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POSITIVE PERIODIC SOLUTIONS FOR NONAUTONOMOUS IMPULSIVE NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEMS WITH TIME-VARYING DELAYS ON TIME SCALES

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Abstract. Using a fixed point theorem of strict-set-contraction, we prove the existence of positive periodic solutions for a class of nonautonomous impulsive neutral functional differential system with time-varying delays on time scales.

1. Introduction

Yang [13] studied the dynamic behavior of bounded solutions for the neutral functional differential equation

$$(x(t) - x(t-r))' = -F(x(t)) + G(x(t-r)), \tag{1.1}$$

and obtained the ω limit set for bounded solutions. Variations of (1.1) have received considerable attention in the literature, because they are used as models for phenomena such as population growth, spread of epidemics, dynamics of capital stocks, etc. For more details and references on this subject, we refer the reader to [6, 10, 12].

As is well known, impulsive differential equations arise naturally in the description of physical and biological phenomena that are subjected to instantaneous changes at some time instants called moments. For a review on this theory, which has seen a significant development over the past decades, we refer the interested reader to the monographs [7, 8].

It is well known that continuous and discrete systems are very important in applications; also that Stefan Hilger introduce time scales to unify the continuous and discrete analysis. Therefore, it is meaningful to study dynamic systems on time scales which can unify differential and difference systems.

To the best of the authors' knowledge, the existence of positive periodic solutions for the following system has not been studied on time scales.

$$\begin{gathered}(x(t) + \lambda c(t)x(t-\tau(t)))^{\Delta} = -\lambda f(t, x(t), (x(t-\tau(t))), \quad t \neq t_k,\ t \in \mathbb{T}, \\ x(t_k^+) = x(t_k^-) - \lambda I_k(x(t_k)),\end{gathered} \tag{1.2}$$

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where $\lambda > 0$, $\mathbb{T}$ is an ω-periodic time scale, $c(t) \in C^1(\mathbb{T}, \mathbb{R}_0)$, $\tau(t) \in C(\mathbb{T}, \mathbb{R}_0)$ are all ω-periodic functions, $I_k(u) \in C(\mathbb{R}, \mathbb{R}_0)$, $f(t, x, y) \in C(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}_0)$ satisfies $f(t + \omega, x, y) = f(t, x, y)$ for all $(t, x, y) \in (\mathbb{T} \times \mathbb{R} \times \mathbb{R})$, $\omega > 0$ is a constant, $\mathbb{R}_0 = [0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$. For each interval I of $\mathbb{R}$, we denote $I_{\mathbb{T}} = I \cap \mathbb{T}$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and the left limit of $x(t_k)$ in the sense of time scales, in addition, if t_k is right-scattered, then $x(t_k^+) = x(t_k)$, whereas, if t_k is left-scattered, then $x(t_k^-) = x(t_k)$, $k \in \mathbb{Z}$. There exists a positive integer p such that $t_{k+p} = t_k + \omega$, $I_{k+p} = I_k$, $k \in \mathbb{Z}$. Without loss of generality, we also assume that $[0, \omega)_{\mathbb{T}} \cap \{t_k : k \in \mathbb{Z}\} = \{t_1, t_2, \dots, t_p\}$.

The main purpose of this article is to establish criteria to guarantee the existence of positive periodic solutions of system (1.2) by using a fixed point theorem of strict-set-contraction. This article is organized as follows: In Section 2, we make some preparations. Section 3, by using a fixed point theorem of strict-set-contraction, we prove the existence of positive periodic solutions of system (1.2). In Section 4, we give an example to illustrate the main results.

2. Preliminaries

In this section, we shall first recall some basic definitions, and lemmas which are used in later.

Definition 2.1 ([2]). A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$ is the graininess function. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0\}$ for all $t \in \mathbb{T}$. Let $p \in \mathcal{R}$. The exponential function is defined by

$$e_p(t, s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big),$$

where $\xi_{h(z)}$ is the so-called cylinder transformation.

Lemma 2.2 ([2]). *Let $p, q \in \mathcal{R}$. Then*

(a) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(b) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(c) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(d) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(e) $e_p^{\Delta}(\cdot, s) = pe_p(\cdot, s)$.

Definition 2.3 ([2]). For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative of f at t, denoted by $f^{\Delta}(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - t]| \leq \epsilon|\sigma(t) - s|, \quad \forall s \in U.$$

Definition 2.4 ([2]). If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Definition 2.5 ([5]). We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np, f(t+T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t+T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t+T) = f(t)$ for all $t \in \mathbb{T}$.

To obtain the existence of a periodic solution of (1.2), we first make the following preparations:

Let E be a Banach space and K be a cone in E. The semi-order induced by the cone K is defined by $x \le y$ if $y - x \in K$. In addition, for a bounded subset $A \subset E$, let $\alpha_E(A)$ denote the (Kuratowski) measure of non-compactness defined by

$$\alpha_E(A) = \inf\Big\{\delta > 0 : \text{there is a finite number of subsets } A_i \subset A \text{ such that } A = \cup_i A_i \text{ and } \operatorname{diam}(A_i) \le \delta\Big\},$$

where $\operatorname{diam}(A_i)$ denotes the diameter of the set A_i.

Let E, F be two Banach spaces and $D \subset E$. A continuous and bounded map $\Phi : \bar{\Omega} \to F$ is called k-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_F(\Phi(S)) \le k\alpha_E(S).$$

A map Φ is called strict-set-contractive if it is k-set-contractive for some $0 \le k < 1$.

The following lemma, cited from [3, 4], is used in the proof of our main results.

Lemma 2.6 ([3, 4]). *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K | r \le \|x\| \le R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \to K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

(i) *$\Phi x \not\le x$ for all $x \in K$, $\|x\| = r$ and $\Phi x \not\ge x$, for all $x \in K$, $\|x\| = R$.*
(ii) *$\Phi x \not\ge x$, for all $x \in K$, $\|x\| = r$ and $\Phi x \not\le x$, for all $x \in K$, $\|x\| = R$.*

Then Φ has at least one fixed point in $K_{r,R}$.

For convenience, we introduce the notation:

$$f^M = \max_{t\in[0,\omega]_{\mathbb{T}}} |f(t)|, \quad f^l = \min_{t\in[0,\omega]_{\mathbb{T}}} |f(t)|, \quad f^L = \max_{t\in[0,\omega]_{\mathbb{T}}} |f^\Delta(t)|.$$

We will use the following assumptions:

(H1) $f(t,0,y) = 0$ and there exist positive constants l_f and L_f such that
$$|f(t,x,y) - f(t,u,v)| \le L_f(|x-u| + |y-v|)$$
for all $(t,x,y), (t,u,v) \in (\mathbb{T}, \mathbb{R}, \mathbb{R})$.

(H2) There exists an ω-periodic function $a(t) \in C(\mathbb{T}, \mathbb{R}^-)$ such that
$$1 + a(t)\mu(t) \ne 0, \quad -\zeta\lambda^{-1}a(t) - |c^\Delta(t)| - \zeta c^\sigma(t) \ge 0 \quad \text{for all } t \in [0,\omega]_{\mathbb{T}},$$
where $\zeta = \frac{\gamma_2\eta_2}{\gamma_1\eta_1}$, $\gamma_1 = \max_{t\in[0,\omega]_{\mathbb{T}}} \big(e_{\ominus a}(t, t-\omega) - 1\big)^{-1}$, $\gamma_2 = \min_{t\in[0,\omega]_{\mathbb{T}}} \big(e_{\ominus a}(t, t-\omega) - 1\big)^{-1}$, $\eta_1 = \max_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a}(t,u)$, $\eta_2 = \min_{u\in[t-\omega,t]_{\mathbb{T}}} e_{\ominus a}(t,u)$.

(H3) $(\ominus a)^M \le 1$.

(H4)
$$\begin{aligned}&\max_{t\in[0,\omega]_{\mathbb{T}}} \big\{ -\lambda^{-1}a(t) + |c^\Delta(t)| + c^\sigma(t) + L_f \big\} \\ &\le \frac{\gamma_2\eta_2}{\gamma_1\kappa}\big(1 + (\ominus a)^l\big) \times \int_0^\omega \big(-\zeta\lambda^{-1}a(s) - |c^\Delta(s)| - \zeta c^\sigma(s)\big)\Delta s,\end{aligned}$$

where $\kappa = \max_{t\in[0,\omega]_{\mathbb{T}}} \big\{e_{\ominus a}(\sigma(t)+\omega,t) - e_{\ominus a}(\sigma(t),t)\big\}$.

(H5) There exist positive constants I_k^l, I_k^M such that
$$I_k^l x^2 \le I_k(x) \le I_k^M x^2 \quad \text{for all } x\in\mathbb{R},\ k\in\mathbb{Z}.$$

(H6) $\gamma_1\kappa c^M < 1$.

(H7) $\gamma_1\eta_1 a^M\omega < 1$,
$$\lambda < \frac{1-\gamma_1\eta_1 a^M\omega}{\gamma_1\eta_1(c^L\omega + \zeta^{-1}c^M\omega + L_f + I^M)},$$
where $I^M = \sum_{k=1}^p I_k^M$.

To apply Lemma 2.6 to system (1.2), we define
$$\begin{gathered} PC(\mathbb{T}) = \big\{x:\mathbb{T}\to\mathbb{R}|x|_{(t_k,t_{k+1})_{\mathbb{T}}} \in C((t_k,t_{k+1})_{\mathbb{T}},\mathbb{R}), \\ \exists x(t_k^-) = x(t_k), x(t_k^+), k\in\mathbb{Z}\big\}, \\ PC^1(\mathbb{T}) = \big\{x:\mathbb{T}\to\mathbb{R}|x|_{(t_k,t_{k+1})_{\mathbb{T}}} \in C((t_k,t_{k+1})_{\mathbb{T}},\mathbb{R}), \\ \exists x^\Delta(t_k^-) = x^\Delta(t_k), x(t_k^+), k\in\mathbb{Z}\big\}. \end{gathered}$$

Set
$$\mathbb{X} = \{x(t)\in PC(\mathbb{T}) : x(t+\omega) = x(t)\}$$
with the norm $|x|_0 = \max_{t\in[0,\omega]_{\mathbb{T}}} |x(t)|$, and
$$\mathbb{Y} = \{x(t)\in PC^1(\mathbb{T}) : x(t+\omega) = x(t)\}$$
with the norm $|x|_1 = \max_{t\in[0,\omega]_{\mathbb{T}}}\{|x|_0, |x^\Delta|_0\}$. Then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. Defined the cone K in $\mathbb{Y}$ by
$$K = \{x\in\mathbb{Y} : x(t) \ge \zeta|x|_1,\ t\in[0,\omega]_{\mathbb{T}}\}.$$

Lemma 2.7. *A function $x\in\mathbb{Y}$ is a solution of (1.2) if and only if*
$$x(t) = \int_{t-\omega}^t \lambda G(t,s)F(s)\Delta s + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda G(t,t_k)I_k(x(t_k)),$$
where
$$G(t,s) = \frac{e_{\ominus a}(t,s)}{e_{\ominus a}(t,t-\omega)-1},$$
$$\begin{aligned} F(s) = -\lambda^{-1}a(s)x^\sigma(s) + c^\Delta(s)x(s-\tau(s) \\ + c^\sigma(s)x^\Delta(s-\tau(s)) + f(s,x(s),x(s-\tau(s))), \quad s\in[t-\omega,t]_{\mathbb{T}}, \end{aligned}$$
and $a(t)$ satisfies (H2).

Proof. Rewrite the first equation of (1.2) in the form
$$x^\Delta(t) = -\lambda c^\Delta(t)x(t-\tau(t) - \lambda c^\sigma(t)x^\Delta(t-\tau(t)) - \lambda f(t,x(t),x(t-\tau(t))),$$
then
$$\begin{aligned} x^\Delta(t) + a(t)x^\sigma(t) &= \lambda^{-1}a(t)x^\sigma(t) - \lambda c^\Delta(t)x(t-\tau(t)) - \lambda c^\sigma(t)x^\Delta(t-\tau(t)) \\ &\quad - \lambda f(t,x(t),x(t-\tau(t))) \\ &:= -\lambda F(t). \end{aligned} \tag{2.1}$$

Let $x\in\mathbb{Y}$ be a solution of system (1.2). Multiply both sides of (2.1) by $e_a(t,0)$ to get
$$\big(x(t)e_a(t,0)\big)^\Delta = -\lambda e_a(t,0)F(t). \tag{2.2}$$

For any $t \in \mathbb{T}$, there exists $k \in \mathbb{Z}$ such that t_k is the first impulsive point after $t-\omega$. For $s \in [t-\omega, t_k]_{\mathbb{T}}$, we integrate (2.2) from $t-\omega$ to s to obtain

$$x(s)e_a(s,0) = x(t-\omega)e_a(t-\omega,0) - \int_{t-\omega}^{s} \lambda e_a(r,0)F(r)\Delta r.$$

Then

$$x(t_k)e_a(t_k,0) = x(t-\omega)e_a(t-\omega,0) - \int_{t-\omega}^{t_k} \lambda e_a(r,0)F(r)\Delta r.$$

For $s \in (t_k, t_{k+1}]_{\mathbb{T}}$, we integrate (2.2) from t_k to s once more to obtain

$$\begin{aligned} x(s)e_a(s,0) &= x(t_k^+)e_a(t_k,0) - \int_{t_k}^{s} \lambda e_a(r,0)F(r)\Delta r \\ &= x(t_k)e_a(t_k,0) - \int_{t_k}^{s} \lambda e_a(r,0)F(r)\Delta r - \lambda e_a(t_k,0)I_k(x(t_k)) \\ &= x(t-\omega)e_a(t-\omega,0) - \int_{t-\omega}^{s} \lambda e_a(r,0)F(r)\Delta r - \lambda e_a(t_k,0)I_k(x(t_k)). \end{aligned}$$

Repeating the above process for $s \in [t-\omega, t]_{\mathbb{T}}$, we have

$$\begin{aligned} x(s)e_a(s,0) = x(t-\omega)e_a(t-\omega,0) &- \int_{t-\omega}^{s} \lambda e_a(r,0)F(r)\Delta r \\ &- \sum_{k:t_k\in[t-\omega,s]_{\mathbb{T}}} \lambda e_a(t_k,0)I_k(x(t_k)). \end{aligned}$$

Let $s=t$ in the above equality, we get

$$\begin{aligned} x(t)e_a(t,0) = x(t-\omega)e_a(t-\omega,0) &- \int_{t-\omega}^{t} \lambda e_a(s,0)F(s)\Delta s \\ &- \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda e_a(t_k,0)I_k(x(t_k)). \end{aligned}$$

Then

$$\begin{aligned} x(t) &= \int_{t-\omega}^{t} \frac{e_{\ominus a}(t,s)}{e_{\ominus a}(\omega,0)-1}\lambda F(s)\Delta s + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \frac{e_{\ominus a}(t,t_k)}{e_{\ominus a}(\omega,0)-1}\lambda I_k(x(t_k)) \\ &= \int_{t-\omega}^{t} \lambda G(t,s)F(s)\Delta s + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda G(t,t_k)I_k(x(t_k)), \end{aligned}$$

where we have used Lemma 2.2 to simplify the exponentials. The proof is complete. $\square$

Let the mapping Φ be defined by

$$(\Phi x)(t) = \int_{t-\omega}^{t} \lambda G(t,s)F(s)\Delta s + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda G(t,t_k)I_k(x(t_k)), \tag{2.3}$$

where $x \in K$, $t \in \mathbb{T}$, $F(s)$, $G(t,s)$ is given by (2.1) and

$$0 < \gamma_2\eta_2 \le G(t,s) \le \gamma_1\eta_1, \quad s \in [t-\omega,t]_{\mathbb{T}}.$$

Lemma 2.8. *Assume that* (H1)–(H4) *hold. Then* $\Phi : K \to K$ *is well defined.*

Proof. For any $x \in K$, we have $\Phi x \in PC^1(\mathbb{T})$. In view of (2.3), for $t \in \mathbb{T}$, we have

$$\begin{aligned}(\Phi x)(t+\omega) &= \frac{1}{e_{\ominus a}(t,t-\omega)-1}\int_t^{t+\omega} \lambda e_{\ominus a}(t+\omega,s)F(s)\Delta s \\ &\quad + \sum_{k:t_k\in[t,t+\omega]_{\mathbb{T}}} \lambda G(t+\omega,t_k)I_k(x(t_k)).\end{aligned} \tag{2.4}$$

Using the periodicity of a, c, τ, f, and letting $u = s-\omega$, by (2.4) we obtain

$$\begin{aligned}(\Phi x)(t+\omega) &= \frac{1}{e_{\ominus a}(t,t-\omega)-1}\int_{t-\omega}^{t} \lambda e_{\ominus a}(t+\omega,u+\omega)F(u)\Delta u \\ &\quad + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda G(t+\omega,t_k+\omega)I_k(x(t_k+\omega)).\end{aligned} \tag{2.5}$$

At the same time, from the definition of $e_a(t,s)$ and the periodicity of a, we have $e_{\ominus a}(t+\omega,u+\omega) = e_{\ominus a}(t,u)$ and $e_{\ominus a}(t+\omega,u) = e_{\ominus a}(t,u-\omega)$. Thus (2.5) becomes $(\Phi x)(t+\omega) = (\Phi x)(t)$. In view of (H2), for $x \in K$, $t_k \in [0,\omega]_{\mathbb{T}}$, we obtain

$$\begin{aligned}F(t) &= -\lambda^{-1}a(t)x^\sigma(t) + c^\Delta(t)x(t-\tau(t) + c^\sigma(t)x^\Delta(t-\tau(t)) + f(t,x(t),x(t-\tau(t)) \\ &\geq -\zeta\lambda^{-1}a(t)|x|_1 - |c^\Delta(t)||x|_1 - \zeta c^\sigma(t)|x|_1 \\ &= \big(-\zeta\lambda^{-1}a(t) - |c^\Delta(t)| - \zeta c^\sigma(t)\big)|x|_1 \geq 0.\end{aligned} \tag{2.6}$$

For $x \in K$, $t \in [0,\omega]_{\mathbb{T}}$, by (2.6) we obtain

$$\begin{aligned}|\Phi x|_0 &= \max_{t\in[0,\omega]_{\mathbb{T}}}\Big\{\int_{t-\omega}^t \lambda G(t,s)F(s)\Delta s + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda G(t,t_k)I_k(x(t_k))\Big\} \\ &\leq \lambda\gamma_1\eta_1\int_0^\omega F(s)\Delta s + \lambda\gamma_1\eta_1\sum_{k=1}^p I_k(x(t_k))\end{aligned}$$

and

$$\begin{aligned}(\Phi x)(t) &= \int_{t-\omega}^t \lambda G(t,s)F(s)\Delta s + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda G(t,t_k)I_k(x(t_k)) \\ &\geq \lambda\gamma_2\eta_2\int_0^\omega F(s)\Delta s + \lambda\gamma_2\eta_2\sum_{k=1}^p I_k(x(t_k)) \\ &\geq \frac{\gamma_1\eta_1}{\gamma_2\eta_2}|\Phi x|_0 = \zeta|\Phi x|_0.\end{aligned} \tag{2.7}$$

Form (2.3) and (H3), we have

$$\begin{aligned}(\Phi x)^\Delta(t) &= \Big(\int_{t-\omega}^{\tilde a} \lambda G(t,s)F(s)\Delta s + \int_{\tilde a}^t \lambda G(t,s)F(s)\Delta s \\ &\quad + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}} \lambda G(t,t_k)I_k(x(t_k))\Big)^\Delta \\ &= \frac{e_{\ominus a}(\sigma(t),t)\big(1-e_a(t-\omega,t)\big)}{e_{\ominus a}(t,t-\omega)-1}\lambda F(t) + \ominus a(\Phi x)(t) \\ &\leq \ominus a(\Phi x)(t) \leq (\ominus a)^M(\Phi x)(t) \leq (\Phi x)(t),\end{aligned}$$

where $\tilde{a} \in [t-\omega, t]_{\mathbb{T}}$ is an arbitrary constant. If $(\Phi x)^{\Delta}(t) \geq 0$, we have

$$|\Phi x|_0 \geq |(\Phi x)^{\Delta}|_0. \tag{2.8}$$

If $(\Phi x)^{\Delta}(t) < 0$, by (2.7) and (H4), we have

$$\begin{aligned}
-(\Phi x)^{\Delta}(t) &= \frac{e_{\ominus a}(\sigma(t)+\omega, t) - e_{\ominus a}(\sigma(t), t)}{e_{\ominus a}(t, t-\omega) - 1}\lambda F(t) - \ominus a(\Phi x)(t) \\
&\leq \lambda\gamma_1\kappa\big(-\lambda^{-1}a(t)x^{\sigma}(t) + c^{\Delta}(t)x(t-\tau(t)) + c^{\sigma}(t)x^{\Delta}(t-\tau(t)) \\
&\quad + f(t, x(t), x(t-\tau(t)))\big) - (\ominus a)^l(\Phi x)(t) \\
&\leq \lambda\gamma_1\kappa\big(-\lambda^{-1}a(t) + |c^{\Delta}(t)| + c^{\sigma}(t) + L_f\big)|x|_1 - (\ominus a)^l(\Phi x)(t) \\
&\leq \lambda\gamma_2\eta_2\big(1 + (\ominus a)^l\big)\int_0^{\omega}\big(-\zeta\lambda^{-1}a(s) - |c^{\Delta}(s)| - \zeta c^{\sigma}(s)\big)|x|_1\Delta s \\
&\quad - (\ominus a)^l(\Phi x)(t) \\
&\leq \lambda\gamma_2\eta_2\big(1 + (\ominus a)^l\big)\int_0^{\omega} F(s)\Delta s - (\ominus a)^l(\Phi x)(t) \\
&\leq \big(1 + (\ominus a)^l\big)(\Phi x)(t) - (\ominus a)^l(\Phi x)(t) \\
&= (\Phi x)(t).
\end{aligned} \tag{2.9}$$

Form (2.8) and (2.9), we have $|(\Phi x)^{\Delta}|_0 \leq |(\Phi x)|_0$. So $|\Phi x|_1 = |\Phi x|_0$. By (2.7) we have $(\Phi x)(t) \geq \zeta|\Phi x|_1$. Hence $\Phi x \in K$. The proof is complete. $\square$

Lemma 2.9. *Assume that* (H1)-(H6) *hold, then* $\Phi : K \cap \Omega_R \to K$ *is strict-set-contractive, where* $\Omega_R = \{x \in \mathbb{Y} : |x|_1 < R\}$.

Proof. Obviously, Φ is continuous and bounded on Ω_R. Now we show $\alpha_{\mathbb{Y}}\big(\Phi(S)\big) \leq \gamma_1\kappa c^M\alpha_{\mathbb{Y}}(S)$ for any bounded set $S \subset \overline{\Omega}_R$. Let $\eta = \alpha_{\mathbb{Y}}(S)$. Then for any positive number $\varepsilon < \gamma_1\kappa c^M\eta$, there exists a finite family of subsets $\{S_i\}$ satisfying $S = \cup_i S_i$ with $\operatorname{diam} S_i \leq \eta + \varepsilon$. Hence,

$$|x - y|_1 \leq \eta + \varepsilon \quad \text{for all } x, y \in S_i. \tag{2.10}$$

As S and S_i are precompact in $\mathbb{X}$, then there exist a finite family of subsets (S_{ij}) of S_i such that $S_i = \cup_{ij} S_{ij}$ and

$$|x - y|_0 \leq \varepsilon \quad \text{for all } x, y \in S_{ij}. \tag{2.11}$$

Furthermore, for any $x \in S$, $t \in [0, \omega]_{\mathbb{T}}$, we have

$$\begin{aligned}
|(\Phi x)(t)| &= \int_{t-\omega}^{t}\lambda G(t,s)F(s)\Delta s + \sum_{k:t_k\in[t-\omega,t]_{\mathbb{T}}}\lambda G(t,t_k)I_k\big(x(t_k)\big) \\
&\leq \lambda\gamma_1\eta_1 R\int_0^{\omega}\big(\lambda^{-1}|a(s)| + |c^{\Delta}(s)| + |c^{\sigma}(s)| + L_f\big)\Delta s + \lambda\gamma_1\eta_1 R^2\sum_{k=1}^{m} I_k^M \\
&:= \Gamma
\end{aligned}$$

and

$$\begin{aligned}
|(\Phi x)^{\Delta}(t)| &= \Big|\frac{e_{\ominus a}(\sigma(t), t) - e_{\ominus a}(\sigma(t)+\omega, t)}{e_{\ominus a}(t, t-\omega) - 1}\lambda F(t) - \ominus a(\Phi x)(t)\Big| \\
&\leq \lambda\gamma_1\kappa R\big(\lambda^{-1}a^M + c^L + c^M + L_f\big) + (\ominus a)^M\Gamma.
\end{aligned}$$

Applying Arzela-Ascoli theorem on time scales [1], we know that $\Phi(S)$ is precompact in $\mathbb{X}$. So there exists a finite family of subsets $\{S_{ijk}\}$ of S_{ij} such that $S_{ij} = \cup_k S_{ijk}$ and

$$|\Phi x - \Phi y|_0 \le \varepsilon \quad \text{for all } x, y \in S_{ijk}. \tag{2.12}$$

From (2.10)-(2.11), for any $x, y \in S_{ijk}$, we have

$$\begin{aligned}
&|(\Phi x)^\Delta - (\Phi y)^\Delta|_0 \\
&= \max_{t\in[0,\omega]_\mathbb{T}} \Big\{ \Big| (\ominus a)(\Phi x(t) - \Phi y(t)) \\
&\quad + \frac{e_{\ominus a}(\sigma(t),t) - e_{\ominus a}(\sigma(t)+\omega,t)}{e_{\ominus a}(t,t-\omega)-1}\big(-\lambda^{-1}a(t)x^\sigma(t) + c^\Delta(t)x(t-\tau(t)) \\
&\quad + c^\sigma(t)x^\Delta(t-\tau(t)) + f(t,x(t),x(t-\tau(t)))\big) \\
&\quad - \frac{e_{\ominus a}(\sigma(t),t) - e_{\ominus a}(\sigma(t)+\omega,t)}{e_{\ominus a}(t,t-\omega)-1}\big(-\lambda^{-1}a(t)y^\sigma(t) + c^\Delta(t)y(t-\tau(t)) \\
&\quad + c^\sigma(t)y^\Delta(t-\tau(t)) + f(t,y(t),y(t-\tau(t)))\big) \Big| \Big\} \\
&\le (\ominus a)^M \big|\Phi x(t) - \Phi y(t)\big|_0 + \gamma_1\kappa\lambda^{-1}a^M \max_{t\in[0,\omega]_\mathbb{T}} |x^\sigma(t) - y^\sigma(t)| \\
&\quad + \gamma_1\kappa c^L \max_{t\in[0,\omega]_\mathbb{T}} |x(t-\tau(t)) - y(t-\tau(t))| \\
&\quad + \gamma_1\kappa c^M \max_{t\in[0,\omega]_\mathbb{T}} |x^\Delta(t-\tau(t)) - y^\Delta(t-\tau(t))| \\
&\quad + \gamma_1\kappa L_f \max_{t\in[0,\omega]_\mathbb{T}} |x(t) - y(t)| + \gamma_1\kappa L_f \max_{t\in[0,\omega]_\mathbb{T}} |x(t-\tau(t)) - y(t-\tau(t))| \\
&\le (\ominus a)^M\varepsilon + \gamma_1\kappa\lambda^{-1}a^M\varepsilon + \gamma_1\kappa c^L\varepsilon + \gamma_1\kappa c^M(\eta+\varepsilon) + 2\gamma_1\kappa L_f\varepsilon \\
&= \gamma_1\kappa c^M\eta + \widehat{H}\varepsilon,
\end{aligned}$$

where $\widehat{H} = (\ominus a)^M + \gamma_1\kappa\lambda^{-1}a^M + \gamma_1\kappa c^L + \gamma_1\kappa c^M + 2\gamma_1\kappa L_f$. From the above inequality and (2.12), we have

$$|\Phi x - \Phi y|_1 \le \gamma_1\kappa c^M\eta + \widehat{H}\varepsilon \quad \text{for all } x, y \in S_{ijk}.$$

Since ε is arbitrary small, we have

$$\alpha_\mathbb{Y}(\Phi(S)) \le \gamma_1\kappa c^M \alpha_\mathbb{Y}(S).$$

Hence, Φ is strict-set-contractive on $K \cap \Omega_R$. The proof is complete. □

3. Main results

Theorem 3.1. *Assume that* (H1)–(H7) *hold. Then* (1.2) *has at least one positive ω-periodic solution.*

Proof. Let

$$R = \frac{1 + \lambda\gamma_2\eta_2\omega(c^L + c^M)}{\lambda\gamma_2\eta_2\omega I^l\zeta^2}$$

and $0 < r < \min\{1, R\}$, where $I^l = \sum_{k=1}^p I_k^l$. Then $0 < r < R$. From Lemmas 2.8 and 2.9, we know that Φ is strict-set-contractive on $K_{r,R}$. In view of Lemma 2.7, we see that if there exists $x^* \in K$ such that $\Phi x^* = x^*$, then x^* is a positive ω-periodic solution of (1.2). Now, we shall prove that condition (ii) of Lemma 2.6 holds.

First, we prove that $\Phi x \ngeq x$, for all $x \in K$, $|x|_1 = r$. Otherwise, there exists $x \in K$, $|x|_1 = r$, such that $\Phi x \geq x$. So $\Phi x - x \in K$, which implies that

$$(\Phi x)(t) - x(t) \geq \zeta |\Phi x - x|_1 \geq 0 \quad \text{for all } t \in [0,\omega]_{\mathbb{T}}. \tag{3.1}$$

In addition, for $t \in [0,\omega]_{\mathbb{T}}$, from (H7) we obtain

$$\begin{aligned}
(\Phi x)(t) &= \int_{t-\omega}^{t} \lambda G(t,s) F(s) \Delta s + \sum_{k: t_k \in [t-\omega, t]_{\mathbb{T}}} \lambda G(t,t_k) I_k(x(t_k)) \\
&= \int_{t-\omega}^{t} \lambda G(t,s) \big(-\lambda^{-1} a(s) x^{\sigma}(s) + c^{\Delta}(s) x(s-\tau(s)) + c^{\sigma}(s) x^{\Delta}(s-\tau(s)) \\
&\quad + f(s, x(s), x(s-\tau(s))) \big) \Delta s + \sum_{k: t_k \in [t-\omega, t]_{\mathbb{T}}} \lambda G(t,t_k) I_k(x(t_k)) \\
&\leq \lambda \gamma_1 \eta_1 \omega \big(\lambda^{-1} a^M |x|_0 + c^L |x|_0 + \zeta^{-1} c^M |x|_0 + L_f |x|_0 \big) + \lambda \gamma_1 \eta_1 I^M |x|_0^2 \\
&\leq \lambda \gamma_1 \eta_1 \big(\lambda^{-1} a^M \omega + c^L \omega + \zeta^{-1} c^M \omega + L_f + I^M \big) |x|_0 \\
&< |x|_0.
\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we find that

$$|x|_0 \leq |\Phi x|_0 < |x|_0,$$

which is a contradiction. Finally, we prove that $\Phi x \nleq x$ for all $x \in K$, $|x|_1 = R$ also holds. For this case, we only prove that $\Phi x \not< x$ for all $x \in K$, $|x|_1 = R$. Otherwise, there exists $x \in K$ and $|x|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K/\{0\}$. Then we have

$$x(t) - (\Phi x)(t) \geq \zeta |x - \Phi x|_1 > 0 \quad \text{for all } t \in [0,\omega]_{\mathbb{T}}. \tag{3.3}$$

At the same time, for any $t \in [0,\omega]_{\mathbb{T}}$, we have

$$\begin{aligned}
(\Phi x)(t) &= \int_{t-\omega}^{t} \lambda G(t,s) F(s) \Delta s + \sum_{k: t_k \in [t-\omega, t]_{\mathbb{T}}} \lambda G(t,t_k) I_k(x(t_k)) \\
&= \int_{t-\omega}^{t} \lambda G(t,s) \big(-\lambda^{-1} a(s) x^{\sigma}(s) + c^{\Delta}(s) x(s-\tau(s)) + c^{\sigma}(s) x^{\Delta}(s-\tau(s)) \\
&\quad + f(s, x(s), x(s-\tau(s))) \big) \Delta s + \sum_{k: t_k \in [t-\omega, t]_{\mathbb{T}}} \lambda G(t,t_k) I_k(x(t_k)) \\
&\geq \lambda \gamma_2 \eta_2 \omega \big(\lambda^{-1} a^l \zeta |x|_1 - c^L |x|_1 - c^M |x|_1 + I^l \zeta^2 |x|_1^2 \big) \\
&\geq \lambda \gamma_2 \eta_2 \omega (I^l \zeta^2 R - c^L - c^M) R = R.
\end{aligned}$$

From (3.3) and above inequality, we obtain that $|x|_0 > |\Phi x|_0 \geq R$, which is a contradiction. Applying Lemma 2.6, we see that there is at least one nonzero fixed point in K. Hence system (1.2) has at least one positive ω-periodic solution. The proof is complete. $\square$

4. An example

When $\mathbb{T}=\mathbb{R}$, consider the system:

$$
\begin{gathered}
(x(t)+\lambda\frac{1}{4}x(t-\tau(t)))'=-\lambda(t^2+\frac{1}{2}\sin^2 x(t)+\frac{1}{4}\cos^2 x(t-1)),\quad t\neq t_k,\quad t\in\mathbb{R},\\
x(t_k^+)=x(t_k^-)-\lambda x^2(t_k),\quad k\in\mathbb{Z},
\end{gathered}
\tag{4.1}
$$

where

$$
\lambda<\min\Big\{\frac{2}{10001\pi},\frac{4\pi-2(e^{\frac{1}{10001}}-1)e^{\frac{1}{10001}}}{10001\pi(2\pi+5(e^{\frac{1}{10001}}-1)e^{\frac{1}{10001}})},\frac{20000e^{\frac{1}{10001}}-20002}{10001(\pi e^{\frac{1}{10001}}+42)e^{\frac{1}{10001}}}\Big\},
$$

$\omega=2\pi$, $p=10$, $L_f=1$, $I^M=20$. Let $a(t)=-\frac{1}{20002\pi}$, so $\gamma_1=\gamma_2=(e^{\frac{1}{10001}}-1)^{-1}$, $\eta_1=e^{\frac{1}{10001}}$, $\eta_2=1$, $\zeta=e^{-\frac{1}{10001}}$. And it is easy to check that (H1)–(H6) are satisfied. By Theorem 3.1, system (4.1) has at least one positive ω-periodic solution.

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