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STABILITY FOR A NON-LOCAL NON-AUTONOMOUS SYSTEM OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH DELAYS

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Abstract. In this article, we establish sufficient conditions for the existence, uniqueness and uniformly stability of solutions for a class of nonlocal non-autonomous system of fractional-order delay differential equations with several delays.

1. Introduction

Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))'$, where $'$ denoted the transpose of the matrix. Let $\alpha \in (0, 1]$ and $i = 1, 2, \dots, n$. Consider the nonlocal problem

$$D^\alpha x_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t - r_j) + h_i(t), \quad t > 0 \tag{1.1}$$

$$x(t) = \Phi(t) \quad \text{for } t < 0, \quad \text{and} \quad \lim_{t\to 0^-} \Phi(t) = O \tag{1.2}$$

$$I^\beta x(t)|_{t=0} = O, \quad \beta \in (0, 1] \tag{1.3}$$

where D^α denoted the Riemann-Liouville derivative of order α; $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times n}$, $H(t) = (h_i(t))_{n\times 1}$, $\Phi(t) = (\phi_i(t))_{n\times 1}$ are given matrices; O is the zero matrix; $r_j \ge 0$ are constants.

Fractional differential equations has been studied by various researchers because they appear in various fields: physics, mechanics, engineering, electrochemistry, economics; see for example [5]-[8], [11]-[14] and references therein.

In this work, we discuss the existence, uniqueness and stability of solution of the non-autonomous time-varying delay system (1.1)-(1.3). Abd El-Salam and El-Sayed [1] proved the existence of a unique uniformly stable solution for the non-autonomous system

$${}^cD_a^\alpha x(t) = A(t)x(t) + f(t) \quad x(0) = x^0, \quad t > 0$$

where ${}^cD_a^\alpha$ is the Caputo fractional derivatives (see [10]-[12]), $A(t)$ and $f(t)$ are continuous matrices. El-Sayed [3] proved the existence and uniqueness of the solution

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$u(t)$ of the problem

$$
\begin{gathered}
{}^cD_a^\alpha u(t) + CD_a^\beta u(t-r) = Au(t) + Bu(t-r), \quad 0 \le \beta \le \alpha \le 1 \\
u(t) = g(t), \quad t \in [a-r, a],\ r > 0
\end{gathered}
$$

by the method of steps, where A, B, C are bounded operators defined on a Banach space X. Zhang [15] established the existence of a unique solution for the delay fractional differential equation

$$
D^\alpha x(t) = A_0 x(t) + A_1 x(t-r) + f(t), \quad t > 0, \quad x(t) = \phi(t), \quad t \in [-r, 0]
$$

by the method of steps, where A_0, A_1 are constant matrices. a study of finite time stability was shown there.

Here we prove the existence of a unique solution for (1.1)-(1.3), of the form

$$
x_i(t) = \begin{cases} \phi_i(t), & t < 0 \\ 0, & t = 0 \\ I^\alpha\{\sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j) + h_i(t)\}, & t > 0. \end{cases}
$$

This solution is in $C((-\infty, T])$, $T < \infty$, and is uniformly stable.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used thought this paper.

Definition 2.1. The fractional (arbitrary) order integral of a function $f \in L_1[a, b]$ of order $\alpha \in R^+$ is defined by

$$
I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.
$$

where Γ is the gamma function; see [9, 10, 11, 12].

Definition 2.2. The Riemann-liouville fractional (arbitrary) order derivatives of order $\alpha \in (n-1, n)$ of the function f is defined by

$$
D_a^\alpha f(t) = \frac{d^n}{dt^n} I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad t \in [a, b];
$$

see [9, 10, 11, 12].

The concept of stability can be related to that of continuous dependence of solution on their initial value. Consider the non-autonomous linear system

$$
x'(t) = A(t)x(t) \tag{2.1}
$$

with the initial condition $x(t_0) = x^0$.

Definition 2.3. The solution $x = 0$ of (2.1) is called stable if for any $\epsilon > 0, t_0 \ge 0$, there exist $\delta(\epsilon, t_0) > 0$ such that $\|x(t, t_0, x^0)\| < \epsilon$ for $t \ge t_0$ as soon as $\|x^0\| < \delta$. And the solution $x = 0$ of (2.1) will be called uniformly stable if $\delta(\epsilon, t_0)$ can be chosen independent of $t_0 : \delta(\epsilon, t_0) \equiv \delta(\epsilon)$; see [2].

3. Existence and Uniqueness

Let $X = (C_n(I), \|\cdot\|_1)$, where $C_n(I)$ be the class of continuous column n-vectors functions. For $x \in C_n[0,T]$, define the norm $\|x\| = \sum_{i=1}^n \sup_{t\in[0,T]}\{e^{-Nt}|x_i(t)|\}$. For a matrix B define the norm $\|B\| = \sum_{i=1}^n |b_i| = \sum_{i=1}^n \sup_{t,j} |b_{ij}|$.

Theorem 3.1. *Let $a_{ij}, b_{ij}, h_i, \phi_i$ be in $C(I)$. Then there exist a unique solution $x \in X$ of* (1.1)-(1.3)

Proof. For $t > 0$, equation (1.1) can be written as

$$\frac{d}{dt} I^{1-\alpha} x_i(t) = \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j) + h_i(t)$$

integrating both sides of the above equation, we obtain

$$I^{1-\alpha}x_i(t) - I^{1-\alpha}x_i(t)|_{t=0} = \int_0^t \{\sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(s)x_j(s-r_j) + h_i(s)\}\, ds$$

then

$$I^{1-\alpha}x_i(t) = \int_0^t \{\sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(s)x_j(s-r_j) + h_i(s)\}\, ds\,.$$

Applying the operator by I^α, on both sides,

$$Ix_i(t) = I^{\alpha+1}\big\{ \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j) + h_i(t)\big\}$$

differentiating both side, we obtain

$$x_i(t) = I^\alpha\{\sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j) + h_i(t)\}, \quad i = 1, 2, \dots, n \tag{3.1}$$

Now let $F : X \to X$, defined by

$$Fx_i = I^\alpha\big\{ \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j) + h_i(t)\big\}$$

then

$$\begin{aligned}
|Fx_i - Fy_i| &= |I^\alpha\{\sum_{j=1}^n a_{ij}(t)\{x_j(t) - y_j(t)\} + \sum_{j=1}^n b_{ij}(t)\{x_j(t-r_j) - y_j(t-r_j)\}\}| \\
&= \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \{\sum_{j=1}^n a_{ij}(s)\{x_j(s) - y_j(s)\} \\
&\quad + \sum_{j=1}^n b_{ij}(s)\{x_j(s-r_j) - y_j(s-r_j)\}\}ds\Big| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^n |a_{ij}(s)||x_j(s) - y_j(s)|ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^n |b_{ij}(s)||x_j(s-r_j) - y_j(s-r_j)|ds
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n} \sup_{t,\forall j} |a_{ij}(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s) - y_j(s)| ds \\
&\quad + \sum_{j=1}^{n} \sup_{t,\forall j} |b_{ij}(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s-r_j) - y_j(s-r_j)| ds \\
&\le \sum_{j=1}^{n} a_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s) - y_j(s)| ds \\
&\quad + \sum_{j=1}^{n} b_i \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s-r_j) - y_j(s-r_j)| ds \\
&\quad + \sum_{j=1}^{n} b_i \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s-r_j) - y_j(s-r_j)| ds \\
&\le a_i \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s) - y_j(s)| ds \\
&\quad + b_i \sum_{j=1}^{n} \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x_j(s-r_j) - y_j(s-r_j)| ds
\end{aligned}
$$

and

$$
\begin{aligned}
&e^{-Nt}|Fx_i - Fy_i| \\
&\le a_i \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} |x_j(s) - y_j(s)| ds \\
&\quad + b_i \sum_{j=1}^{n} \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+r_j)} e^{-N(s-r_j)} |x_j(s-r_j) - y_j(s-r_j)| ds \\
&\le a_i \sum_{j=1}^{n} \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} ds \\
&\quad + b_i \sum_{j=1}^{n} \int_0^{t-r_j} \frac{(t-\theta-r_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} e^{-N\theta} |x_j(\theta) - y_j(\theta)| d\theta \\
&\le a_i \sum_{j=1}^{n} \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \frac{1}{N^\alpha} \int_0^{Nt} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du \\
&\quad + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \int_0^{t-r_j} \frac{(t-\theta-r_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} d\theta \\
&\le \frac{a_i}{N^\alpha} \|x-y\| + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \int_0^{t-r_j} \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-Nu} e^{-Nr_j} du \\
&\le \frac{a_i}{N^\alpha} \|x-y\| + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \frac{e^{-Nr_j}}{N^\alpha} \int_0^{N(t-r_j)} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du \\
&\le \frac{a_i}{N^\alpha} \|x-y\| + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \frac{e^{-Nr_j}}{N^\alpha}
\end{aligned}
$$

$$\le \frac{a_i}{N^\alpha}\|x-y\| + \frac{b_i}{N^\alpha}\sum_{j=1}^{n}\sup_t e^{-Nt}|x_j(t)-y_j(t)|$$
$$\le \frac{a_i+b_i}{N^\alpha}\|x-y\|\,.$$

Then

$$\begin{aligned}\|Fx-Fy\| &= \sum_{i=1}^{n}\sup_t e^{-Nt}|Fx_i-Fy_i| \\ &\le \sum_{i=1}^{n}\frac{a_i+b_i}{N^\alpha}\|x-y\| \\ &\le \frac{\|A\|+\|B\|}{N^\alpha}\|x-y\|.\end{aligned}$$

Now choose N large enough such that $\frac{\|A\|+\|B\|}{N^\alpha}<1$, so the map $F: X\to X$ is a contraction and it has a fixed point $x=Fx$ and hence, there exist a unique column vector $x\in X$ which is the solution of the integral equation (3.1).

We now prove the equivalence between the integral equation (3.1) and the non-local problem (1.1)-(1.3). Indeed, since $x\in C_n(I)$ and $I^{1-\alpha}x(t)\in C_n(I)$ applying the operator $I^{1-\alpha}$ on both sides of (3.1), we obtain

$$\begin{aligned}I^{1-\alpha}x_i(t) &= I^{1-\alpha}I^{\alpha}\{\sum_{j=1}^{n}a_{ij}(t)x_j(t)+\sum_{j=1}^{n}b_{ij}(t)x_j(t-r_j)+h_i(t)\},\quad i=1,2,\dots,n \\ &= I\{\sum_{j=1}^{n}a_{ij}(t)x_j(t)+\sum_{j=1}^{n}b_{ij}(t)x_j(t-r_j)+h_i(t)\}\,.\end{aligned}$$

Differentiating both sides,

$$DI^{1-\alpha}x_i(t) = DI\{\sum_{j=1}^{n}a_{ij}(t)x_j(t)+\sum_{j=1}^{n}b_{ij}(t)x_j(t-r_j)+h_i(t)\},.$$

Then

$$D^{\alpha}x_i(t) = \sum_{j=1}^{n}a_{ij}(t)x_j(t)+\sum_{j=1}^{n}b_{ij}(t)x_j(t-r_j)+h_i(t),\quad t>0$$

which proves the equivalence of (3.1) and (1.1).

We want to prove that $\lim_{t\to 0^+}x_i=0$. Since $x_j(s), a_{ij}(s), h_i(s)$ are continuous on $[0,T]$, there exist constants l_j, L_j, m_i, M_i such that $l_j\le a_{ij}(s)x_j(s)\le L_j$ and $m_i\le h_i(s)\le M_i$. We have

$$I^{\alpha}\{\sum_{j=1}^{n}a_{ij}(t)x_j(t)+h_i(t)\} = \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big\{\sum_{j=1}^{n}a_{ij}(s)x_j(s)+h_i(s)\big\}ds$$

which implies

$$\begin{aligned}\{\sum_{j=1}^{n}l_j+m_i\}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds &\le I^{\alpha}\{\sum_{j=1}^{n}a_{ij}(t)x_j(t)+h_i(t)\} \\ &\le \{\sum_{j=1}^{n}L_j+M_i\}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\end{aligned}$$

which in turn implies

$$\{\sum_{j=1}^{n} l_j + m_i\}\frac{t^\alpha}{\Gamma(\alpha+1)} \leq I^\alpha\{\sum_{j=1}^{n} a_{ij}(t)x_j(t) + h_i(t)\} \leq \{\sum_{j=1}^{n} L_j + M_i\}\frac{t^\alpha}{\Gamma(\alpha+1)}$$

and

$$\lim_{t\to 0^+} I^\alpha\{\sum_{j=1}^{n} a_{ij}(t)x_j(t) + h_i(t)\} = 0.$$

Since $b_{ij}(s), \phi_j(s-r_j)$ are continuous on $[0, r_j]$, there exist constants k_j, K_j such that $k_j \leq b_{ij}(s)\phi_j(s-r_j) \leq K_j$. Also $b_{ij}(s), x_j(s-r_j)$ are continuous on $[r_j, T]$, then there exist a constants k_j^*, K_J^* such that $k_j^* \leq b_{ij}(s)x_j(s-r_j) \leq K_j^*$. Let $k = \min_{\forall j}\{k_j, k_j^*\}$ and $K = \max_{\forall j}\{K_j, K_j^*\}$, we have

$$\begin{aligned}
&I^\alpha \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j)\\
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} b_{ij}(s)x_j(s-r_j)ds\\
&= \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} b_{ij}(s)\phi_j(s-r_j)ds + \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} b_{ij}(s)x_j(s-r_j)ds
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{j=1}^{n} k_j \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \sum_{j=1}^{n} k_j^* \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\\
&\leq I^\alpha \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j)\\
&\leq \sum_{j=1}^{n} K_j \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \sum_{j=1}^{n} K_j^* \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{j=1}^{n} k_j\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{(t-r_j)^\alpha}{\Gamma(\alpha+1)}\Big) + \sum_{j=1}^{n} k_j^*\frac{(t-r_j)^\alpha}{\Gamma(\alpha+1)} \leq I^\alpha \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j)\\
&\leq \sum_{j=1}^{n} K_j\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{(t-r_j)^\alpha}{\Gamma(\alpha+1)}\Big) + \sum_{j=1}^{n} K_j^*\frac{(t-r_j)^\alpha}{\Gamma(\alpha+1)}\,.
\end{aligned}$$

Then

$$k\frac{t^\alpha}{\Gamma(\alpha+1)} \leq I^\alpha \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j) \leq K\frac{t^\alpha}{\Gamma(\alpha+1)}$$

and

$$\lim_{t\to 0^+} I^\alpha \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j) = 0.$$

Then from (3.1) $\lim_{t\to 0^+} x_i = 0$. □

Now for $t \in (-\infty, T], T < \infty$, the solution of (1.1)-(1.3) takes the form

$$x_i(t) = \begin{cases} \phi_i(t), & t < 0 \\ 0, & t = 0 \\ I^{\alpha}\{\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j) + b_i(t)\}, & t > 0 \end{cases}$$

4. Stability

In this section we study the stability of the solution of the nonlocal problem (1.1)-(1.3).

Definition 4.1. The solution of the non-autonomous linear system (1.1) is stable if for any $\epsilon > 0$, there exist $\delta > 0$ such that for any two solutions $x(t) = (x_1(t), x_2(t), \dots, x_n(t))'$ and $\widetilde{x}(t) = (\widetilde{x}_1(t), \widetilde{x}_2(t), \dots, \widetilde{x}_n(t))'$ with the initial conditions (1.2)-(1.3) and $\{I^{\beta}\widetilde{x}(t)|_{t=0} = 0,\ \beta \in (0,1], \widetilde{x}(t) = \widetilde{\Phi}(t)$ for $t < 0$ and $\lim_{t\to 0}\widetilde{\Phi}(t) = O\}$, respectively, one has $\|\Phi(t) - \widetilde{\Phi}(t)\| \le \delta$, then $\|x(t) - \widetilde{x}(t)\| < \epsilon$ for all $t \ge 0$.

Theorem 4.2. *The solution of the nonlocal delay system* (1.1)-(1.3) *is uniformly stable.*

Proof. Let $x(t)$ and $\widetilde{x}(t)$ be two solutions of the system (1.1) under the conditions (1.2)-(1.3) and $I^{\beta}\widetilde{x}(t)|_{t=0} = 0$, $\widetilde{x}(t) = \widetilde{\Phi}(t)$, $t < 0$ and $\lim_{t\to 0}\widetilde{\Phi}(t) = O$, respectively. Then for $t > 0$, from (3.1), we have

$$\begin{aligned}
|x_i - \widetilde{x}_i| &= |I^{\alpha}\{\sum_{j=1}^{n} a_{ij}(t)(x_j(t) - \widetilde{x}_j(t)) + \sum_{j=1}^{n} b_{ij}(t)(x_j(t-r_j) - \widetilde{x}_j(t-r_j)\}| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\{\sum_{j=1}^{n} |a_{ij}(s)||x_j(s) - \widetilde{x}_j(s)| \\
&\quad + \sum_{j=1}^{n} |b_{ij}(s)||x_j(s-r_j) - \widetilde{x}_j(s-r_j)|\}ds \\
&\le \sum_{j=1}^{n} \sup_{t,\forall j} |a_{ij}(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_j(s) - \widetilde{x}_j(s)|ds \\
&\quad + \sum_{j=1}^{n} \sup_{t,\forall j} |b_{ij}(t)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_j(s-r_j) - \widetilde{x}_j(s-r_j)|ds \\
&\le \sum_{j=1}^{n} a_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\phi_j(s) - \widetilde{\phi}_j(s)|\,ds \\
&\quad + \sum_{j=1}^{n} b_i \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\phi_j(s-r_j) - \widetilde{\phi}_j(s-r_j)|ds \\
&\quad + \sum_{j=1}^{n} b_i \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x_j(s-r_j) - \widetilde{x}_j(s-r_j)|ds
\end{aligned}$$

and

$$e^{-Nt}|x_i - \widetilde{x}_i|$$

$$
\begin{aligned}
&\le a_i \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} |x_j(s) - \widetilde{x}_j(s)| ds \\
&\quad + b_i \sum_{j=1}^{n} \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+r_j)} e^{-N(s-r_j)} |\phi_j(s-r_j) - \widetilde{\phi}_j(s-r_j)| ds \\
&\quad + b_i \sum_{j=1}^{n} \int_{r_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+r_j)} e^{-N(s-r_j)} |x_j(s-r_j) - \widetilde{x}_j(s-r_j)| ds \\
&\le a_i \sum_{j=1}^{n} \sup_t \{e^{-Nt} |x_j(t) - \widetilde{x}_j(t)|\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} ds \\
&\quad + b_i \sum_{j=1}^{n} \int_{-r_j}^{0} \frac{(t-\theta-r_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} e^{-N\theta} |\phi_j(\theta) - \widetilde{\phi}_j(\theta)| d\theta \\
&\quad + b_i \sum_{j=1}^{n} \int_0^{t-r_j} \frac{(t-\theta-r_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} e^{-N\theta} |x_j(\theta) - \widetilde{x}_j(\theta)| d\theta \\
&\le \frac{a_i}{N^\alpha} \|x_j(t) - \widetilde{x}_j(t)\| \int_0^{Nt} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du \\
&\quad + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt} |\phi_j(t) - \widetilde{\phi}_j(t)|\} \int_{-r_j}^{0} \frac{(t-\theta-r_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} d\theta \\
&\quad + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt} |x_j(t) - \widetilde{x}_j(t)|\} \int_0^{t-r_j} \frac{(t-\theta-r_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} d\theta \\
&\le \frac{a_i}{N^\alpha} \|x_j(t) - \widetilde{x}_j(t)\| \\
&\quad + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt} |\phi_j(t) - \widetilde{\phi}_j(t)|\} \frac{e^{-Nr_j}}{N^\alpha} \int_{N(t-r_j)}^{Nt} \frac{u^{\alpha-1} e^{-Nu}}{\Gamma(\alpha)} du \\
&\quad + b_i \sum_{j=1}^{n} \sup_t \{e^{-Nt} |x_j(t) - \widetilde{x}_j(t)|\} \frac{e^{-Nr_j}}{N^\alpha} \int_0^{N(t-r_j)} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du \\
&\le \frac{a_i}{N^\alpha} \|x_j(t) - \widetilde{x}_j(t)\| + \frac{b_i}{N^\alpha} \sum_{j=1}^{n} e^{-Nr_j} \sup_t \{e^{-Nt} |x_j(t) - \widetilde{x}_j(t)|\} \\
&\quad + \frac{b_i}{N^\alpha} \sum_{j=1}^{n} e^{-Nr_j} \sup_t \{e^{-Nt} |\phi_j(t) - \widetilde{\phi}_j(t)|\} \\
&\le \frac{a_i}{N^\alpha} \|x_j(t) - \widetilde{x}_j(t)\| + \frac{b_i}{N^\alpha} \sum_{j=1}^{n} \sup_t \{e^{-Nt} |x_j(t) - \widetilde{x}_j(t)|\} \\
&\quad + \frac{b_i}{N^\alpha} \sum_{j=1}^{n} \sup_t \{e^{-Nt} |\phi_j(t) - \widetilde{\phi}_j(t)|\} \\
&\le \frac{a_i + b_i}{N^\alpha} \|x - \widetilde{x}\| + \frac{b_i}{N^\alpha} \|\Phi - \widetilde{\Phi}\| .
\end{aligned}
$$

Then

$$\begin{aligned}\|x-\widetilde{x}\| &= \sum_{i=1}^{n}\sup_t e^{-Nt}|x_i-\widetilde{x}_i\\ &\le \sum_{i=1}^{n}\frac{a_i+b_i}{N^\alpha}\|x-\widetilde{x}\|+\sum_{i=1}^{n}\frac{b_i}{N^\alpha}\|\Phi-\widetilde{\Phi}\|\\ &\le \frac{\|A\|+\|B\|}{N^\alpha}\|x-\widetilde{x}\|+\frac{\|B\|}{N^\alpha}\|\Phi-\widetilde{\Phi}\|;\end{aligned}$$

i.e.,

$$\Big(1-\frac{\|A\|+\|B\|}{N^\alpha}\Big)\|x-\widetilde{x}\| \le \frac{\|A\|}{N^\alpha}\|\Phi-\widetilde{\Phi}\|$$

and

$$\|x-\widetilde{x}\| \le \Big(1-\frac{\|A\|+\|B\|}{N^\alpha}\Big)^{-1}\|\Phi-\widetilde{\Phi}\|;$$

therefore, for $\delta>0$ such that $\|\Phi-\widetilde{\Phi}\|<\delta$, we can find $\epsilon=\big(1-\frac{\|A\|+\|B\|}{N^\alpha}\big)^{-1}\delta$ such that $\|x-\widetilde{x}\|\le\epsilon$ which proves that the solution $x(t)$ is uniformly stable. □

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