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HOMOCLINIC SOLUTIONS FOR SECOND-ORDER NON-AUTONOMOUS HAMILTONIAN SYSTEMS WITHOUT GLOBAL AMBROSETTI-RABINOWITZ CONDITIONS

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Abstract. This article studies the existence of homoclinic solutions for the second-order non-autonomous Hamiltonian system

$$\ddot{q} - L(t)q + W_q(t,q) = 0,$$

where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$. The function $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ is not assumed to satisfy the global Ambrosetti-Rabinowitz condition. Assuming reasonable conditions on L and W, we prove the existence of at least one nontrivial homoclinic solution, and for $W(t,q)$ even in q, we prove the existence of infinitely many homoclinic solutions.

1. Introduction

The purpose of this work is to study the existence of homoclinic solutions for the second-order non-autonomous Hamiltonian system

$$\ddot{q} - L(t)q + W_q(t,q) = 0, \tag{1.1}$$

where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. We say that a solution $q(t)$ of (1.1) is homoclinic (to 0) if $q \in C^2(\mathbb{R}, \mathbb{R}^n)$, $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $t \to \pm\infty$. If $q(t) \not\equiv 0$, $q(t)$ is called a nontrivial homoclinic solution.

The existence of homoclinic solutions is one of the most important problems in the history of Hamiltonian systems, and has been studied intensively by many mathematicians. Assuming that $L(t)$ and $W(t,q)$ are independent of t, or T-periodic in t, many authors have studied the existence of homoclinic solutions for Hamiltonian system (1.1) via critical point theory and variational methods, see for instance [2, 4, 6, 8, 3, 15, 17] and the references therein. More general cases are considered in the recent papers [9, 19]. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems.

In this paper we are interested in the case where (1.1) is non-autonomous. If $L(t)$ and $W(t,q)$ are neither autonomous nor periodic in t, this problem is quite

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different from the ones just described, because of the lack of compactness of the Sobolev embedding, see for instance [10, 14, 18] and the references therein. In [18], the authors considered (1.1) without periodicity assumptions on L and W and showed that (1.1) possesses one homoclinic solution by using a variant of the Mountain Pass Theorem without the (PS) condition. In [14], under the same assumptions of [18], the authors, by employing a new compact embedding theorem, obtained the existence and multiplicity of homoclinic solution of (1.1). The authors in [10] removed the technical coercivity condition on L and proved the existence of homoclinic solutions when L and W are even in t by the method of approximating the homoclinic orbits from solutions of boundary problems. By the way, a concentration-compactness principle by Lions [11, 12] has also been used to find one homoclinic solution of (1.1), for example [1, 5].

Here, we must point that all the results in the works mentioned above are obtained under the assumption that W satisfies the global Ambrosetti-Rabinowitz condition on q; i.e., there is a constant $\theta > 2$ such that, for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \backslash \{0\}$,

$$0 < \theta W(t,q) \le \big(W_q(t,q), q\big), \tag{1.2}$$

where $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot|$ is the induced norm. More recently, in [13], the authors discussed the existence of even homoclinic solutions under a class of superquadratic conditions on W which is weaker than the global Ambrosetti-Rabinowitz condition (see Corollary 2 in [13]). More precisely, they assumed that W and L are even in $t \in \mathbb{R}$ and satisfy:

- (W1) there exist constants $d_1 > 0$ and $\vartheta \ge 2$ such that $W(t,q) \le d_1|q|^{\vartheta}$ for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$;
- (W2) $W(t,q) = o(|q|^2)$ as $|q| \to 0$ uniformly with respect to $t \in \mathbb{R}$;
- (W3) $W_q(t,q) \to 0$ as $|q| \to 0$ uniformly with respect to $t \in \mathbb{R}$;
- (W4) there exist constants $d_2 > 0$ and $\delta > \vartheta - 2$ and $\tau \in L^1(\mathbb{R}, \mathbb{R}^+)$ such that
$$\big(W_q(t,q), q\big) - 2W(t,q) \ge d_2|q|^{\delta} - \tau(t)$$
for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$;

and some other reasonable assumptions on L and W. They also gave an example which satisfies their conditions but do not satisfy the global Ambrosetti-Rabinowitz condition ([13, Remark 2]). On the other hand, compared with the case that $W(t,q)$ is superquadratic as $|q| \to +\infty$, the literature is smaller which is available for the case that $W(t,q)$ is subquadratic as $|q| \to +\infty$. As far as the authors are aware, only the papers [7, 20, 21] dealt with this case.

Motivated by the works mentioned above, in this paper we give some more general conditions on L and W to guarantee that (1.1) has nontrivial homoclinic solutions. Explicitly, assuming that $W(t,q)$ satisfies some superquadratic conditions which are weaker than the global Ambrosetti-Rabinowitz condition and different from in [13], and some other reasonable assumptions on L and W, we give a new result to guarantee that (1.1) has at least one nontrivial homoclinic solution and infinitely many homoclinic solutions if $W(t,q)$ is even in q, which generalizes and improves the previous results in the literature.

To state our main result, we state the basic hypotheses on L and W:

- (H1) $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$; there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $\big(L(t)q, q\big) \ge \alpha(t)|q|^2$ and $\alpha(t) \to +\infty$ as $|t| \to +\infty$.

From this condition, we see that there is a positive constant $\beta > 0$ such that

$$\big(L(t)q, q\big) \geq \beta |q|^2 \quad \text{for all } t \in \mathbb{R},\ q \in \mathbb{R}^n. \tag{1.3}$$

(H2) $W(t,q) \geq 0$ for all $(t,q) \in \mathbb{R} \times \mathbb{R}^n$ and there exist constants $M > 0$ and $R_1 > 0$ such that
$$W(t,q) \leq M|q|^2 \quad \text{for all } (t,q) \in \mathbb{R} \times \mathbb{R}^n,\ |q| \leq R_1,$$
where $2M < \beta$, with β defined in (1.3);

(H3) there exist $\alpha_0(t) > 0$ and constants $\alpha_1 > 2$, $R_2 > 0$ such that
$$W(t,q) \geq \alpha_0(t)|q|^{\alpha_1} \quad \text{for all } (t,q) \in \mathbb{R} \times \mathbb{R}^n,\ |q| \geq R_2;$$

(H4) there exist constants $\mu > 2$ and α_2 with $0 \leq \alpha_2 < (\mu - 2)/2$ such that
$$\mu W(t,q) - \big(W_q(t,q), q\big) \leq \alpha_2 \big(L(t)q, q\big) \quad \text{for all } (t,q) \in \mathbb{R} \times \mathbb{R}^n;$$

(H5) $W_q(t,q) = O(|q|)$ as $|q| \to 0$ uniformly with respect to $t \in \mathbb{R}$;

(H6) there exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that
$$|W_q(t,q)| \leq |\overline{W}(q)|$$
for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$.

Our main result reads as follows.

Theorem 1.1. *Suppose that the conditions* (H1)-(H6) *are satisfied, then* (1.1) *possesses one nontrivial homoclinic solution. Moreover, if we assume that $W(t,q)$ is even in q, i.e.,*

(H7) *$W(t,-q) = W(t,q)$ for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$,*

then (1.1) *has infinitely many homoclinic solutions.*

Remark 1.2. Note that if (1.2) holds, so does (H4), however the reverse is not true. Now, we give an example of W which satisfies (H2)-(H6) but does not satisfy the conditions (1.2), (W1) and (W2). This does not say anything about (W4).

Example 1.3.
$$W(t,q) = a(t)\,|q|^2 \exp(|q|^\gamma), \tag{1.4}$$
where $\gamma > 0$ is a constant and $a(t)$ is a positive, continuous, bounded function with $\inf_{t\in\mathbb{R}} a(t) > 0$.

Then we have
$$W(t,q) \leq \sup_{t\in\mathbb{R}} a(t) \exp(R_1^\gamma)|q|^2 := M|q|^2 \quad \text{for all } (t,q) \in \mathbb{R} \times \mathbb{R}^n,\ |q| \leq R_1,$$
where $R_1 > 0$ is any given constant, which implies that (H2) holds if $\sup_{t\in\mathbb{R}} a(t)$ is small enough. Moreover, it is easy to check that
$$\begin{gathered} W(t,q) \geq a(t)|q|^{2+\gamma}, \\ W_q(t,q) = 2a(t)\exp(|q|^\gamma)q + \gamma a(t)\exp(|q|^\gamma)|q|^\gamma q, \\ \big(W_q(t,q), q\big) = 2a(t)|q|^2\exp(|q|^\gamma) + \gamma a(t)|q|^{\gamma+2}\exp(|q|^\gamma). \end{gathered} \tag{1.5}$$
So, for any constant $\mu > 2$, we have
$$\mu W(t,q) - \big(W_q(t,q), q\big) = a(t)|q|^2 \exp(|q|^\gamma)\big(\mu - 2 - \gamma|q|^\gamma\big),$$
which yields
$$0 < \mu W(t,q) - \big(W_q(t,q), q\big) \leq (\mu - 2)\sup_{t\in\mathbb{R}} a(t) \exp\big(\frac{\mu-2}{\gamma}\big)|q|^2 \tag{1.6}$$

for all $(t,q)\in\mathbb{R}\times\mathbb{R}^n$ and $0<|q|\le\big(\frac{\mu-2}{\gamma}\big)^{1/\gamma}$; i.e., (1.2) does not hold for every $t\in\mathbb{R}$ and $q\in\mathbb{R}^n\setminus\{0\}$; and

$$\mu W(t,q)-\big(W_q(t,q),q\big)\le 0\quad\text{for all } (t,q)\in\mathbb{R}\times\mathbb{R}^n,\ |q|\ge\big(\frac{\mu-2}{\gamma}\big)^{1/\gamma},$$

which, combining with (1.6), implies that, for some $\mu>2$, if $\sup_{t\in\mathbb{R}}a(t)$ is small enough (note that (1.3)), (H4) holds. On the other hand, by (1.4), we have

$$\lim_{q\to 0}\frac{W(t,q)}{|q|^2}=a(t)\ge\inf_{t\in\mathbb{R}}a(t)>0,$$

and, by (1.5), we have

$$\begin{aligned}|W_q(t,q)|&=2a(t)\exp(|q|^\gamma)q+\gamma a(t)\exp(|q|^\gamma)|q|^\gamma q\\&\le\sup_{t\in\mathbb{R}}a(t)\exp(|q|^\gamma)(2+\gamma|q|^\gamma)|q|\end{aligned}$$

and

$$2\inf_{t\in\mathbb{R}}a(t)\le\lim_{q\to 0}\frac{|W_q(t,q)|}{|q|}=2a(t)\le 2\sup_{t\in\mathbb{R}}a(t).$$

The remainder of the paper is organized as follows. In section 2, some preliminary results are given. In section 3, we give the proof of Theorem 1.1.

2. Preliminary Results

To establish our result via the critical point theory, we firstly describe some properties of the space on which the variational framework associated with (1.1) is defined. Let

$$E=\big\{q\in H^1(\mathbb{R},\mathbb{R}^n):\int_{\mathbb{R}}\big[|\dot q(t)|^2+\big(L(t)q(t),q(t)\big)\big]dt<+\infty\big\},$$

then the space E is a Hilbert space with the inner product

$$(x,y)=\int_{\mathbb{R}}\big[\big(\dot x(t),\dot y(t)\big)+\big(L(t)x(t),y(t)\big)\big]dt$$

and the corresponding norm $\|x\|^2=(x,x)$. Note that

$$E\subset H^1(\mathbb{R},\mathbb{R}^n)\subset L^p(\mathbb{R},\mathbb{R}^n)$$

for all $p\in[2,+\infty]$ with the embedding being continuous. In particular, for $p=+\infty$, there exists a constant $C>0$ such that

$$\|q\|_\infty\le C\|q\|,\quad\forall q\in E. \tag{2.1}$$

Here $L^p(\mathbb{R},\mathbb{R}^n)$ $(2\le p<+\infty)$ and $H^1(\mathbb{R},\mathbb{R}^n)$ denote the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norms

$$\|q\|_p:=\Big(\int_{\mathbb{R}}|q(t)|^p dt\Big)^{1/p}\quad\text{and}\quad\|q\|_{H^1}:=\big(\|q\|_2^2+\|\dot q\|_2^2\big)^{1/2}$$

respectively. $L^\infty(\mathbb{R},\mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|q\|_\infty:=\operatorname{ess\,sup}\{|q(t)|:t\in\mathbb{R}\}.$$

Lemma 2.1 ([14, Lemma 1])**.** *Suppose that L satisfies* (H1)*. Then the embedding of E in $L^2(\mathbb{R},\mathbb{R}^n)$ is compact.*

Similar to Lemma 2 of [14], we can get the following result. For the reader's convenience, we give the details of its proof.

Lemma 2.2. *Suppose that* (H1), (H5), (H6) *are satisfied. If* $q_k \rightharpoonup q_0$ *(weakly) in* E*, then* $W_q(t, q_k) \to W_q(t, q_0)$ *in* $L^2(\mathbb{R}, \mathbb{R}^n)$*.*

Proof. Assume that $q_k \rightharpoonup q_0$ in E. Then, by Banach-Steinhaus Theorem and (2.1), there is a constant $b_1 > 0$ such that

$$\sup_{k\in\mathbb{N}} \|q_k\|_\infty \le b_1.$$

Assumptions (H5) and (H6) imply the existence of $b_2 > 0$ such that

$$|W_q(t, q_k(t))| \le b_2|q_k(t)|$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence

$$|W_q(t, q_k(t)) - W_q(t, q_0(t))| \le b_2(|q_k(t)| + |q_0(t)|) \le b_2(|q_k(t) - q_0(t)| + 2|q_0(t)|).$$

Since, by Lemma 2.1, $q_k \to q_0$ in L^2, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|q_k - q_0\|_2 < +\infty,$$

which implies that $q_k(t) \to q_0(t)$ for almost every $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |q_k(t) - q_0(t)| = \zeta(t) \in L^2(\mathbb{R}, \mathbb{R}^n).$$

Therefore,

$$|W_q(t, q_k(t)) - W_q(t, q_0(t))| \le b_2(\zeta(t) + 2|q_0(t)|).$$

Then, using the Lebesgue's Convergence Theorem, the lemma is readily proved. $\square$

Now we introduce more notations and some necessary definitions. Let $\mathcal{B}$ be a real Banach space, $I \in C^1(\mathcal{B}, \mathbb{R})$, which means that I is a continuously Fréchet-differentiable functional defined on $\mathcal{B}$.

Definition 2.3 ([16])**.** $I \in C^1(\mathcal{B}, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{B}$, for which $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$, possesses a convergent subsequence in $\mathcal{B}$.

Moreover, let B_r be the open ball in $\mathcal{B}$ with the radius r and centered at 0 and ∂B_r denote its boundary. We obtain the existence and multiplicity of homoclinic solutions of (1.1) by use of the following well-known Mountain Pass Theorems, see [16].

Lemma 2.4 ([16, Theorem 2.2])**.** *Let $\mathcal{B}$ be a real Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$ satisfying the (PS) condition. Suppose that $I(0) = 0$ and that*

- (A1) *there exist constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$,*
- (A2) *there exists $e \in \mathcal{B} \setminus \overline{B}_\rho$ such that $I(e) \le 0$.*

Then I possesses a critical value $c \ge \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0, 1], \mathcal{B}) : g(0) = 0, g(1) = e\}.$$

Lemma 2.5 ([16, Theorem 9.12])**.** *Let $\mathcal{B}$ be an infinite dimensional real Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$ be even satisfying the (PS) condition and $I(0) = 0$. If $\mathcal{B} = V \oplus X$, where V is finite dimensional, and I satisfies:*

- (A3) *there exist constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$ and*
- (A4) *for each finite dimensional subspace $\tilde{E} \subset \mathcal{B}$, there is an $R = R(\tilde{E})$ such that $I \leq 0$ on $\tilde{E} \backslash B_{R(\tilde{E})}$.*

Then I has an unbounded sequence of critical values.

3. Proof of Theorem 1.1

Now we establish the corresponding variational framework to obtain homoclinic solutions of (1.1). Define the functional $I : \mathcal{B} = E \to \mathbb{R}$ as follows

$$\begin{aligned} I(q) &= \int_{\mathbb{R}} \big[\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2}\big(L(t)q(t), q(t)\big) - W(t, q(t))\big]dt \\ &= \frac{1}{2}\|q\|^2 - \int_{\mathbb{R}} W(t, q(t))dt. \end{aligned} \tag{3.1}$$

Lemma 3.1. *Under the conditions of Theorem 1.1, we have*

$$I'(q)v = \int_{\mathbb{R}} \big[\big(\dot{q}(t), \dot{v}(t)\big) + \big(L(t)q(t), v(t)\big) - \big(W_q(t, q(t)), v(t)\big)\big]dt, \tag{3.2}$$

for all q, $v \in E$, which yields that

$$I'(q)q = \|q\|^2 - \int_{\mathbb{R}} \big(W_q(t, q(t)), q(t)\big)dt. \tag{3.3}$$

Moreover, I is a continuously Fréchet-differentiable functional defined on E, i.e., $I \in C^1(E, \mathbb{R})$ and any critical point of I on E is a classical solution of (1.1) *with $q(\pm\infty) = 0 = \dot{q}(\pm\infty)$.*

Proof. We firstly show that $I : E \to \mathbb{R}$. By (H2), there exist constants $M > 0$ and $R_1 > 0$ such that

$$W(t, q) \leq M|q|^2 \quad \text{for all } (t, q) \in \mathbb{R} \times \mathbb{R}^n,\ |q| \leq R_1. \tag{3.4}$$

Let $q \in E$, then $q \in C^0(\mathbb{R}, \mathbb{R}^n)$ (the space of continuous functions q on $\mathbb{R}$ such that $q(t) \to 0$ as $|t| \to +\infty$). Therefore there is a constant $R > 0$ such that $|t| \geq R$ implies that $|q(t)| \leq R_1$. Hence, by (3.4), we have

$$\int_{\mathbb{R}} W(t, q(t))dt \leq \int_{-R}^{R} W(t, q(t))dt + M\int_{|t|\geq R} |q(t)|^2 dt < +\infty. \tag{3.5}$$

Combining (3.1) and (3.5), we show that $I : E \to \mathbb{R}$.

Next we prove that $I \in C^1(E, \mathbb{R})$. Rewrite I as $I = I_1 - I_2$, where

$$I_1 := \int_{\mathbb{R}} \big[\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2}\big(L(t)q(t), q(t)\big)\big]dt \quad \text{and} \quad I_2 := \int_{\mathbb{R}} W(t, q(t))dt.$$

It is easy to check that $I_1 \in C^1(E, \mathbb{R})$ and

$$I_1'(q)v = \int_{\mathbb{R}} \big[\big(\dot{q}(t), \dot{v}(t)\big) + \big(L(t)q(t), v(t)\big)\big]dt. \tag{3.6}$$

Therefore it is sufficient to show this is the case for I_2. In the process we will see that

$$I_2'(q)v = \int_{\mathbb{R}} \big(W_q(t, q(t)), v(t)\big)dt, \tag{3.7}$$

which is defined for all q, $v \in E$. For any given $q \in E$, let us define $J(q) : E \to \mathbb{R}$ as

$$J(q)v = \int_{\mathbb{R}} \big(W_q(t, q(t)), v(t)\big) dt, \quad v \in E.$$

It is obvious that $J(q)$ is linear. Now we show that $J(q)$ is bounded. Indeed, for any given $q \in E$, there exists a constant $M_1 > 0$ such that $\|q\| \le M_1$ and, by (2.1), $\|q\|_\infty \le CM_1$. So according to (H5) and (H6), there is a constant $b_3 > 0$ such that

$$|W_q(t, q(t))| \le b_3 |q(t)|, \quad \text{for all} \quad t \in \mathbb{R},$$

which yields that, by (1.3) and the Hölder inequality,

$$|J(q)v| = \Big| \int_{\mathbb{R}} \big(W_q(t, q(t), v(t))\big) dt\Big| \le b_3 \|q\|_2 \|v\|_2 \le \frac{b_3}{\beta} \|q\| \|v\|. \tag{3.8}$$

Moreover, for q and $v \in E$, by the Mean Value Theorem, we have

$$\int_{\mathbb{R}} W(t, q(t) + v(t)) dt - \int_{\mathbb{R}} W(t, q(t)) dt = \int_{\mathbb{R}} \Big(W_q(t, q(t) + h(t)v(t)), v(t)\Big) dt,$$

where $h(t) \in (0, 1)$. Therefore, by Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} \big(W_q(t, q(t) + h(t)v(t)), v(t)\big) dt - \int_{\mathbb{R}} \big(W_q(t, q(t)), v(t)\big) dt \\ &= \int_{\mathbb{R}} \Big(W_q(t, q(t) + h(t)v(t)) - W_q(t, q(t)), v(t)\Big) dt \to 0 \end{aligned} \tag{3.9}$$

as $\|v\| \to 0$. Combining (3.8) and (3.9), we see that (3.7) holds. It remains to prove that I_2' is continuous. Suppose that $q \to q_0$ in E and note that

$$\begin{aligned} \sup_{\|v\|=1} |I_2'(q)v - I_2'(q_0)v| &= \sup_{\|v\|=1} \Big| \int_{\mathbb{R}} \big(W_q(t, q(t)) - W_q(t, q_0(t)), v(t)\big) dt \Big| \\ &\le \|W_q(\cdot, q(\cdot)) - W_q(\cdot, q_0(\cdot))\|_2 \|v\|_2 \\ &\le \frac{\|v\|}{\sqrt{\beta}} \|W_q(\cdot, q(\cdot)) - W_q(\cdot, q_0(\cdot))\|_2. \end{aligned}$$

By Lemma 2.2, we obtain $I_2'(q)v - I_2'(q_0)v \to 0$ as $\|q\| \to \|q_0\|$ uniformly with respect to v, which implies the continuity of I_2' and we show that $I \in C^1(E, \mathbb{R})$.

Lastly, we check that critical points of I are classical solutions of (1.1) satisfying $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $|t| \to +\infty$. We have known that $E \subset C^0(\mathbb{R}, \mathbb{R}^n)$. Moreover if q is one critical point of I, by (3.2), we have $L(t)q - W_q(t, q)$ is the weak derivative of $\dot{q}$. Since $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $E \subset C^0(\mathbb{R}, \mathbb{R}^n)$, which yields that $q \in C^2(\mathbb{R}, \mathbb{R}^n)$; i.e., q is a classical solution of (1.1). Moreover, it is easy to check that $\dot{q}(t) \to 0$ as $|t| \to +\infty$. $\square$

Lemma 3.2. *Under the conditions* (H1), (H4), (H5), (H6)*, I satisfies the (PS) condition.*

Proof. Assume that $\{u_j\}_{j \in \mathbb{N}} \subset E$ is a sequence such that $\{I(u_j)\}_{j \in \mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_j)| \le C_1, \quad \|I'(u_j)\|_{E^*} \le C_1 \tag{3.10}$$

for every $j \in \mathbb{N}$.

We firstly prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. By (3.1), (3.3) and (H4), we have

$$\begin{aligned}
&\big(\frac{\mu}{2}-1\big)\|u_j\|^2\\
&= \mu I(u_j) - I'(u_j)u_j + \int_{\mathbb{R}} \big(\mu W(t,u_j(t)) - \big(W_q(t,u_j(t)),u_j(t)\big)\big)dt \\
&\le \mu I(u_j) - I'(u_j)u_j + \alpha_2 \int_{\mathbb{R}} \big(L(t)u_j(t),u_j(t)\big)dt.
\end{aligned} \tag{3.11}$$

Let us define

$$\eta(q) = \int_{\mathbb{R}} \Big[\big(\frac{\mu-2}{2}\big)|\dot q(t)|^2 + \big(\frac{\mu-2}{2}-\alpha_2\big)\big(L(t)q(t),q(t)\big)\Big]dt,$$

then we have

$$\mu_1\|q\|^2 \le \eta(q) \le \mu_2\|q\|^2, \tag{3.12}$$

where $\mu_1 = \frac{\mu-2}{2}-\alpha_2$ and $\mu_2 = \frac{\mu-2}{2}$. Thus, combining (3.10), (3.11) with (3.12), we obtain

$$\mu_1\|u_j\|^2 \le \eta(u_j) \le \mu I(u_j) - I'(u_j)u_j \le \mu C_1 + C_1\|u_j\|. \tag{3.13}$$

Since $\mu_1 > 0$, the inequality (3.13) shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E. By Lemma 2.1, the sequence $\{u_j\}_{j\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_j\}_{j\in\mathbb{N}}$, and there exists $u \in E$ such that

$$\begin{gathered}
u_j \rightharpoonup u, \quad \text{weakly in } E,\\
u_j \to u, \quad \text{strongly in } L^2(\mathbb{R},\mathbb{R}^n).
\end{gathered}$$

Hence

$$\big(I'(u_j) - I'(u)\big)(u_j - u) \to 0,$$

and by Lemma 2.2 and the Hölder inequality, we have

$$\int_{\mathbb{R}} \Big(W_q(t,u_j(t)) - W_q(t,u(t)), u_j(t) - u(t)\Big)dt \to 0$$

as $j \to +\infty$. On the other hand, an easy computation shows that

$$\begin{aligned}
&\big(I'(u_j) - I'(u), u_j - u\big)\\
&= \|u_j - u\|^2 - \int_{\mathbb{R}} \Big(W_q(t,u_j(t)) - W_q(t,u(t)), u_j(t) - u(t)\Big)dt.
\end{aligned}$$

Consequently, $\|u_j - u\| \to 0$ as $j \to +\infty$. □

Now, we present the proof of Theorem 1.1, divided into several steps.

Proof of Theorem 1.1. Step 1. It is clear that $I(0) = 0$ by (H2) and $I \in C^1(E,\mathbb{R})$ satisfies the (PS) condition by Lemmas 3.1 and 3.2.

Step 2. We now show that there exist constants $\rho > 0$ and $\alpha > 0$ such that I satisfies the condition (A1) of Lemma 2.4. In fact, assume that $q \in E$ and $0 < \|q\|_\infty \le R_1$. Then, by (1.3) and (H2), we have

$$\int_{\mathbb{R}} W(t,q(t))dt \le M\int_{\mathbb{R}} |q(t)|^2 dt \le M\|q\|_2^2 \le \frac{M}{\beta}\|q\|^2,$$

and in consequence, combining this with (3.1), we obtain

$$I(q) \ge \frac{1}{2}\|q\|^2 - \frac{M}{\beta}\|q\|^2 = \frac{1}{2}\big(1 - \frac{2M}{\beta}\big)\|q\|^2. \tag{3.14}$$

Note that (H2) implies $1-\frac{2M}{\beta}>0$. Set

$$\rho=\frac{R_1}{C},\quad \alpha=\frac{(\beta-2M)R_1^2}{2\beta C^2}>0. \tag{3.15}$$

By (2.1), if $\|q\|=\rho$, then $0<\|q\|_\infty\le R_1$ and (3.14) gives that $I|_{\partial B_\rho}\ge\alpha$.
Step 3. It remains to prove that there exists $e\in E$ such that $\|e\|>\rho$ and $I(e)\le 0$, where ρ is defined in (3.15). By (3.1), we have, for every $m\in\mathbb{R}\backslash\{0\}$ and $q\in E\backslash\{0\}$,

$$I(m\,q)=\frac{m^2}{2}\|q\|^2-\int_{\mathbb{R}}W(t,m\,q(t))dt.$$

Take some $Q\in E$ such that $\|Q\|=1$. Then there exists a subset Ω of positive measure of $\mathbb{R}$ such that $Q(t)\neq 0$ for $t\in\Omega$. Take $m>0$ such that $m|Q(t)|\ge R_2$ for $t\in\Omega$. Then, by (H2) and (H3), we obtain that

$$I(m\,Q)\le\frac{m^2}{2}-m^{\alpha_1}\int_{\Omega}\alpha_0(t)|Q(t)|^{\alpha_1}dt. \tag{3.16}$$

Since $\alpha_0(t)>0$ and $\alpha_1>2$, (3.16) implies that $I(m\,Q)<0$ for some $m>0$ such that $m|Q(t)|\ge R_2$ for $t\in\Omega$ and $\|m\,Q\|>\rho$, where ρ is defined in (3.15). By Lemma 2.4, I possesses a critical value $c\ge\alpha>0$ given by

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}I(g(s)),$$

where

$$\Gamma=\{g\in C([0,1],E):g(0)=0,\,g(1)=e\}.$$

Hence there is $q\in E$ such that

$$I(q)=c,\quad I'(q)=0.$$

Step 4. Now suppose that $W(t,q)$ is even in q; i.e., (H7) holds, which implies that I is even. Moreover, we already know that $I(0)=0$ and $I\in C^1(E,\mathbb{R})$ satisfies the (PS) condition in Step 1.

To apply Lemma 2.5, it suffices to prove that I satisfies the conditions (A3) and (A4) of Lemma 2.5. (A3) is identically the same as in Step 2, so it is already proved. Now we prove that (A4) holds. Let $\tilde{E}\subset E$ be a finite dimensional subspace. From Step 3 we know that, for any $Q\in\tilde{E}\subset E$ such that $\|Q\|=1$, there is $m_Q>0$ such that

$$I(m\,Q)<0,\quad \text{for every } |m|\ge m_Q>0.$$

Since $\tilde{E}\subset E$ is a finite dimensional subspace, we can choose an $\tilde{R}=R(\tilde{E})>0$ such that

$$I(q)<0,\quad \forall q\in\tilde{E}\backslash B_{\tilde{R}}.$$

Hence, by Lemma 2.5, I possesses an unbounded sequence of critical values $\{c_j\}_{j\in\mathbb{N}}$ with $c_j\to+\infty$. Let q_j be the critical point of I corresponding to c_j, then (1.1) has infinitely many homoclinic solutions.

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