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SINGULAR ELLIPTIC EQUATIONS INVOLVING A CONCAVE TERM AND CRITICAL CAFFARELLI-KOHN-NIRENBERG EXPONENT WITH SIGN-CHANGING WEIGHT FUNCTIONS

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Abstract. In this article we establish the existence of at least two distinct solutions to singular elliptic equations involving a concave term and critical Caffarelli-Kohn-Nirenberg exponent with sign-changing weight functions.

1. Introduction

This article shows the existence of at least two solutions to the problem

$$\begin{gathered} -\operatorname{div}\Big(\frac{\nabla u}{|x|^{2a}}\Big)-\mu\frac{u}{|x|^{2(a+1)}}=\lambda h(x)\frac{|u|^{q-2}u}{|x|^{c}}+k(x)\frac{|u|^{2_*-2}u}{|x|^{2_*b}}\quad \text{in } \Omega\backslash\{0\} \\ u=0 \quad \text{on } \partial\,\Omega \end{gathered} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain, $N\geq 3$, $0\in\Omega$, $a<(N-2)/2$, $a\leq b<a+1$, $1<q<2$, $c\leq q(a+1)+N(1-q/2)$, $2_*:=2N/(N-2+2(b-a))$ is the critical Caffarelli-Kohn-Nirenberg exponent, $\mu<\bar{\mu}_a:=(N-2(a+1))^2/4$, λ is a positive parameter and h, k are continuous functions which change sign in $\bar{\Omega}$.

It is clear that degeneracy and singularity occur in problem (1.1). In these situations, the classical methods fail to be applied directly so that the existence results may become a delicate matter that is closely related to some phenomena due to the degenerate (or singular) character of the differential equation. The starting point of the variational approach to these problems is the following Caffarelli-Kohn-Nirenberg inequality in [6]: there is a positive constant $C_{a,b}$ such that

$$\Big(\int_{\mathbb{R}^N}|x|^{-2_*b}|u|^{2_*}dx\Big)^{1/2_*}\leq C_{a,b}\Big(\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx\Big)^{1/2}\quad \forall u\in C_0^\infty(\Omega), \tag{1.2}$$

where $-\infty<a<(N-2)/2$, $a\leq b<a+1$, $2_*=2N/(N-2+2(b-a))$. For sharp constants and extremal functions, see [7,9]. In (1.2), as $b=a+1$, then $2_*=2$ and we have the following weighted Hardy inequality [9]:

$$\int_{\mathbb{R}^N}|x|^{-2(a+1)}u^2dx\leq\frac{1}{\bar{\mu}_a}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx\quad \text{for all } u\in C_0^\infty(\Omega). \tag{1.3}$$

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We introduce a weighted Sobolev space $D_a^{1,2}(\Omega)$ which is the completion of the space $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{0,a} = \Big(\int_\Omega |x|^{-2a}|\nabla u|^2 dx\Big)^{1/2}.$$

Define H_μ as the completion of the space $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mu,a} := \Big(\int_\Omega \big(|x|^{-2a}|\nabla u|^2 - \mu|x|^{-2(a+1)}u^2\big)dx\Big)^{1/2} \quad \text{for } -\infty < \mu < \bar{\mu}_a.$$

By weighted Hardy inequality $\|\cdot\|_{\mu,a}$ is equivalent to $\|\cdot\|_{0,a}$; i.e.,

$$\Big(1 - \frac{1}{\bar{\mu}_a}\max(\mu,0)\Big)^{1/2}\|u\|_{0,a} \le \|u\|_{\mu,a} \le \Big(1 - \frac{1}{m\bar{u}_a}\min(\mu,0)\Big)^{1/2}\|u\|_{0,a},$$

for all $u \in H_\mu$. From the boundedness of Ω and the standard approximation arguments, it is easy to see that (1.2) hold for any $u \in H_\mu$ in the sense

$$\Big(\int_\Omega |x|^{-c}|u|^p dx\Big)^{1/p} \le C\Big(\int_\Omega |x|^{-2a}|\nabla u|^2 dx\Big)^{1/2}, \tag{1.4}$$

for $1 \le p \le 2N/(N-2)$, $c \le p(a+1) + N(1-p/2)$, and in [15] if $p < 2N/(N-2)$ the embedding $H_\mu \hookrightarrow L_p(\Omega,|x|^{-c})$ is compact, where $L_p(\Omega,|x|^{-c})$ is the weighted L_p space with norm

$$|u|_{p,c} = \Big(\int_\Omega |x|^{-c}|u|^p dx\Big)^{1/p}.$$

We start by giving a brief historic point of view. It is known that the number of nontrivial solutions of problem (1.1) is affected by the concave and convex terms. This study has been the focus of a great deal of research in recent years.

The case $h \equiv 1$ and $k \equiv 1$ has been studied extensively by many authors, we refer the reader to [1], [2], [8], [14] and the references therein. In [1] Ambrosetti et al. studied the problem (1.1) for $\mu = 0$, $a = b = c = 0, 2_* = 2^* = 2N/(N-2)$ replaced by p, where $1 < p \le 2_*$. They establish the existence of $\Lambda_0 > 0$ such that $(\mathcal{P}_{\lambda,0})$ for λ fixed in $(0,\Lambda_0)$ has at least two positive solutions by using sub-super method and the Mountain Pass Theorem, problem (1.1) for $\lambda = \Lambda_0$ has also a positive solution and no positive solution for $\lambda > \Lambda_0$. When $\mu > 0$, $a = b = c = 0$, Chen [8] studied the asymptotic behavior of solutions to problem (1.1) by using the Moser's iteration. By applying the Ekeland Variational Principle he obtained a first positive solution, and by the Mountain Pass Theorem he proved the existence of a second positive solution. Recently, Bouchekif and Matallah [2] extended the results of [8] to problem $(\mathcal{P}_{\lambda,\mu})$ with $a = c = 0$, $0 \le b < 1$, they established the existence of two positive solutions under some sufficient conditions for λ and μ. Lin [14] considered a more general problem (1.1) with $0 \le a < (N-2)/2$, $a \le b < a+1$, $c = 0, 1 < q < 2$ and $\mu > 0$.

For the case $h \not\equiv 1$ or $k \not\equiv 1$, we refer the reader to [3, 12, 17, 18] and the references therein. Tarantello [17] studied the problem (1.1) for $\mu = 0$, $a = b = c = 0$, $q = \lambda = 1$, $k \equiv 1$ and h not necessarily equals to 1, satisfying some conditions. Recently, problem (1.1) in $\Omega = \mathbb{R}^N$ with $q = 1$ has considered in [3].

Wu [18] showed the existence of multiple positive solutions for problem (1.1) with $a = b = c = 0$, $1 < q < 2$, $k \equiv 1, h$ is a continuous function which changes sign in $\bar{\Omega}$. In [12], Hsu and Lin established the existence of multiple nontrivial solutions to problem (1.1) with $a = b = c = 0$, $1 < q < 2$, h and k are smooth functions which change sign in $\bar{\Omega}$.

The operator $L_{\mu,a}u := -\operatorname{div}(|x|^{-2a}\nabla u) - \mu|x|^{-2(a+1)}u$ has been the subject of many papers, we quote, among others [11] for $a = 0$ and $\mu < \bar{\mu}_0$, and [10] or [16] for general case i.e $-\infty < a < (N-2)/2$ and $\mu < \bar{\mu}_a$.

Xuan et al. [16] proved that under the conditions

$$N \geq 3, \quad a < (N-2)/2, \quad 0 < \sqrt{\bar{\mu}_a} - \sqrt{\bar{\mu}_a - \mu} + a < (N-2)/2,$$
$$a \leq b < a+1, \quad \mu < \bar{\mu}_a - b^2,$$

for $\varepsilon > 0$, the function

$$u_\varepsilon(x) = C_0\varepsilon^{\frac{2}{2_*-2}}\Big(\varepsilon^{\frac{2\sqrt{\bar{\mu}_a-\mu}}{\sqrt{\bar{\mu}_a-\mu}-b}}|x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})}+|x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})}\Big)^{-\frac{2}{2_*-2}} \tag{1.5}$$

with a suitable positive constant C_0, is a weak solution of

$$-\operatorname{div}\big(|x|^{-2a}\nabla u\big) - \mu|x|^{-2(a+1)}u = |x|^{-2_*b}|u|^{2_*-2}u \quad \text{in } \mathbb{R}^N\backslash\{0\}.$$

Furthermore,

$$\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u_\varepsilon|^2dx - \mu\int_{\mathbb{R}^N}|x|^{-2(a+1)}u_\varepsilon^2dx = \int_{\mathbb{R}^N}|x|^{-2_*b}|u_\varepsilon|^{2_*}dx = A_{a,b,\mu}, \tag{1.6}$$

where $A_{a,b,\mu}$ is the best constant,

$$A_{a,b,\mu} = \inf_{u\in H_\mu\backslash\{0\}} E_{a,b,\mu}(u) = E_{a,b,\mu}(u_\varepsilon), \tag{1.7}$$

with

$$E_{a,b,\mu}(u) := \frac{\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2dx - \mu\int_{\mathbb{R}^N}|x|^{-2(a+1)}u^2dx}{(\int_{\mathbb{R}^N}|x|^{-2_*b}|u|^{2_*}dx)^{2/2_*}}.$$

Also in [13] and [14], they proved that for $0 \leq a < (N-2)/2$, $a \leq b < a+1$, $0 \leq \mu < \bar{\mu}_a$, the function defined for $\varepsilon > 0$ as

$$v_\varepsilon(x) = (2.2_*\varepsilon^2(\bar{\mu}_a-\mu))^{\frac{1}{2_*-2}}\Big(\varepsilon^2|x|^{\frac{(2_*-2)(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})}{2}} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})}\Big)^{-\frac{2}{2_*-2}} \tag{1.8}$$

is a weak solution of

$$-\operatorname{div}(|x|^{-2a}\nabla u) - \mu|x|^{-2(a+1)}u = |x|^{-2_*b}|u|^{2_*-2}u \quad \text{in } \mathbb{R}^N\backslash\{0\},$$

and satisfies

$$\int_{\mathbb{R}^N}|x|^{-2a}|\nabla v_\varepsilon|^2dx - \mu\int_{\mathbb{R}^N}|x|^{-2(a+1)}v_\varepsilon^2dx = \int_{\mathbb{R}^N}|x|^{-2_*b}|v_\varepsilon|^{2_*}dx = B_{a,b,\mu}, \tag{1.9}$$

where $B_{a,b,\mu}$ is the best constant,

$$B_{a,b,\mu} := \inf_{u\in H_\mu\backslash\{0\}} E_{a,b,\mu}(u) = E_{a,b,\mu}(v_\varepsilon). \tag{1.10}$$

A natural question that arises in concert applications is to see what happens if these elliptic (degenerate or non-degenerate) problems are affected by a certain singular perturbations. In our work we prove the existence of at least two distinct nonnegative critical points of energy functional associated to problem (1.1) by splitting the Nehari manifold (see for example Tarantello [17] or Brown and Zhang [5]).

In this work we consider the following assumptions:

(H) h is a continuous function defined in $\bar{\Omega}$ and there exist h_0 and ρ_0 positive such that $h(x) \geq h_0$ for all $x \in B(0, 2\rho_0)$, where $B(a, r)$ is a ball centered at a with radius r;

(K) k is a continuous function defined in $\bar{\Omega}$ and satisfies $k(0) = \max_{x\in\bar{\Omega}} k(x) > 0$, $k(x) = k(0) + o(x^{\beta})$ for $x \in B(0, 2\rho_0)$ with $\beta > 2_*\sqrt{\bar{\mu}_a - \mu}$;

and one of the following two assumptions

(A1) $N > 2(|b| + 1)$ and

$$(a, \mu) \in]-1, 0[\times]0, \bar{\mu}_a - b^2[\cup[0, \tfrac{N-2}{2}[\times]a(a - N + 2), \bar{\mu}_a - b^2[,$$

(A2) $N \geq 3$, $(a, \mu) \in [0, \frac{N-2}{2}[\times[0, \bar{\mu}_a[$.

Following the method introduced in [17, 12], we obtain the following existence result.

Theorem 1.1. *Suppose that $a < (N - 2)/2$, $a \leq b < a + 1$, $1 < q < 2$, $c \leq q(a + 1) + N(1 - q/2)$,* (H), (K) *hold and* (A1) *or* (A2) *are satisfy. Then there exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$ problem* (1.1) *has at least two nonnegative solutions in H_μ.*

This paper is organized as follows. In section 2 we give some preliminaries. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary results

We start by giving the following definitions.

Let E be a Banach space and a functional $I \in \mathcal{C}^1(E, \mathbb{R})$. We say that (u_n) is a Palais Smale sequence at level l ($(PS)_l$ in short) if $I(u_n) \to l$ and $I'(u_n) \to 0$ in E' (dual of E) as $n \to \infty$. We say also that I satisfies the Palais Smale condition at level l if any $(PS)_l$ sequence has a subsequence converging strongly in E.

Define

$$w_\varepsilon := \begin{cases} u_\varepsilon & \text{if } (a, \mu) \in]-1, 0[\times]0, \bar{\mu}_a - b^2[\cup[0, \frac{N-2}{2}[\times]a(a - N + 2), \bar{\mu}_a - b^2[, \\ v_\epsilon & \text{if } (a, \mu) \in [0, \frac{N-2}{2}[\times[0, \bar{\mu}_a[, \end{cases} \tag{2.1}$$

and

$$S_{a,b,\mu} := \begin{cases} A_{a,b,\mu} & \text{if } (a, \mu) \in]-1, 0[\times]0, \bar{\mu}_a - b^2[\cup[0, \frac{N-2}{2}[\times]a(a - N + 2), \bar{\mu}_a - b^2[, \\ B_{a,b,\mu} & \text{if } (a, \mu) \in [0, \frac{N-2}{2}[\times[0, \bar{\mu}_a[,. \end{cases} \tag{2.2}$$

Since our approach is variational, we define the functional $I_{\lambda,\mu}$ as

$$I_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2_{\mu,a} - \frac{\lambda}{q}\int_\Omega h(x)|x|^{-c}|u|^q dx - \frac{1}{2_*}\int_\Omega k(x)|x|^{-2_*b}|u|^{2_*}dx,$$

for $u \in H_\mu$. By (1.2) and (1.4) we can guarantee that $I_{\lambda,\mu}$ is well defined in H_μ and $I_{\lambda,\mu} \in C^1(H_\mu, \mathbb{R})$.

$u \in H_\mu$ is said to be a weak solution of (1.1) if it satisfies

$$\int_\Omega (|x|^{-2a}\nabla u\nabla v - \mu|x|^{-2(a+1)}uv - \lambda h(x)|x|^{-c}|u|^{q-2}uv - k(x)|x|^{-2_*b}|u|^{2_*-2}uv)dx = 0$$

for all $v \in H_\mu$. By the standard elliptic regularity argument, we have that $u \in C^2(\Omega\backslash\{0\})$.

In many problems as (1.1), $I_{\lambda,\mu}$ is not bounded below on H_μ but is bounded below on an appropriate subset of H_μ and a minimizer in this set (if it exists) may give rise to solutions of the corresponding differential equation.

A good candidate for an appropriate subset of H_μ is the so called Nehari manifold
$$\mathcal{N}_\lambda = \{u \in H_\mu \backslash \{0\}, \langle I'_{\lambda,\mu}(u), u\rangle = 0\}.$$
It is useful to understand $\mathcal{N}_\lambda$ in terms of the stationary points of mappings of the form
$$\Psi_u(t) = I_{\lambda,\mu}(tu), \quad t > 0,$$
and so
$$\Psi'_u(t) = \langle I'_{\lambda,\mu}(tu), u\rangle = \frac{1}{t}\langle I'_{\lambda,\mu}(tu), tu\rangle.$$
An immediate consequence is the following proposition.

Proposition 2.1. *Let $u \in H_\mu\backslash\{0\}$ and $t > 0$. Then $tu \in \mathcal{N}_\lambda$ if and only if $\Psi'_u(t) = 0$.*

Let u be a local minimizer of $I_{\lambda,\mu}$, then Ψ_u has a local minimum at $t = 1$. So it is natural to split $\mathcal{N}_\lambda$ into three subsets $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$ corresponding respectively to local minimums, local maximums and points of inflexion.

We define
$$\begin{aligned}\mathcal{N}_\lambda^+ &= \big\{u \in \mathcal{N}_\lambda : (2-q)\|u\|^2_{\mu,a} - (2_* - q)\int_\Omega k(x)\frac{|u|^{2_*}}{|x|^{2_*b}}dx > 0\big\}\\ &= \{u \in \mathcal{N}_\lambda : (2-2_*)\|u\|^2_{\mu,a} + (2_* - q)\lambda\int_\Omega h(x)\frac{|u|^q}{|x|^c}dx > 0\}.\end{aligned}$$
Note that $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^0$ similarly by replacing $>$ by $<$ and $=$ respectively.
$$c_\lambda := \inf_{u\in\mathcal{N}_\lambda} I_{\lambda,\mu}(u);\ c_\lambda^+ := \inf_{u\in\mathcal{N}_\lambda^+} I_{\lambda,\mu}(u); \quad c_\lambda^- := \inf_{u\in\mathcal{N}_\lambda^-} I_{\lambda,\mu}(u). \tag{2.3}$$
The following lemma shows that minimizers on $\mathcal{N}_\lambda$ are critical points for $I_{\lambda,\mu}$.

Lemma 2.2. *Assume that u is a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_\lambda$ and that $u \notin \mathcal{N}_\lambda^0$. Then $I'_{\lambda,\mu}(u) = 0$.*

The proof of the above lemma is essentially the same as that of [5, Theorem 2.3].

Lemma 2.3. *Let*
$$\Lambda_1 := \Big(\frac{2-q}{2_*-q}\Big)^{\frac{2-q}{2_*-q}}\Big(\frac{2_*-2}{(2_*-q)C_1}\Big)|h^+|^{-1}_\infty|k^+|_\infty(S_{a,b,\mu})^{\frac{N(2-q)}{4(a+1-b)}},$$
where $\eta^+(x) = \max(\eta(x), 0)$, and $|\eta^+|_\infty = \sup_{x\in\Omega} ess|\eta^+(x)|$. Then $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \Lambda_1)$.

Proof. Suppose that $\mathcal{N}_\lambda^0 \neq \emptyset$. Then for $u \in \mathcal{N}_\lambda^0$, we have
$$\begin{aligned}\|u\|^2_{\mu,a} &= \frac{2_*-q}{2-q}\int_\Omega k(x)\frac{|u|^{2_*}}{|x|^{2_*b}}dx,\\ \|u\|^2_{\mu,a} &= \lambda\frac{2_*-q}{2_*-2}\int_\Omega h(x)\frac{|u|^q}{|x|^c}dx.\end{aligned}$$
Moreover by (H), (K), Caffarelli-Kohn-Nirenberg and Hölder inequalities, we obtain
$$\begin{aligned}\|u\|^2_{\mu,a} &\geq \Big(\frac{2-q}{(2_*-2)|k^+|_\infty}(S_{a,b,\mu})^{2_*/2}\Big)^{2/(2_*-2)},\\ \|u\|^2_{\mu,a} &\leq \Big(\lambda\frac{2_*-q}{2_*-2}(S_{a,b,\mu})^{-q/2}C_1|h^+|_\infty\Big)^{2/(2-q)}.\end{aligned}$$
Thus $\lambda \geq \Lambda_1$. From this, we can conclude that $\mathcal{N}_\lambda^0 = \emptyset$ if $\lambda \in (0, \Lambda_1)$. $\square$

Thus we conclude that $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all $\lambda \in (0, \Lambda_1)$.

Lemma 2.4. *Let c_λ^+, c_λ^- defined in* (2.1). *Then there exists $\delta_0 > 0$ such that*

$$c_\lambda^+ < 0 \ \forall \lambda \in (0, \Lambda_1) \quad \text{and} \quad c_\lambda^- > \delta_0 \ \forall \lambda \in (0, \frac{q}{2}\Lambda_1).$$

Proof. Let $u \in \mathcal{N}_\lambda^+$. Then

$$\int_\Omega k(x) \frac{|u|^{2_*}}{|x|^{2_* b}} dx < \frac{2-q}{2_* - q} \|u\|_{\mu,a}^2,$$

which implies

$$\begin{aligned} c_\lambda^+ &\le I_{\lambda,\mu}(u) \\ &= (\frac{1}{2} - \frac{1}{q}) \|u\|_{\mu,a}^2 + (\frac{1}{q} - \frac{1}{2_*}) \int_\Omega k(x) \frac{|u|^{2_*}}{|x|^{2_* b}} dx \\ &< -\frac{(2-q)(2_* - 2)}{2.2_* q} \|u\|_{\mu,a}^2 < 0. \end{aligned}$$

Let $u \in \mathcal{N}_\lambda^-$. Then

$$\frac{2-q}{2_* - q} \|u\|_{\mu,a}^2 < \int_\Omega k(x) \frac{|u|^{2_*}}{|x|^{2_* b}} dx.$$

Moreover by (H), (K) and Caffarelli-Kohn-Nirenberg inequality, we have

$$\int_\Omega k(x) \frac{|u|^{2_*}}{|x|^{2_* b}} dx \le (S_{a,b,\mu})^{-2_*/2} \|u\|_{\mu,a}^{2_*} |k^+|_\infty.$$

This implies

$$\|u\|_{\mu,a} > \big(\frac{2-q}{(2_* - 2)|k^+|_\infty} \big)^{1/(2_* - 2)} (S_{a,b,\mu})^{2_*/(2(2_* - 2))}.$$

On the other hand,

$$I_{\lambda,\mu}(u) \ge \|u\|_{\mu,a}^q \Big((\frac{1}{2} - \frac{1}{2_*}) \|u\|_{\mu,a}^{2-q} - \lambda \frac{2_* - q}{2_* q} (S_{a,b,\mu})^{-q/2} C_1 |h^+|_\infty \Big)$$

Thus, if $\lambda \in (0, \frac{q}{2}\Lambda_1)$ we get $I_{\lambda,\mu}(u) \ge \delta_0$, where

$$\begin{aligned} \delta_0 := \Big(\tfrac{2-q}{(2_* - 2)|k^+|_\infty} \Big)^{\frac{q}{2_* - 2}} (S_{a,b,\mu})^{\frac{2_* q}{2(2_* - 2)}} \Big((\frac{1}{2} - \frac{1}{2_*}) (S_{a,b,\mu})^{\frac{2_*(2-q)}{2(2_* - 2)}} \big(\tfrac{2-q}{(2_* - q)|k^+|_\infty} \big)^{\frac{2-q}{2_* - 2}} \\ - \lambda \tfrac{2_* - q}{2_* - 2} (S_{a,b,\mu})^{-q/2} C_1 |h^+|_\infty \Big). \end{aligned}$$

□

As in [18, Proposition 9], we have the following result.

Lemma 2.5. (i) *If $\lambda \in (0, \Lambda_1)$, then there exists a $(PS)_{c_\lambda}$ sequence $(u_n) \subset \mathcal{N}_\lambda$ for $I_{\lambda,\mu}$.*
(ii) *If $\lambda \in (0, \frac{q}{2}\Lambda_1)$, then there exists a $(PS)_{c_\lambda^-}$ sequence $(u_n) \subset \mathcal{N}_\lambda^-$ for $I_{\lambda,\mu}$.*

We define

$$K^+ := \big\{ u \in \mathcal{N}_\lambda : \int_\Omega k(x) \frac{|u|^{2_*}}{|x|^{2_* b}} dx > 0 \big\}, \quad K_0^- := \big\{ u \in \mathcal{N}_\lambda : \int_\Omega k(x) \frac{|u|^{2_*}}{|x|^{2_* b}} dx \le 0 \big\},$$

$$H^+ := \{ u \in \mathcal{N}_\lambda : \int_\Omega h(x) \frac{|u|^q}{|x|^c} dx > 0 \}, \quad H_0^- := \{ u \in \mathcal{N}_\lambda : \int_\Omega h(x) \frac{|u|^q}{|x|^c} dx \le 0 \},$$

and

$$t_{\max} = t_{\max}(u) := \Big(\frac{2-q}{2_*-2}\Big)^{1/(2_*-2)} \|u\|_{\mu,a}^{2/(2_*-2)} \Big(\int_\Omega k(x)\frac{|u|^{2_*}}{|x|^{2_*b}}dx\Big)^{-1/(2_*-2)},$$

for $u \in K^+$. Then we have the following result.

Proposition 2.6. *For $\lambda \in (0, \Lambda_1)$ we have*

(1) *If $u \in K^+ \cap H_0^-$ then there exists unique $t^+ > t_{\max}$ such that $t^+u \in \mathcal{N}_\lambda^-$ and*
$$I_{\lambda,\mu}(t^+u) \geq I_{\lambda,\mu}(tu) \quad \text{for } t \geq t_{\max};$$
(2) *If $u \in K^+ \cap H^+$, then there exist unique t^-, t^+ such that $0 < t^- < t_{\max} < t^+$, $t^-u \in \mathcal{N}_\lambda^+$, $t^+u \in \mathcal{N}_\lambda^-$ and*
$$I_{\lambda,\mu}(t^+u) \geq I_{\lambda,\mu}(tu) \text{ for } t \geq t^- \text{ and } I_{\lambda,\mu}(t^-u) \leq I_{\lambda,\mu}(tu) \text{ for } t \in [0, t^+].$$
(3) *If $u \in K^- \cap H^-$, then does not exist $t > 0$ such that $tu \in \mathcal{N}_\lambda$.*
(4) *If $u \in K_0^- \cap H^+$, then there exists unique $0 < t^- < +\infty$ such that $t^-u \in \mathcal{N}_\lambda^+$ and*
$$I_{\lambda,\mu}(t^-u) = \inf_{t\geq 0} I_{\lambda,\mu}(tu).$$

Proof. For $u \in H_\mu$, we have

$$\Psi_u(t) = I_{\lambda,\mu}(tu) = \frac{t^2}{2}\|u\|_{\mu,a}^2 - \lambda\frac{t^q}{q}\int_\Omega h(x)\frac{|u|^q}{|x|^c}dx - \frac{t^{2_*}}{2_*}\int_\Omega k(x)\frac{|u|^{2_*}}{|x|^{2_*b}}dx$$

and

$$\Psi_u'(t) = t^{q-1}\Big(\varphi_u(t) - \lambda\int_\Omega h(x)\frac{|u|^q}{|x|^c}\Big),$$

where

$$\varphi_u(t) = t^{2-q}\|u\|_{\mu,a}^2 - t^{2_*-q}\int_\Omega k(x)\frac{|u|^{2_*}}{|x|^{2_*b}}.$$

Easy computations show that φ_u is concave and achieves its maximum at

$$t_{\max} := \Big(\frac{2-q}{2_*-2}\Big)^{1/(2_*-2)} \|u\|_{\mu,a}^{2/(2_*-2)} \Big(\int_\Omega k(x)\frac{|u|^{2_*}}{|x|^{2_*b}}dx\Big)^{-1/(2_*-2)}$$

for $u \in K^+$; that is,

$$\Psi(t_{\max}) = C_{a,b,q,N}\|u\|_{\mu,a}^{(2_*-q)/(2_*-2)} \Big(\int_\Omega k(x)\frac{|u|^{2_*}}{|x|^{2_*b}}dx\Big)^{(q-2)/(2_*-2)},$$

where

$$C_{a,b,q,N} = \frac{2_*+q-4}{2_*-2}\Big(\frac{2-q}{2_*-2}\Big)^{(2-q)/(2_*-2)}.$$

Then we can get the conclusion of our proposition easily. □

3. Proof of Theorem 1.1

Existence of a local minimum for $I_{\lambda,\mu}$ on $\mathcal{N}_\lambda^+$. We want to prove that $I_{\lambda,\mu}$ can achieve a local minimizer on $\mathcal{N}_\lambda^+$.

Proposition 3.1. *Let $\lambda \in (0, \Lambda_1)$, then $I_{\lambda,\mu}$ has a minimizer u_λ in $\mathcal{N}_\lambda^+$ such that*

$$I_{\lambda,\mu}(u_\lambda) = c_\lambda^+ < 0.$$

Proof. By Lemma 2.5, there exists a minimizing sequence $(u_n) \subset \mathcal{N}_\lambda$ such that

$$I_{\lambda,\mu}(u_n) \to c_\lambda \text{ and } I'_{\lambda,\mu}(u_n) \to 0 \text{ in } H_\mu^{-1} \text{ (dual of } H_\mu).$$

Since

$$I_{\lambda,\mu}(u_n) = (\frac{1}{2} - \frac{1}{2_*})\|u_n\|^2_{\mu,a} - \lambda(\frac{1}{q} - \frac{1}{2_*}) \int_\Omega h(x)\frac{|u_n|^q}{|x|^c},$$

by Caffarelli-Kohn-Nirenberg inequality, we have

$$c_\lambda + \circ_n(1) \geq (\frac{1}{2} - \frac{1}{2_*})\|u_n\|^2_{\mu,a} - \lambda\frac{2_* - q}{2_* q}(S_{a,b,\mu})^{-q/2}C_1|h^+|_\infty \|u_n\|^q_{\mu,a},$$

where $\circ_n(1)$ denotes that $\circ_n(1) \to 0$ *as* $n \to \infty$.Thus (u_n) is bounded in H_μ, then passing to a subsequence if necessary, we have the following convergence:

$$\begin{gathered} u_n \rightharpoonup u_\lambda \quad \text{in } H_\mu, \\ u_n \rightharpoonup u_\lambda \quad \text{in } L_{2_*}(\Omega, |x|^{-2_* b}), \\ u_n \to u_\lambda \quad \text{in } L_q(\Omega, |x|^{-c}), \\ u_n \to u_\lambda \quad \text{a.e. in } \Omega. \end{gathered}$$

Thus $u_\lambda \in \mathcal{N}_\lambda$ is a weak solution of (1.1). As $c_\lambda < 0$ and $I_{\lambda,\mu}(0) = 0$, then $u_\lambda \not\equiv 0$. Now we show that $u_n \to u_\lambda$ in H_μ. Suppose otherwise, then $\|u_\lambda\|_\mu < \liminf_{n\to\infty} \|u_n\|_\mu$, and we obtain

$$\begin{aligned} c_\lambda &\leq I_{\lambda,\mu}(u_\lambda) \\ &= (\frac{1}{2} - \frac{1}{2_*})\|u_\lambda\|^2_{\mu,a} - \lambda\frac{2_* - q}{2_* q}\int_\Omega h(x)\frac{|u_\lambda|^q}{|x|^c} \\ &< \liminf_{n\to\to\infty}\Big((\frac{1}{2} - \frac{1}{2_*})\|u_n\|^2_{\mu,a} - \lambda\frac{2_* - q}{2_* q}\int_\Omega h(x)\frac{|u_n|^q}{|x|^c}\Big) \\ &= c_\lambda. \end{aligned}$$

We obtain a contradiction. Consequently $u_n \to u_\lambda$ strongly in H_μ. Moreover, we have $u_\lambda \in \mathcal{N}^+_\lambda$. If not $u_\lambda \in \mathcal{N}^-_\lambda$, thus $\Psi'_u(1) = 0$ and $\Psi''_u(1) < 0$, which implies that $I_{\lambda,\mu}(u_\lambda) > 0$, contradiction. □

Existence of a local minimum for $I_{\lambda,\mu}$ on $\mathcal{N}^-_\lambda$. To prove the existence of a second nonnegative solution we need the following results.

Lemma 3.2. *Let (u_n) is a $(PS)_l$ sequence with $u_n \rightharpoonup u$ in H_μ. Then there exists positive constant $\tilde{C} := C(a, b, N, q, |h^+|_\infty, S_{a,b,\mu})$ such that*

$$I'_{\lambda,\mu}(u) = 0 \quad \textit{and} \quad I_{\lambda,\mu}(u) \geq -\tilde{C}\lambda^{2/(2-q)}.$$

Proof. It is easy to prove that $I'_{\lambda,\mu}(u) = 0$, which implies that $\langle I'_{\lambda,\mu}(u), u\rangle = 0$, and

$$I_{\lambda,\mu}(u) - \frac{1}{2_*}\langle I'_{\lambda,\mu}(u), u\rangle = (\frac{1}{2} - \frac{1}{2_*})\|u\|^2_{\mu,a} - \lambda(\frac{1}{q} - \frac{1}{2_*})\int_\Omega h(x)\frac{|u|^q}{|x|^c}dx.$$

By Caffarelli-Kohn-Nirenberg, Hölder and Young inequalities we find that

$$I_{\lambda,\mu}(u) \geq (\frac{1}{2} - \frac{1}{2_*})\|u\|^2_{\mu,a} - \lambda\frac{2_* - q}{2_* q}(S_{a,b,\mu})^{-q/2}C_1|h^+|_\infty\|u\|^q_{\mu,a}.$$

There exists $\tilde{C}$ such that

$$(\frac{1}{2} - \frac{1}{2_*})t^2 - \lambda\frac{2_* - q}{2_* q}(S_{a,b,\mu})^{-q/2}C_1|h^+|_\infty t^q \geq -\tilde{C}\ \lambda^{2/(2-q)} \quad \text{for all } t \geq 0.$$

Then we conclude that $I_{\lambda,\mu}(u) \geq -\tilde{C}\ \lambda^{2/(2-q)}$. □

Lemma 3.3. *Let (u_n) in H_μ be such that*

$$I_{\lambda,\mu}(u_n) \to l < l^* := (\frac{1}{2} - \frac{1}{2_*})|k^+|_\infty (S_{a,b,\mu})^{2_*/(2_*-2)} - \tilde{C}\lambda^{2/(2-q)}, \tag{3.1}$$

$$I'_{\lambda,\mu}(u_n) \to 0 \text{ in } H_\mu^{-1}. \tag{3.2}$$

Then there exists a subsequence strongly convergent.

Proof. From (3.1) and (3.2) we deduce that (u_n) is bounded. Thus up a subsequence, we have the following convergence:

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } H_\mu, \\ u_n \rightharpoonup u \quad \text{in } L_{2_*}(\Omega, |x|^{-2_* b}), \\ u_n \to u \quad \text{in } L_q(\Omega, |x|^{-c}), \\ u_n \to u \quad \text{a.e. in } \Omega. \end{gathered}$$

Then u is a weak solution of problem (1.1).

Denote $v_n = u_n - u$. As k is continuous on Ω, then the Brézis - Lieb [4] leads to

$$\int_\Omega k(x)\frac{|u_n|^{2_*}}{|x|^{2_* b}}dx = \int_\Omega k(x)\frac{|v_n|^{2_*}}{|x|^{2_* b}}dx + \int_{\mathbb{R}^N} k(x)\frac{|u|^{2_*}}{|x|^{2_* b}}dx, \tag{3.3}$$

and

$$\|u_n\|^2_{\mu,a} = \|v_n\|^2_{\mu,a} + \|u\|^2_{\mu,a} + \circ_n(1). \tag{3.4}$$

Using the Lebesgue theorem, it follows that

$$\lim_{n\to\infty}\int_\Omega h(x)\frac{|u_n|^q}{|x|^c}dx = \int_\Omega h(x)\frac{|u|^q}{|x|^c}dx. \tag{3.5}$$

From (3.3), (3.4) and (3.5), we deduce that

$$I_{\lambda,\mu}(u_n) = I_{\lambda,\mu}(u) + \frac{1}{2}\|v_n\|^2_{\mu,a} - \frac{1}{2_*}\int_\Omega k(x)\frac{|v_n|^{2_*}}{|x|^{2_* b}}dx + \circ_n(1),$$

and

$$\langle I'_{\lambda,\mu}(u_n), u_n\rangle = \langle I'_{\lambda,\mu}(u), u\rangle + \|v_n\|^2_{\mu,a} - \int_\Omega k(x)\frac{|v_n|^{2_*}}{|x|^{2_* b}}dx + \circ_n(1),$$

using the fact that $v_n \rightharpoonup 0$ in H_μ, we can assume that

$$\|v_n\|^2_{\mu,a} \to \theta \quad \text{and} \quad \int_\Omega k(x)\frac{|v_n|^{2_*}}{|x|^{2_* b}}dx \to \theta \geq 0.$$

By the definition of $S_{a,b,\mu}$ we have

$$\|v_n\|^2_{\mu,a} \geq S_{a,b,\mu}\Big(\int_\Omega \frac{|v_n|^{2_*}}{|x|^{2_* b}}dx\Big)^{2/2_*},$$

and so $\theta \geq |k^+|_\infty S_{a,b,\mu}\theta^{2/2_*}$.

Assume $\theta \neq 0$, then $\theta \geq |k^+|_\infty (S_{a,b,\mu})^{2_*/(2_*-2)}$, and we get by Lemma 3.3 that

$$\begin{aligned} l &= I_{\lambda,\mu}(u) + (\frac{1}{2} - \frac{1}{2_*})\theta \\ &\geq -\tilde{C}\lambda^{2/(2-q)} + (\frac{1}{2} - \frac{1}{2_*})|k^+|_\infty (S_{a,b,\mu})^{2_*/(2_*-2)} = l^* \end{aligned}$$

which is a contradiction. So $l = 0$; i.e., $u_n \to u$ in H_μ. □

In the following, we shall give some estimates for the extremal functions defined in (2.1). Let $\Psi(x) \in C_0^\infty(\Omega)$ such that $0 \le \Psi(x) \le 1$, $\Psi(x) = 1$ for $|x| \le \rho_0$, $\Psi(x) = 0$ for $|x| \ge 2\rho_0$, where ρ_0 is a small positive number. Set

$$\tilde{u}(x) = \Big(|x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})}\Big)^{-\frac{2}{2_*-2}}$$

$$\tilde{v}_\varepsilon(x) = \begin{cases} \Psi(x)\Big(\varepsilon^{\frac{2\sqrt{\bar{\mu}_a-\mu}}{\sqrt{\bar{\mu}_a-\mu}-b}}|x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})}\Big)^{-\frac{2}{2_*-2}} \\ \quad \text{if (A1) holds,} \\ \Psi(x)\Big(\varepsilon^2|x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})}\Big)^{-\frac{2}{2_*-2}} \\ \quad \text{if (A2) holds.} \end{cases}$$

By a straightforward computation, one finds

$$\int_\Omega k(x)\frac{|\tilde{v}_\varepsilon|^{2_*}}{|x|^{2_*b}}dx = \varepsilon^{-\frac{N-2(a+1-b)}{2(a+1-b)}}|k^+|_\infty \int_\Omega \frac{|\tilde{u}|^{2_*}}{|x|^{2_*b}}dx + O(\varepsilon),$$

where $O(\varepsilon^\zeta)$ denotes $|O(\varepsilon^\zeta)|/\epsilon^\zeta \le C$,

$$\|\tilde{v}_\varepsilon\|^2_{\mu,a} = \varepsilon^{-\frac{N-2(a+1-b)}{2(a+1-b)}}\|\tilde{u}\|^2_{\mu,a} + O(1),$$
$$\frac{\|\tilde{v}_\varepsilon\|^2_{\mu,a}}{\int_\Omega k(x)\frac{|\tilde{v}_\varepsilon|^{2_*}}{|x|^{2_*b}}dx} = O(\varepsilon^{\frac{N-2(a+1-b)}{2(a+1-b)}}).$$

Lemma 3.4. *Let l^* be defined in Lemma 3.3, then there exists $\Lambda_4 > 0$ such that for all $\lambda \in (0, \Lambda_4)$ we have $l^* > 0$ and $\sup_{t\ge 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon) < l^*$.*

Proof. We consider the following two functions

$$f(t) = I_{\lambda,\mu}(t\tilde{v}_\varepsilon) = \frac{t^2}{2}\|\tilde{v}_\varepsilon\|^2_{\mu,a} - \frac{t^{2_*}}{2_*}\int_{\mathbb{R}^N} k(x)\frac{|\tilde{v}_\varepsilon|^{2_*}}{|x|^{2_*b}}dx - \lambda\frac{t^q}{q}\int_{\mathbb{R}^N} h(x)\frac{|\tilde{v}_\varepsilon|^q}{|x|^c}dx,$$

and

$$\tilde{f}(t) = \frac{t^2}{2}\|\tilde{v}_\varepsilon\|^2_{\mu,a} - \frac{t^{2_*}}{2_*}|k^+|_\infty\int_{\mathbb{R}^N}\frac{|\tilde{v}_\varepsilon|^{2_*}}{|x|^{2_*b}}dx.$$

Let $\Lambda_2 > 0$ be such that

$$(\frac{1}{2} - \frac{1}{2_*})|k^+|_\infty(S_{a,b,\mu})^{2_*/(2_*-2)} - \tilde{C}\lambda^{2/(2-q)} > 0 \quad \text{for all } \lambda \in (0, \Lambda_2).$$

Then

$$f(0) = 0 < (\frac{1}{2} - \frac{1}{2_*})|k^+|_\infty(S_{a,b,\mu})^{2_*/(2_*-2)} - \tilde{C}\lambda^{2/(2-q)} \quad \text{for all } \lambda \in (0, \Lambda_2).$$

By the continuity of $f(t)$, there exists $t_1 > 0$ small enough such that

$$f(t) < (\frac{1}{2} - \frac{1}{2_*})|k^+|_\infty(S_{a,b,\mu})^{2_*/(2_*-2)} - \tilde{C}\lambda^{2/(2-q)} \quad \text{for all } t \in (0, t_1).$$

On the other hand,

$$\max_{t\ge 0}\tilde{f}(t) = (\frac{1}{2} - \frac{1}{2_*})|k^+|_\infty(S_{a,b,\mu})^{2_*/(2_*-2)} + O(\varepsilon^{\frac{N-2(a+1-b)}{2(a+1-b)}}).$$

Then

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon) < (\frac{1}{2}-\frac{1}{2_*})|k^+|_\infty (S_{a,b,\mu})^{2_*/(2_*-2)} + O(\varepsilon^{\frac{N-2(a+1-b)}{2(a+1-b)}}) \\ - \lambda \frac{t_1^q}{q} h_0 \int_{B(0,\rho_0)} \frac{|\tilde{v}_\varepsilon|^q}{|x|^c} dx.$$

Let $0<\varepsilon<\rho_0^{(2_*-2)\sqrt{\bar{\mu}_a-\mu}}$ then

$$\begin{aligned}&\int_{B(0,\rho_0)} \frac{|\tilde{v}_\varepsilon|^q}{|x|^c} dx \\ &= \int_{B(0,\rho_0)} |x|^{-c} (\varepsilon^{\frac{2\sqrt{\bar{\mu}_a-\mu}}{\sqrt{\bar{\mu}_a-\mu}-b}} |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}-\sqrt{\bar{\mu}_a-\mu})} + |x|^{\frac{2_*-2}{2}(\sqrt{\bar{\mu}_a}+\sqrt{\bar{\mu}_a-\mu})})^{-\frac{2q}{2_*-2}} dx \\ &\geq C_2.\end{aligned}$$

Now, taking $\varepsilon=\lambda^{\frac{2(2_*-2)}{2_*-q}}$ we get $\lambda<\rho_0^{(2-q)\sqrt{\bar{\mu}_a-\mu}}$ and

$$\sup_{t\geq 0} I_{\lambda,\mu}(t\tilde{v}_\varepsilon) < (\frac{1}{2}-\frac{1}{2_*})|k^+|_\infty (S_{a,b,\mu})^{2_*/(2_*-2)} + O(\lambda^{2/(2-q)}) - \lambda\frac{t_1^q}{q} h_0 C_2.$$

Choosing $\Lambda_3>0$ such that

$$O(\lambda^{2/(2-q)}) - \lambda \frac{t_1^q}{q} h_0 C_2 < -\tilde{C}\lambda^{2/(2-q)} \quad \text{for all } \lambda\in(0,\Lambda_3).$$

Then if we take $\Lambda_4=\min\{\Lambda_2,\Lambda_3,\rho_0^{(2-q)\sqrt{\bar{\mu}_a-\mu}}\}$ we deduce that

$$\sup_{t\geq 0} J_\lambda(t\tilde{v}_\varepsilon) < l^* \quad \text{for all } \lambda\in(0,\Lambda_4).$$

□

Now, we prove that $I_{\lambda,\mu}$ can achieve a local minimizer on $\mathcal{N}_\lambda^-$.

Proposition 3.5. *Let $\Lambda^*=\min\{q\Lambda_1/2,\Lambda_4\}$. Then for all $\lambda\in(0,\Lambda^*)$, $I_{\lambda,\mu}$ has a minimizer v_λ in $\mathcal{N}_\lambda^-$ such that $I_{\lambda,\mu}(v_\lambda)=c_\lambda^-$.*

Proof. By Lemma 2.5, there exists a minimizing sequence $(u_n)\subset\mathcal{N}_\lambda^-$ for all $\lambda\in(0,q\Lambda_1/2)$ such that $I_{\lambda,\mu}(u_n)\to c_\lambda^-$ and $I'_{\lambda,\mu}(u_n)\to 0$ in H_μ^{-1}. Since $I_{\lambda,\mu}$ is coercive on $\mathcal{N}_\lambda^-$ thus (u_n) bounded. Then, passing to a subsequence if necessary, we have the following convergence:

$$\begin{gathered} u_n \rightharpoonup v_\lambda \quad \text{in } H_\mu, \\ u_n \rightharpoonup v_\lambda \quad \text{in } L_{2_*}(\Omega,|x|^{-2_*b}), \\ u_n \to v_\lambda \quad \text{in } L_q(\Omega,|x|^{-c}), \\ u_n \to v_\lambda \quad \text{a.e. in } \Omega. \end{gathered}$$

By Lemma 3.4, $c_\lambda^-<l^*$, thus from Lemma 3.3 we deduce that $u_n\to v_\lambda$ in H_μ. Then we conclude that $I_{\lambda,\mu}(v_\lambda)=c_\lambda^->0$. Similarly as the proof of Proposition 3.1, we conclude that $I_{\lambda,\mu}$ has a minimizer v_λ in $\mathcal{N}_\lambda^-$ for all $\lambda\in(0,\Lambda^*)$ such that $I_{\lambda,\mu}(v_\lambda)=c_\lambda^->0$. □

Proof of Theorem 1.1. By Propositions 2.6 and 3.5, there exists $\Lambda^*>0$ such that (1.1) has two nonnegative solutions $u_\lambda\in\mathcal{N}_\lambda^+$ and $v_\lambda\in\mathcal{N}_\lambda^-$ since $\mathcal{N}_\lambda^+\cap\mathcal{N}_\lambda^-=\emptyset$. □

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