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# EXISTENCE OF MULTIPLE SOLUTIONS FOR A $p(x)$-LAPLACE EQUATION

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Abstract. This article shows the existence of at least three nontrivial solutions to the quasilinear elliptic equation

$$-\Delta_{p(x)}u + |u|^{p(x)-2}u = f(x, u)$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, with the nonlinear boundary condition $|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = g(x, u)$ or the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. In addition, this paper proves that one solution is positive, one is negative, and the last one is a sign-changing solution. The method used here is based on Nehari results, on three sub-manifolds of the space $W^{1,p(x)}(\Omega)$.

## 1. Introduction

The study of variational problems with nonstandard growth conditions is an interesting topic. $p(x)$-growth conditions can be regarded as an important case of nonstandard $(p, q)$-growth conditions. Many results have been obtained on this kind of problems; see for example [1, 2, 8, 9, 12]. We refer the reader to the overview papers [10, 15] for advances and references in this area.

In this article, we consider the non-homogeneous nonlinear Neumann boundary-value problem

$$\begin{gathered}-\Delta_{p(x)}u + |u|^{p(x)-2}u = f(x, u), \quad x \in \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = g(x, u), \quad x \in \partial\Omega;\end{gathered} \tag{1.1}$$

and the problem

$$\begin{gathered}-\Delta_{p(x)}u = f(x, u), \quad x \in \Omega,\\ u = 0, \quad x \in \partial\Omega,\end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative, $p(x) \in C(\overline{\Omega})$, $\inf_{x\in\Omega} p(x) > 1$.

The operator $-\Delta_{p(x)}u := -div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, which becomes $p$-Laplacian when $p(x) \equiv p$ (a constant). It possesses more complicated nonlinearities than the $p$-Laplacian. For related results involving the Laplace operator, see [1, 10, 15].

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In this article, we construct three sub-manifolds of the space $W^{1,p(x)}(\Omega)$ based on Naheri ideas. And under some assumptions, there exist three different, nontrivial solutions of (1.1) or eqrefePprime on sub-manifold respectively. Moreover these solutions are, one positive, one negative and the other one has non-constant sign. This result extends the conclusions of [14], and in [3], the author also do some research with the similar method.

Throughout this paper, by (weak) solutions of (1.1) or (1.2) we understand critical points of the associated energy functional $\Phi$ or $\Psi$ acting on the Sobolev space $W^{1,p(x)}(\Omega)$:

$$\Phi(v) = \int_\Omega \frac{1}{p(x)}(|\nabla v|^{p(x)} + |v|^{p(x)})\,dx - \int_\Omega F(x,v)\,dx - \int_{\partial\Omega} G(x,v)dS;$$
$$\Psi(v) = \int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx - \int_\Omega F(x,v)\,dx,$$

where $F(x,u) = \int_0^u f(x,z)dz$, $G(x,u) = \int_0^u g(x,z)dz$ and $dS$ is the surface measure. We also denote $\mathcal{F}(v) = \int_\Omega F(x,v)\,dx$ and $\mathcal{G}(v) = \int_{\partial\Omega} G(x,v)dS$.

For a reflexive manifold $M$, $T_uM$ denotes the tangential space at a point $u \in M$, $T_u^*M$ denotes the co-tangential space.

In this paper span$\{v_1,\dots,v_k\}$ denotes the vector space generated by the vectors $v_1,\dots,v_k$.

## 2. The space $W^{1,p(x)}(\Omega)$

To discuss problem (1.1) or (1.2), we need some results on the space $W^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly, we state some basic properties of the space $W^{1,p(x)}(\Omega)$ which will be used later (for more details, see [5, 6, 7]).

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, denote:

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}); h(x) > 1\, \forall x \in \overline{\Omega}\},$$
$$h^+ = \max_{x\in\overline{\Omega}} h(x), \quad h^- = \min{}_{x\in\overline{\Omega}} h(x), \quad h \in C(\overline{\Omega}),$$
$$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_\Omega |u|^{p(x)}\,dx < \infty\}.$$

We introduce a norm on $L^{p(x)}(\Omega)$:

$$|u|_{p(x)} = \inf\big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)}\,dx \le 1\big\}.$$

Then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, we call it a generalized Lebesgue space.

**Proposition 2.1** ([8, 9]).

(1) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniformly convex Banach space, and has conjugate space $L^{q(x)}(\Omega)$, where $1/q(x) + 1/p(x) = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*
$$\big|\int_\Omega uv\,dx\big| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)};$$

(2) *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$, for any $x \in \Omega$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and the imbedding is continuous.*

**Proposition 2.2** ([8, 18]). *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies*

$$|f(x,s)| \le a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \quad \text{for any } x \in \Omega, s \in \mathbb{R},$$

*where $p_1(x)$, $p_2(x) \in C_+(\overline{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $a(x) \ge 0$, and $b \ge 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_f(u))(x) = f(x,u(x))$ is a continuous and bounded operator.*

**Proposition 2.3** ([8, 19]). *Denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(x)}(\Omega).$$

*Then*

(1) $|u(x)|_{p(x)} < 1 (= 1; > 1)$ *if and only if* $\rho(u) < 1 (= 1; > 1)$;
(2) $|u(x)|_{p(x)} > 1$ *implies* $|u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$; $|u(x)|_{p(x)} < 1$ *implies* $|u|^{p^-}_{p(x)} \ge \rho(u) \ge |u|^{p^+}_{p(x)}$;
(3) $|u(x)|_{p(x)} \to 0$ *if and only if* $\rho(u) \to 0$; $|u(x)|_{p(x)} \to \infty$ *if and only if* $\rho(u) \to \infty$.

**Proposition 2.4** ([8, 19]). *If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \dots$, then the following statements are equivalent:*

(1) $\lim_{k\to\infty} |u_k - u|_{p(x)} = 0$;
(2) $\lim_{k\to\infty} \rho(u_k - u) = 0$;
(3) $u_k \to u$ *in measure in* $\Omega$ *and* $\lim_{k\to\infty} \rho(u_k) = \rho(u)$.

Let us define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $p^*(x) = np(x)/(n-p(x))$, $p^*_\partial(x) = (n-1)p(x)/(n-p(x))$, when $p(x) < n$.

**Proposition 2.5** ([4, 8]). (1) *$W_{(0)}^{1,p(x)}(\Omega)$ is separable reflexive Banach space;*
(2) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous;*
(3) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*_\partial(x)$ for any $x \in \overline{\Omega}$, then the trace imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ is compact and continuous;*
(4) *(Poincaré) There is a constant $C > 0$, such that*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

## 3. Assumptions and statement of main result

The assumptions on the source terms $f$ and $g$ are as follows:

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with respect to the first argument and continuous differentiable with respect to the second argument for a.e. $x \in \Omega$. Moreover, $f(x,0) = 0$ for every $x \in \overline{\Omega}$.

(F2) There exist $q(x), s(x), t(x)$, such that $p(x) \le p^+ < q^- \le q(x) < p^*(x)$, $s(x) > p^*(x)/(p^*(x) - q(x))$,

$$t(x) = s(x)q(x)/(2 + (q(x) - 2)s(x)) > p^*(x)/(p^*(x) - 2)$$

and there exist functions $a(x) \in L^{s(x)}(\Omega)$, $b(x) \in L^{t(x)}(\Omega)$, such that for $x \in \Omega$, $u \in \mathbb{R}$,

$$|f_u(x,u)| \le a(x)|u|^{q(x)-2} + b(x).$$

(F3) There exist constants $c_1 \in (0, 1/(p^+ - 1))$, $c_2 > p^+$, $0 < c_3 < c_4$, such that for any $u \in L^{q(x)}(\Omega)$,

$$\begin{aligned} c_3 \int_\Omega |u|^{q(x)}\,dx \le c_2 \int_\Omega F(x,u)\,dx \le \int_\Omega f(x,u)u\,dx \\ \le c_1 \int_\Omega f_u(x,u)u^2\,dx \le c_4 \int_\Omega |u|^{q(x)}\,dx. \end{aligned}$$

(G1) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with respect to the first argument and continuous differentiable with respect to the second argument for a.e. $x \in \partial\Omega$. Moreover, $g(x, 0) = 0$ for every $x \in \partial\Omega$.

(G2) There exist functions $r(x), \sigma(x), \tau(x)$, such that $p(x) \le p^+ < r^- \le r(x) < p^*_\partial(x)$, $\sigma(x) > p^*_\partial(x)/(p^*_\partial(x) - r(x))$, $\tau(x) = \sigma(x)r(x)/(2+(r(x)-2)\sigma(x)) > p^*_\partial(x)/(p^*_\partial(x) - 2)$ and there exist functions $\alpha(x) \in L^{\sigma(x)}(\partial\Omega)$, $\beta(x) \in L^{\tau(x)}(\partial\Omega)$, such that for $x \in \partial\Omega$, $u \in \mathbb{R}$,

$$|g_u(x,u)| \le \alpha(x)|u|^{r(x)-2} + \beta(x).$$

(G3) There exist constants $k_1 \in (0, 1/(p^+ - 1))$, $k_2 > p^+$, $0 < k_3 < k_4$, such that for any $u \in L^{r(x)}(\partial\Omega)$ satisfies

$$\begin{aligned} k_3 \int_\Omega |u|^{r(x)}\,dx \le k_2 \int_{\partial\Omega} G(x,u)dS \le \int_{\partial\Omega} g(x,u)u dS \\ \le k_1 \int_{\partial\Omega} g_u(x,u)u^2 dS \le k_4 \int_\Omega |u|^{r(x)}\,dx. \end{aligned}$$

We remark that by assumptions (F1), (F2) there exists $\pi(x) < p^*(x)$ such that the operators $u \to f(u)$ is bounded and Lipschitz continuous from $L^{\pi(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^{-1}$. Moreover, by the compactness of the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{\pi(x)}(\Omega)$ the operator is compact if viewed as acting on $W^{1,p(x)}(\Omega)$. Hence the expression

$$\psi(u, \cdot) = \langle \nabla\Phi(u), \cdot \rangle$$

is well defined and Lipschitz continuously differentiable in $W^{1,p(x)}(\Omega)$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $(W^{1,p(x)}(\Omega))^{-1}$ and $W^{1,p(x)}(\Omega)$.

The main result of the paper reads as follows:

**Theorem 3.1.** *Under assumptions* (F1)-(F3), (G1)-(G3)*, there exist three different, non trivial solutions of* (1.1) *and* (1.2)*. Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

## 4. The proof for the Neumann problem

The proof uses the similar approach as in [14]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that $\Phi$ has a critical point in $K_i$. These sets will be subsets of smooth manifolds $N_i \in W^{1,p(x)}(\Omega)$ which we will obtain by imposing a sign condition and a normalization condition on admissible functions. Let

$$\begin{gathered}
N_1 = \{u \in W^{1,p(x)}(\Omega) : \int_\Omega u_+ \, dx > 0, \langle \nabla\Phi(u), u^+\rangle = 0\}, \\
N_2 = \{u \in W^{1,p(x)}(\Omega) : \int_\Omega u_- \, dx > 0, \langle \nabla\Phi(u), u^-\rangle = 0\}, \\
N_3 = N_1 \cap N_2,
\end{gathered}$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of $u$.

In fact, following Nehari [13], conditions in $N_i$, $(i = 1, 2, 3)$, are *norming conditions* for the positive and negative parts of $u$.

Also we define

$$K_1 = \{u \in N_1 | u \geq 0\}, \quad K_2 = \{u \in N_2 | u \leq 0\}, K_3 = N_3.$$

To prove that the sets $N_i, K_i$ possess the properties stated at the beginning of the paragraph we establish the following estimate:

**Lemma 4.1.** *There exist constants $c_j > 0, j = 1, 2$, such that for any $u \in K_i$,*

$$\begin{aligned}
c_1 \min\{\|u\|^{p^-}, \|u\|^{p^+}\} &\leq \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx \\
&\leq \int_\Omega f(x, u) u \, dx + \int_{\partial\Omega} g(x, u) u dS \\
&\leq c_2 \Phi(u).
\end{aligned}$$

*Proof.* Since $u \in K_i$, we have

$$\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)}) \, dx = \int_\Omega f(x, u) u \, dx + \int_{\partial\Omega} g(x, u) u dS.$$

This proves the second inequality. Now, by (F3) and (G3)

$$\begin{gathered}
\int_\Omega F(x, u) \, dx \leq \frac{1}{c_2} \int_\Omega f(x, u) u \, dx, \\
\int_{\partial\Omega} G(x, u) dS \leq \frac{1}{k_2} \int_{\partial\Omega} g(x, u) u dS.
\end{gathered}$$

So for $c = \max\{1/c_2, 1/k_2\} < 1/p^+$,

$$\Phi(u) \geq (\frac{1}{p^+} - c) \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} \, dx.$$

This proves the third inequality. The first inequality is easily obtained by proposition 2.3. $\square$

**Lemma 4.2.** *There exists $c > 0$ such that*
$$\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \ge c \quad \text{for } u \in K_1,$$
$$\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx \ge c \quad \text{for } u \in K_2,$$
$$\int_\Omega (|\nabla u_\pm|^{p(x)} + |u_\pm|^{p(x)})\,dx \ge c \quad \text{for } u \in K_3.$$

*Proof.* By the definitions of $K_i$, conditions (F3), (G3) and proposition 2.3,
$$\begin{aligned}
&\int_\Omega (|\nabla u_\pm|^{p(x)} + |u_\pm|^{p(x)})\,dx \\
&= \int_\Omega f(x,u_\pm)u_\pm\,dx + \int_{\partial\Omega} g(x,u_\pm)u_\pm dS \\
&\le c(\int_\Omega |u_\pm|^{q(x)}\,dx + \int_\Omega |u_\pm|^{r(x)}\,dx) \\
&\le c(\|u_\pm\|^{q^\pm} + \|u_\pm\|^{r^\pm}) \\
&\le c(|\nabla u_\pm|_{p(x)}^{q^\pm} + |u_\pm|_{p(x)}^{q^\pm} + |\nabla u_\pm|_{p(x)}^{r^\pm} + |u_\pm|_{p(x)}^{r^\pm}) \\
&\le c[(\int_\Omega |\nabla u_\pm|^{p(x)} + |u_\pm|^{p(x)}\,dx)^{\frac{q^\pm}{p^\pm}} + (\int_\Omega |\nabla u_\pm|^{p(x)} + |u_\pm|^{p(x)}\,dx)^{\frac{r^\pm}{p^\pm}}].
\end{aligned}$$
Here by $U^{p^\pm}$ we mean $\max\{U^{p^+}, U^{p^-}\}$. Then obviously the conclusion holds since $p^+ < q^-, r^-$. $\square$

**Lemma 4.3.** *There exists $c > 0$ such that $\Phi(u) \ge c\int_\Omega(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx$ for every $u \in W^{1,p(x)}(\Omega)$* s.t. *$\|u\| \le c$.*

*Proof.* By (F3), (G3) and the Sobolev inequalities in proposition 2.5 we have
$$\begin{aligned}
\Phi(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx - \mathcal{F}(u) - \mathcal{G}(u) \\
&\ge \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx - c(\int_\Omega |u|^{q(x)}\,dx + \int_\Omega |u|^{r(x)}\,dx) \\
&\ge \frac{1}{p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx - c(\int_\Omega |u|^{q(x)}\,dx + \int_\Omega |u|^{r(x)}\,dx) \\
&\ge c\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx,
\end{aligned}$$
if $\|u\|$ is sufficiently small, since $p^+ < q^-, r^-$ ,for all $x \in \Omega$, and the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$. $\square$

**Lemma 4.4.** *Put $P^+(u) = u_+$, $P^-(u) = u_-$, then the mapping $P^+$ and $P^-$: $W^{1,p(x)}(\Omega) \to W^{1,p(x)}(\Omega)$ is continuous.*

*Proof.* Here we prove only that the projection $P^+ : W^{1,p(x)}(\Omega) \to W^{1,p(x)}(\Omega)$ is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$, we prove $(u_n)_+ \to u_+$ in $W^{1,p(x)}(\Omega)$. It is obvious the following inequality holds:
$$|(u_n)_+(x) - u_+(x)| \le |u_n(x) - u(x)|, \quad \text{for a.e. } x \in \Omega,$$

which implies

$$|(u_n)_+(x) - u_+(x)|^{p(x)} \le |u_n(x) - u(x)|^{p(x)}, \quad \text{for a.e. } x \in \Omega.$$

So $\int_\Omega |u_n(x) - u(x)|^{p(x)}\,dx \to 0$ implies $\int_\Omega |(u_n)_+(x) - u_+(x)|^{p(x)}\,dx \to 0$. Then $(u_n)_+ \to u_+$ in $L^{p(x)}(\Omega)$.

Next we prove $|\nabla(u_n)_+ - \nabla u_+|_{p(x)} \to 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \begin{cases} 1, & t > 0,\\ 0, & t \le 0, \end{cases}$$

then

$$\begin{aligned} &|\nabla(u_n)_+(x) - \nabla u_+(x)|\\ &= |g(u_n(x))\nabla u_n(x) - g(u(x))\nabla u(x)|\\ &\le |g(u_n(x))(\nabla u_n(x) - \nabla u(x))| + |(g(u_n(x)) - g(u(x)))\nabla u(x)|\\ &\le |\nabla u_n(x) - \nabla u(x)| + |(g(u_n(x)) - g(u(x)))\nabla u(x)|, \end{aligned}$$

so

$$|\nabla(u_n)_+(x) - \nabla u_+(x)|_{p(x)} \le |\nabla u_n - \nabla u|_{p(x)} + |(g(u_n) - g(u))\nabla u|_{p(x)}.$$

We already know that $|\nabla u_n - \nabla u|_{p(x)} \to 0$. And by $|(g(u_n) - g(u))\nabla u| \to 0$ and the Lebesgue Dominated Convergence Theorem, we conclude that $\int_\Omega |(g(u_n) - g(u))\nabla u|^{p(x)}\,dx \to 0$. That is $|(g(u_n) - g(u))\nabla u|_{p(x)} \to 0$. So $(|\nabla u_n)_+ - \nabla u_+|_{p(x)} \to 0$, which ends the proof. $\square$

**Remark.** By the above Lemma, for any $c > 0$, the set $\{u \in W^{1,p(x)}(\Omega) : \|u_\pm\| < c\}$ is open in $W^{1,p(x)}(\Omega)$, and the set $\{u \in W^{1,p(x)}(\Omega) : \|u_\pm\| \le c\}$ is closed in $W^{1,p(x)}(\Omega)$.

The regularity properties of the sets $N_i$ are stated in the following Lemma.

**Lemma 4.5.**
(1) *$N_i$ is a $C^{1,1}$ sub-manifold of $W^{1,p(x)}(\Omega)$ of co-dimension $1 (i = 1, 2), 2 (i = 3)$ respectively;*
(2) *The sets $K_i$ are complete;*
(3) *For any $u \in N_i$, we have the direct decomposition*

$$\begin{gathered} T_u W^{1,p(x)}(\Omega) = T_u N_1 \oplus \operatorname{span}\{u_+\},\\ T_u W^{1,p(x)}(\Omega) = T_u N_2 \oplus \operatorname{span}\{u_-\},\\ T_u W^{1,p(x)}(\Omega) = T_u N_3 \oplus \operatorname{span}\{u_+, u_-\}, \end{gathered}$$

*where $T_u N_i$ is the tangent space at $u$ of the Banach manifold $N_i$. Finally, the projection onto the first component in this decomposition is uniformly continuous on bounded sets of $N_i$.*

*Proof.* (1) Denote

$$\begin{aligned} N_1' &= \{u \in W^{1,p(x)}(\Omega) | \int_\Omega u_+\,dx > 0\},\\ N_2' &= \{u \in W^{1,p(x)}(\Omega) | \int_\Omega u_-\,dx > 0\},\\ N_3' &= N_1' \cap N_1'. \end{aligned}$$

By the continuous embedding of $W^{1,p(x)}(\Omega) \to L^1(\Omega)$, the set $N_i'$ is open in $W^{1,p(x)}(\Omega)$. Therefore it suffices to show that $N_i$ is a smooth sub-manifold of $N_i'$. By the implicit function theorem, part (1) of the lemma will be followed from a representation of $N_i$ as the inverse image of a regular value of a $C^{1,1}$-function $\psi_i : N_i' \to \mathbb{R}^m$ with $m = 1 (i = 1, 2), m = 2 (i = 3)$ respectively. In fact we define: for $u \in N_1'$

$$\psi_1(u) = \int_\Omega (|\nabla u_+|^{p(x)} + |u_+|^{p(x)})\,dx - \langle \mathcal{F}'(u), u_+\rangle - \langle \mathcal{G}'(u), u_+\rangle;$$

for $u \in N_2'$

$$\psi_2(u) = \int_\Omega (|\nabla u_-|^{p(x)} + |u_-|^{p(x)})\,dx - \langle \mathcal{F}'(u), u_-\rangle - \langle \mathcal{G}'(u), u_-\rangle;$$

For $u \in N_3'$

$$\psi_3(u) = (\psi_1(u), \psi_2(u)).$$

Then $N_i = \psi_i^{-1}(0)$ and that 0 is a regular value of $\psi_i$ may be seen from the estimates:

$$\begin{aligned}
\langle \nabla\psi_1(u), u_+\rangle &= \int_\Omega p(x)(|u_+|^{p(x)} + |\nabla u_+|^{p(x)})\,dx - \int_\Omega f_u(x,u)u_+^2 \\
&\quad + f(x,u)u_+\,dx - \int_{\partial\Omega} g_u(x,u)u_+^2 + g(x,u)u_+ dS \\
&\le (p^+ - 1)\int_\Omega f(x,u)u_+\,dx - \int_\Omega f_u(x,u)u_+^2\,dx \\
&\quad + (p^+ - 1)\int_{\partial\Omega} g(x,u)u_+ dS - \int_{\partial\Omega} g_u(x,u)u_+^2 dS.
\end{aligned}$$

By (F3), (G3) the last term is bounded by

$$(p^+ - 1 - c_1^{-1})\int_\Omega f(x,u)u_+\,dx + (p^+ - 1 - k_1^{-1})\int_{\partial\Omega} g(x,u)u_+ dS.$$

Recall that $c_1, k_1 < 1/(p^+ - 1)$, and by Lemma 4.1 the above formula is bounded by

$$-c\int_\Omega (|\nabla u_+|^{p(x)} + |u_+|^{p(x)})\,dx,$$

which is strictly negative by Lemma 4.2. Therefore, $N_1$ is a smooth sub-manifold of $W^{1,p(x)}$. The exact same argument applies to $N_2$. Since, trivially,

$$\langle \nabla\psi_1(u), u_-\rangle = \langle \nabla\psi_2(u), u_+\rangle = 0$$

for $u \in N_3$, the same argument holds for $N_3$.

(2) Let $\{u_j\}$ be a Cauchy sequence in $K_i$. Then $u_j \to u \in W^{1,p(x)}(\Omega)$ and also $u_{j\pm} \to u_\pm \in W^{1,p(x)}(\Omega)$. By Lemma 4.2, $u \in \{u \in W^{1,p(x)}(\Omega) | u \ge 0, u \ne 0\}$. Since continuity of $\psi_i$ that $u \in K_i$.

(3) By (1), we have the direct decomposition $T_u W^{1,p(x)}(\Omega) = T_u N_1 \oplus \operatorname{span}\{u_+\}$, $T_u W^{1,p(x)}(\Omega) = T_u N_2 \oplus \operatorname{span}\{u_-\}$, $T_u W^{1,p(x)}(\Omega) = T_u N_3 \oplus \operatorname{span}\{u_+, u_-\}$. Let $v \in T_u W^{1,p(x)}(\Omega)$ be a unit tangential vector. Then $v = v' + v''$ where $v', v''$ are given by

$$v'' = (\langle \nabla\psi_1(u)\big|_{\operatorname{span}\{u_+\}}, \cdot\rangle)^{-1}\langle \nabla\psi_1(u), v\rangle \in \operatorname{span}\{u_+\}, \quad v' = v - v'' \in T_u N_1.$$

Obviously the mapping $\nabla\psi_1$ is uniformly bounded on bounded subsets of $K_1$ and the uniform boundedness of $(\langle \nabla\psi_1(u)\big|_{\mathrm{span}\{u_+\}}, \cdot\rangle)^{-1}$ on such sets is a consequence of the estimate proved in part (1) of this proof. So we have the conclusion of the lemma. The similar results hold for $i = 2, 3$. ☐

**Lemma 4.6.** *The function $\Phi\big|_{N_i}$ satisfies the Palais-Smale condition.*

*Proof.* Let $\{u_k\} \in N_1$ be a Palais-Smale sequence, that is $\Phi(u_k)$ is uniformly bounded and $\nabla\Phi\big|_{N_1}(u_k) \to 0$ strongly. We need to show that there exists a subsequence $u_{k_i}$, that converges strongly.
In fact, the assumptions imply that $\nabla\Phi(u_k) \to 0$. To see this let $v_j \in T_{u_j} W^{1,p(x)}(\Omega)$ be a unit tangential vector such that

$$\langle \nabla\Phi(u_j), v_j\rangle = \|\nabla\Phi(u_j)\|_{(W^{1,p(x)}(\Omega))^{-1}}.$$

By Lemma 4.5 (3), $v_j = v_j' + v_j'' \in T_{u_j}N_1 + \mathrm{span}\{(u_j)_+\}$, since $\Phi((u_j)_+) \le \Phi(u_j) \le c$ and by Lemma 4.1, then the sequence $\{(u_j)_+\}$ is uniformly bounded. Hence $\|v_j\| \ge \|v_j'\| - \|v_j''\|$ implies $v_j'$ is uniformly bounded in $W^{1,p(x)}(\Omega)$. $(\langle \nabla\Phi\big|_{\mathrm{span}\{(u_j)_+\}}(u_j), (u_j)_+\rangle = \psi_1(u_j) = 0.)$ Hence

$$\|\nabla\Phi(u_j)\|_{(W^{1,p(x)}(\Omega))^{-1}} = \langle \nabla\Phi(u_j), v_j\rangle = \langle \nabla\Phi\big|_{N_1}(u_j), v_j'\rangle \to 0.$$

As $u_j$ is bounded in $W^{1,p(x)}(\Omega)$, there exists $u \in W^{1,p(x)}(\Omega)$ such that $u_j \rightharpoonup u$, weakly in $W^{1,p(x)}(\Omega)$. By condition (F2) and (G2), it is well known that the unrestricted functional $\Phi$ satisfies the Palais-Smale condition, the lemma then follows. Similaly when $i = 2, 3$ the lemma also holds. ☐

From the proof of lemma 4.6, we immediately obtain the following result.

**Lemma 4.7** (Nehari result)**.** *Let $u \in N_i$ be a critical point of the restricted functional $\Phi\big|_{N_i}$. Then $u$ is also a critical point of the unrestricted functional $\Phi$.*

**Lemma 4.8.** *There exists a critical point of the energy functional $\Phi$ in $K_i$.*

*Proof.* From Lemma 4.3 we know that there exists a sufficient small constant $\tau > 0$, such that

$$\Phi(u) \ge \tau \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}\,dx \ge \tau\|u\|^{p_+}, \tag{4.1}$$

when $\|u\| \le \tau$. Let

$$U = \{u \in N_1 : \|u^-\| < \tau\},$$

then $U$ is an open set in $N_1$ which contains $K_1$, and $\overline{U}$ is complete. As $\Phi$ is bounded from below, we denote $c = \inf_{u\in\overline{U}} \Phi(u)$. Let $\{u_n\}$ be the sequence minimizing $\Phi$ to $c$; i.e.m $u_n \in \overline{U}$, such that $\Phi(u_n) \to c$ as $n \to \infty$. From Lemma 4.3 we know that $\Phi((u_n)_+) \le \Phi(u_n)$, so $\{(u_n)_+\}$ is also the minimizing sequence tending $\Phi$ to $c$. (In fact, $(u_n)_+ \in K_1$.) Now we have $\Phi((u_n)_+) < c + \varepsilon_n$, in which $\tau^{p_+ + 1} > \varepsilon_n$ for $\forall n \in N^+$. Put $\delta_n = \sqrt{\varepsilon_n}$, from Ekeland's variational principle we know there exists a sequence $\{v_n\} \subset \overline{U}$ such that the following holds,

$$\begin{gathered} \|(u_n)_+ - v_n\| \le \delta_n, \quad \Phi(v_n) \le \Phi((u_n)_+) < c + \varepsilon_n, \\ \Phi(v_n) < \Phi(w) + \frac{\varepsilon_n}{\delta_n}\|v_n - w\|, \quad \forall w \in \overline{U},\ w \ne v_n. \end{gathered} \tag{4.2}$$

We assert that

$$v_n \in U, \quad \text{i.e., } \|(v_n)_-\| < \tau. \tag{4.3}$$

In fact, if the above assertion doesn't hold, i.e. $\|(v_n)_-\| = \tau$, then from (4.1) we have

$$\Phi((v_n)_-) \geq \tau\|(v_n)_-\|^{p_+} = \tau^{p_+ +1} > \varepsilon_n.$$

Observe that $(v_n)_+ \in K_1 \subset U$, we have

$$\Phi(v_n) = \Phi((v_n)_+) + \Phi((v_n)_-) \geq c + \varepsilon_n,$$

which contradicts $\Phi(v_n) < c + \varepsilon_n$. So assertion (4.3) holds.

Since $v_n \in U$ and (4.2) imply $\nabla\Phi\big|_{N_1}(v_n) \to 0$ as $n \to \infty$. By Lemma 4.6, $\{v_n\}$ contains a convergence subsequence, which we also denote as $\{v_n\}$, and $v_n \to v_0 \in W^{1,p(x)}(\Omega)$ as $n \to \infty$. From $\|(u_n)_+ - v_n\| \to 0$, we have $(u_n)_+ \to v_0$ in $W^{1,p(x)}(\Omega)$. And from the completeness of $K_1$ we get $v_0 \in K_1$. At the same time $\nabla\Phi\big|_{N_1}(v_0) = 0$. Then from Lemma 4.7, $\nabla\Phi(v_0) = 0$. Similarly, we can prove the lemma when $i = 2, 3$. $\square$

Finally since the sets $K_i$ are disjoint and $0 \notin K_i$ the proof of Theorem 3.1 is complete.

**Remark** For the Dirichlet problem (1.2), let

$$\begin{gathered}
N_1 = \{u \in W_0^{1,p(x)}(\Omega) : \int_\Omega u_+ \,dx > 0, \langle\nabla\Psi(u), u^+\rangle = 0\}, \\
N_2 = \{u \in W_0^{1,p(x)}(\Omega) : \int_\Omega u_- \,dx > 0, \langle\nabla\Psi(u), u^-\rangle = 0\}, \\
N_3 = N_1 \cap N_2, \quad K_1 = \{u \geq 0 : u \in N_1\}, \\
K_2 = \{u \leq 0 : u \in N_2\}, \quad K_3 = N_3,
\end{gathered}$$

with a similar approach, we can prove that (1.2) has three nontrivial solutions.

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