1+\mu}\big\}; \\
W = \phi\Big(\frac{e_2\Big(1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{m} b_i\Big)}{L_2}\Big)&; \\
E = \phi\Big(\frac{e_1\Big(1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{m} b_i\Big)}{L_1}\Big)&.
\end{aligned}
$$

It is easy to see that If $Mc > e_2 > \frac{e_1}{k} > e_1 > 0$ and

$$
\begin{gathered}
f(t,u,v) \le Q \quad \text{for all } t\in[0,1], u\in[0,Mc], v\in[-20000,20000]; \\
f(t,u,v) \ge W \quad \text{for all } t\in[0,1/2], u\in[100,200], v\in[-20000,20000]; \\
f(t,u,v) \le E \quad \text{for all } t\in[0,1], u\in[0,2], v\in[-20000,20000];
\end{gathered}
$$

Then (A6), (A7) and (A8) hold. Theorem 2.10 implies that (3.1) has at least three positive solutions such that

$$
x_1(0) < 2, \quad x_2(k) > 100, \quad x_3(0) > 2, \quad x_3(k) < 100.
$$

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Jianye Xia

Department of Mathematics, Guangdong University of Finance, Guangzhou 510320, China

*E-mail address*: `JianyeXia@sohu.com`

Yuji Liu

Department of Mathematics, Hunan Institute of Science and Technology, Yueyang 414000, China

*E-mail address*: `liuyuji888@sohu.com`