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NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS OF SECOND-ORDER WITH INFINITE DELAYS

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Abstract. This work shows the existence of mild solutions to neutral functional differential equations of second-order with infinite delay. The Hausdorff measure of noncompactness and fixed point theorem are used, without assuming compactness on the associated family of operators.

1. Introduction

Differential equations with delays are often more realistic to describe natural phenomena than those without delays, and neutral differential equations arise in many areas of applied mathematics. These two reasons may explain, why they have received much attention in the previous decades. Among the published works, we have [1, 4, 5, 12, 13, 14, 16, 21] and references therein. Existence and stability have been studied by Hale [9, 10], Travis and Webb [19], and Webb [20]. second-order differential equations and integrodifferential equations in Banach spaces have been studied in [2, 11] and [15], respectively.

In this article, we investigate the existence of mild solutions for the neutral functional differential equation

$$\frac{d}{dt}(x'(t) + g(t, x_t)) = Ax(t) + f(t, x_t), \quad t \in J = [0, b], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = z \in X\,. \tag{1.2}$$

We also consider the second order problem

$$\frac{d}{dt}(x'(t) + g(t, x_t, x'(t))) = Ax(t) + f(t, x_t, x'(t)), \quad t \in J = [0, b], \tag{1.3}$$

$$x_0 = \varphi \in \mathcal{B}, \quad x'(0) = z \in X, \tag{1.4}$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators on a Banach space X. In both cases, the history $x_t : (-\infty, 0] \to X, x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; g, f are appropriate functions.

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In this paper, we prove the existence of mild solution of the initial value problems (1.1)-(1.2) and (1.3)-(1.4) under the conditions under assumptions on Hausdorff's measure of noncompactness.

2. Preliminaries

Now we introduce some definitions, notation and preliminary facts which are used throughout this paper.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in X);
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$;
(iii) The map $t \to C(t)x$ is strongly continuous for each $x \in X$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)x = \int_0^t C(s)xds, x \in X, t \in \mathbb{R}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books by Goldstein [7] and Fattorini [6].

The operator A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators, $(C(t))_{t\in R}$, on X and $S(t)$ is the sine function associated with $(C(t))_{t\in R}$. We designate by N, $\widetilde{N}$ certain constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \in J$. We refer the reader to [6] for the necessary concepts about cosine functions. Next we only mention a few results and notations needed to establish our results. As usual we denote by $D(A)$ the domain of A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$.

In this work we employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that introduced by Hale and Kato [10] and it is appropriate to treat retarded differential equations with infinite delay.

Definition 2.1 ([10]). Let $\mathcal{B}$ be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and that satisfies the following cinditions:

(A) If $x : (-\infty, \sigma+b] \to X, b > 0$, such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma+b] : X)$, then for every $t \in [\sigma, \sigma+b)$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$,
  (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t+\sigma)\|x_\sigma\|_{\mathcal{B}}$,
  where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.
(A1) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma+b)$.
(B) The space $\mathcal{B}$ is complete.

Definition 2.2 ([3]). The Hausdorff's measure of noncompactness is defined as $\chi_Y(B) = \inf\{r > 0, B$ can be covered by finite number of balls with radius $r\}$. for bounded set B in any Banach space Y.

Lemma 2.3 ([3]). *Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:*

(1) *B is pre-compact if and only if $\chi_Y(B) = 0$;*
(2) *$\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(convB)$, where $\overline{B}$ and $convB$ are the closure and the convex hull of B respectively;*
(3) *$\chi_Y(B) \leq \chi_Y(C)$ when $B \subseteq C$;*
(4) *$\chi_Y(B + C) \leq \chi_Y(B) + \chi_Y(C)$ where $B + C = \{x + y : x \in B, y \in C\}$;*
(5) *$\chi_Y(B \cup C) \leq \max\{\chi_Y(B), \chi_Y(C)\}$;*
(6) *$\chi_Y(\lambda B) = |\lambda|\chi_Y(B)$ for any $\lambda \in R$;*
(7) *If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k, then $\chi_Z(QB) \leq k\chi_Y(B)$ for any bounded subset $B \subseteq D(Q)$, where Z is a Banach space;*
(8) *If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subsets of Y and $\lim_{n\to\infty} \chi_Y(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.*

Definition 2.4 ([3]). The map $Q : W \subseteq Y \to Y$ is said to be a $\chi_Y -$ *contraction* if Q is bounded continuous and there exists a positive constant $k < 1$ such that $\chi_Y(Q(C)) \leq k\chi_Y(C))$ for any bounded closed subset $C \subseteq W$, where Y is a Banach space.

Lemma 2.5 (Darbo-Sadovskii [3]). *If $W \subseteq Y$ is bounded closed and convex, the map $Q : W \to W$ is a $\chi_Y -$ contraction, then the map Q has at least one fixed point in W.*

In this paper we denote χ the Hausdorff's measure of noncompactness of X, χ_C the Hausdorff's measure of noncompactness of $C([0, b]; X)$ and χ_{C^1} the Hausdorff's measure of noncompactness of $C^1([0, b]; X)$. To discuss the existence results we need the following auxiliary results.

Lemma 2.6 ([3]).

(1) *If $W \subset C([a, b]; X)$ is bounded, then $\chi(W(t)) \leq \chi_C(W)$, for $t \in [a, b]$, where $W(t) = \{u(t) : u \in W\} \subseteq X$;*
(2) *If W is equicontinuous on $[a, b]$, then $\chi(W(t))$ is continuous for $t \in [a, b]$, and*
$$\chi_C(W) = \sup\{\chi(W(t)), t \in [a, b]\};$$
(3) *If $W \subset C([a, b]; X)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous for $t \in [a, b]$, and*
$$\chi\Big(\int_a^t W(s)ds\Big) \leq \int_a^t \chi W(s)ds$$
for all $t \in [a, b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.

The following lemmas are easy to prove.

Lemma 2.7. *If the semigroup $S(t)$ is equicontinuous and $\eta \in L([0, b]; \mathbb{R}^+)$, then the set $\{\int_0^t S(t - s)u(s)ds, \|u(s)\| \leq \eta(s)$ for a.e. $s \in [0, b]\}$ is equicontinuous for $t \in [0, b]$.*

Lemma 2.8 ([8]). *Let $W \subset C^1(J; X)$ be bounded and W' be equicontinuous, then*
$$\chi_{C^1}(W) = \max\{\chi_C(W), \chi_C(W')\} = \max\{\max_{t\in J} \chi_C(W(t)), \max_{t\in J} \chi_C(W'(t))\},$$
where $W' = \{u' : u \in W\}$, $J = [a, b]$.

3. Main results

Now we define the mild solution for the initial value problem(1.1)-(1.2).

Definition 3.1. A function $x : (-\infty, b] \to X$ is a mild solution of the initial value problem (1.1)-(1.2), if $x_0 = \varphi$, $x(\cdot)|_J \in C(J; X)$ and for $t \in J$,

$$x(t) = C(t)\varphi(0) + S(t)(z + g(0, \varphi)) - \int_0^t C(t-s)g(s, x_s)ds + \int_0^t S(t-s)f(s, x_s)ds.$$

For (1.1)-(1.2), we assume the following hypotheses:

(H1f) $f : J \times \mathcal{B} \to X$ satisfies the following two conditions:
  (1) For each $x : (-\infty, b] \to X$, $x_0 \in \mathcal{B}$ and $x|_J \in C(J; X)$, the function $t \to f(t, x_t)$ is strongly measurable and $f(t, \cdot)$ is continuous for a.e. $t \in J$;
  (2) There exist an integrable function $\alpha : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Omega : [0, +\infty) \to (0, +\infty)$, such that $\|f(t, v)\| \le \alpha(t)\Omega(\|v\|_{\mathcal{B}})$, for all $t \in J, v \in \mathcal{B}$;
  (3) There exists an integrable function $\eta : J \to [0, +\infty)$, such that
  $$\chi(S(s)f(t, D)) \le \eta(t) \sup_{-\infty \le \theta \le 0} \chi(D(\theta)) \quad \text{for a.e. } s, t \in J,$$
  where $D(\theta) = \{v(\theta) : v \in D\}$.

(H1g) The function $g(\cdot)$ is continuous and $g(t, \cdot)$ satisfies a Lipschitz condition; that is, there exists a positive constant L_g, such that
$$\|g(t, v_1) - g(t, v_2)\| \le L_g\|v_1 - v_2\|_{\mathcal{B}}, \quad (t, v_i) \in J \times \mathcal{B},\ i = 1, 2.$$

(H1) (1) $K_b(NbL_g + \widetilde{N} \int_0^b \alpha(s)ds \limsup_{\tau\to\infty} \frac{\Omega(\tau)}{\tau}) < 1$
  (2) $K_bNL_gb + \int_0^b \eta(s)ds < 1$.

In this section, $y : (-\infty, b] \to X$ is the function defined by $y_0 = \varphi$ and $y(t) = C(t)\varphi(0) + S(t)(z + g(0, \varphi))$ on J. Clearly, $\|y_t\|_{\mathcal{B}} \le K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}}$, where

$$K_b = \sup_{0\le t\le b} K(t), \quad M_b = \sup_{0\le t\le b} M(t), \|y\|_b = \sup_{0\le t\le b} \|y(t)\|.$$

Now we are in position to estate our main results.

Theorem 3.2. *If the hypotheses* (H1f), (H1g), (H1) *are satisfied, then the initial value problem* (1.1)-(1.2) *has at least one mild solution.*

Proof. Let $S(b)$ be the space $S(b) = \{x : (-\infty, b] \to X \mid x_0 = 0,\ x|_J \in C(J; X)\}$ endowed with supremum norm $\|\cdot\|_b$. Let $\Gamma : S(b) \to S(b)$ be the map defined by

$$(\Gamma x)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ -\int_0^t C(t-s)g(s, x_s + y_s)ds & \\ +\int_0^t S(t-s)f(s, x_s + y_s)ds, & t \in J. \end{cases} \tag{3.1}$$

It is easy to see that $\|x_t + y_t\|_{\mathcal{B}} \le K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t$, where $\|x\|_t = \sup_{0\le s\le t} \|x(s)\|$. Thus, Γ is well defined and with values in $S(b)$. In addition, from the axioms of phase space, the Lebesgue dominated convergence theorem and the conditions (H1f) (H1g), we can show that Γ is continuous.

Step 1. There exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(b) : \|x\|_b \le k\}$. In fact, if we assume that the assertion is false, then for $k > 0$ there exist $x_k \in B_k$

and $t_k \in I$ such that $k < \|\Gamma x_k(t_k)\|$. This yields

$$\begin{aligned}
k &< \|\Gamma x_k(t_k)\| \\
&\le N\int_0^{t_k}(L_g\|x_{ks}+y_s\|_{\mathcal{B}}+\|g(s,0)\|)ds+\widetilde{N}\int_0^{t_k}\alpha(s)\Omega(\|x_{ks}+y_s\|_{\mathcal{B}})ds \\
&\le N\int_0^b L_g(K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}+K_bk+\|g(s,0)\|)ds \\
&\quad +\widetilde{N}\int_0^b\alpha(s)ds\Omega(K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}+K_bk)
\end{aligned}$$

which implies

$$\begin{aligned}
1 &< K_bNbL_g+\widetilde{N}\int_0^b\alpha(s)ds\limsup_{k\to\infty}\frac{\Omega(K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}+K_bk)}{k} \\
&\le K_b(NbL_g+\widetilde{N}\int_0^b\alpha(s)ds\limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau})<1,
\end{aligned}$$

which is a contradiction.

Step 2. Next, we show that Γ is $\chi-$ *contraction*. To clarify this, we decompose Γ in the form $\Gamma=\Gamma_1+\Gamma_2$, for $t\ge 0$, where

$$\begin{aligned}
\Gamma_1x(t) &= -\int_0^t C(t-s)g(s,x_s+y_s)ds, \\
\Gamma_2x(t) &= \int_0^t S(t-s)f(s,x_s+y_s)ds.
\end{aligned}$$

First, we show the Γ_1 is Lipschitz continuous. For arbitrary $x_1,x_2\in B_k$, from Definition 2.1 and hypotheses, we obtain

$$\begin{aligned}
\|\Gamma_1x_1(t)-\Gamma_1x_2(t)\| &\le \|\int_0^t C(t-s)(g(s,x_{1s}+y_s)-g(s,x_{2s}+y_s))ds\| \\
&\le NL_gb\|x_{1t}-x_{2t}\|_{\mathcal{B}}\le K_bNL_gb\|x_{1t}-x_{2t}\|_b;
\end{aligned}$$

that is, $\|\Gamma_1x_1(t)-\Gamma_1x_2(t)\|_b\le K_bNL_gb\|x_{1t}-x_{2t}\|_b$; hence, Γ_1 is Lipschitz continuous, with Lipschitz constant $L'=K_bNL_gb$.

Next, taking $W\subset\Gamma(B_k)$. Obviously, $S(t)$ is equicontinuous. From Lemma 2.7, W is equicontinuous. As $\chi_C(W)=\sup\{\chi(W(t)),t\in J\}$, we have

$$\begin{aligned}
\chi(\Gamma_2W(t)) &= \chi(\int_0^t S(t-s)f(s,W_s+y_s)ds) \\
&\le \int_0^t\eta(s)\sup_{-\infty<\theta\le 0}\chi(W(s+\theta)+y(s+\eta))ds \\
&\le \int_0^t\eta(s)\sup_{0\le\tau\le s}\chi W(\tau)ds \\
&\le \chi_C(W)\int_0^t\eta(s)ds,
\end{aligned}$$

for each bounded set $W\in C(J;X)$. Since

$$\begin{aligned}
\chi_C(\Gamma W) &= \chi_C(\Gamma_1W+\Gamma_2W) \\
&\le \chi_C(\Gamma_1W)+\chi_C(\Gamma_2W)
\end{aligned}$$

$$\leq (L' + \int_0^t \eta(s)ds))\chi_C(W) \leq \chi_C(W),$$

the function Γ is χ-contraction. In view of Lemma 2.5, Darbo-Sadovskii fixed point theorem, we conclude that Γ has at least one fixed point in W. Let x be a fixed of Γ on $S(b)$, then $z = x + y$ is a mild solution of (1.1)-(1.2). So we deduce the existence of a mild solution of (1.1)-(1.2). $\square$

For (1.3)-(1.4), it is possible to establish similar results as those given in the first part of this section. Furthermore, we denote by C^1 the space of smooth functions in the sense above described endowed with the norm $\|u\|_1 = \|u\| + \|u'\|$.

Now we define the mild solution for the initial value problem (1.3)-(1.4).

Definition 3.3. A function $x : (-\infty, b] \to X$ is a mild solution of the initial value problem (1.3)-(1.4) if $x_0 = \varphi$, $x(\cdot)|_J \in C^1(J; X)$ and for $t \in J$,

$$\begin{aligned} x(t) = C(t)\varphi(0) + S(t)(z + g(0, \varphi, z)) - \int_0^t C(t-s)g(s, x_s, x'(t))ds \\ + \int_0^t S(t-s)f(s, x_s, x'(t))ds. \end{aligned}$$

For (1.3)-(1.4), we assume the following hypotheses:

(H2f) $f : J \times \mathcal{B} \times X \to X$ satisfies the following conditions:
  (1) For each $x : (-\infty, b] \to X$, $x_0 = \varphi \in \mathcal{B}$ and $x|_J \in C^1$, the function $t \to f(t, x_t, x'(t))$ is strongly measurable and $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in J$;
  (2) There exist an integrable function $\alpha : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Omega : [0, +\infty) \to (0, +\infty)$, such that
  $$\|f(t, v, w)\| \leq \alpha(t)\Omega(\|v\|_{\mathcal{B}} + \|w\|), \quad t \in J, \ (v, w) \in \mathcal{B} \times X;$$
  (3) There exist integrable functions $\eta_i : J \to [0, +\infty)$, $i = 1, 2$, such that
  $$\begin{gathered} \chi(S(s)f(t, D_1, D_2)) \leq \eta_1(t) \sup_{-\infty \leq \theta \leq 0} \chi(D_1(\theta)), \\ \chi(C(s)f(t, D_1, D_2)) \leq \eta_2(t) \sup_{-\infty \leq \theta \leq 0} \chi(D_2(\theta)) \quad \text{for a.e. } s, t \in J, \end{gathered}$$
  where $D_i(\theta) = \{D_i(\theta) : v \in D\}$, $i = 1, 2$.

(H2g) There exists a positive constant L_g such that
  $$\|g(t, v_1, w_1) - g(t, v_2, w_2)\| \leq L_g(\|v_1 - v_2\|_{\mathcal{B}} + \|w_1 - w_2\|),$$
  $(t, v_i, w_i) \in J \times \mathcal{B} \times X$, $i = 1, 2$.

(H2) (1) $(K_b + 1)(L_g(Nb + 1 + \|A\|\widetilde{N}b) + (N + \widetilde{N}) \int_0^b \alpha(s)ds \limsup_{\tau \to \infty} \frac{\Omega(\tau)}{\tau}) < 1$;
  (2) $L_g(Nb + 1 + \|A\|\widetilde{N}b)(K_b + 1) + \max\{\int_0^b \eta_1(s)ds, \int_0^b \eta_2(s)ds\} < 1$.

In this section, $y : (-\infty, b] \to X$ is the function defined by $y_0 = \varphi$ and $y(t) = C(t)\varphi(0) + S(t)(z + g(0, \varphi, z))$ on J. Clearly, $\|y_t\|_{\mathcal{B}} \leq K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}}$, where $K_b = \sup_{0 \leq t \leq b} K(t)$, $M_b = \sup_{0 \leq t \leq b} M(t)$, $\|y\|_b = \sup_{0 \leq t \leq b} \|y(t)\|$.

Theorem 3.4. *If the hypotheses* (H2f) (H2g), (H2) *are satisfied, then the initial value problem* (1.3)-(1.4) *has at least one mild solution.*

Proof. Let $S^1(b)$ be the space

$$S^1(b) = \{x : (-\infty, b] \to X : x_0 = 0,\ x|_J \in C^1(J; X),\ x'(0) = -g(0, \varphi, z)\}$$

endowed with supremum norm $\|\cdot\|_{1b}$. Let $\Gamma : S^1(b) \to S^1(b)$ be the map defined by

$$(\Gamma x)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ -\int_0^t C(t-s)g(s, x_s + y_s, x'(s) + y'(s))ds & \\ +\int_0^t S(t-s)f(s, x_s + y_s, x'(s) + y'(s))ds, & t \in J, \end{cases} \tag{3.2}$$

where $y_0 = \varphi$ and $y(t) = C(t)\varphi(0) + S(t)(z + g(0, \varphi, z))$ on J. It is easy to see that

$$\|x_t + y_t\|_{\mathcal{B}} \le K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t,$$

where $\|x\|_t = \sup_{0\le s\le t}\|x(s)\|$. Thus, Γ is well defined and with values in $S^1(b)$, and

$$(\Gamma x)'(t) = -g(t, x_t + y_t, x'(t) + y'(t)) - \int_0^t AS(t-s)g(s, x_s + y_s, x'(s) + y'(s))ds$$
$$+ \int_0^t C(t-s)f(s, x_s + y_s, x'(s) + y'(s))ds, \quad t \in J.$$

In addition, from the axioms of phase space, the Lebesgue dominated convergence theorem and the conditions (H2f), (H2g), we can show that Γ and Γ' are continuous.
Step 1. There exists $k > 0$ such that $\Gamma(B_k) \subset B_k := \{x \in S^1(b) : \|x\|_{1b} \le k\}$. In fact, if we assume that the assertion are false, then for $k > 0$ there exist $x_k \in B_k$ and $t_k \in J$ such that $k < \|\Gamma x_k(t_k)\|_1$. This yields

$$\begin{aligned}
k &< \|\Gamma x_k(t_k)\|_1 \\
&= \|\Gamma x_k(t_k)\| + \|(\Gamma x_k)'(t_k)\| \\
&\le N\int_0^{t_k} (L_g(\|x_{ks} + y_s\|_{\mathcal{B}} + \|x_k'(s) + y'(t)\|) + \|g(s, 0, 0)\|)ds \\
&\quad + \int_0^{t_k} \widetilde{N}\alpha(s)\Omega(\|x_{ks} + y_s\|_{\mathcal{B}} + \|x_k'(s) + y'(t)\|)ds \\
&\quad + L_g(\|x_{kt_k} + y_{t_k}\|_{\mathcal{B}} + \|x_k'(t_k) + y'(t_k)\|) + \|g(t_k, 0, 0)\| \\
&\quad + \int_0^{t_k} \|A\|\widetilde{N}(L_g(\|x_{ks} + y_s\|_{\mathcal{B}} + \|x_k'(s) + y'(s)\|) + \|g(s, 0, 0)\|)ds \\
&\quad + \int_0^{t_k} N\alpha(s)\Omega(\|x_{ks} + y_s\|_{\mathcal{B}} + \|x_k'(s) + y'(s)\|)ds \\
&\le bNL_g(K_bk + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b) + N\int_0^b \|g(s, 0, 0)\|ds \\
&\quad + \widetilde{N}\int_0^b \alpha(s)ds\Omega(K_bk + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b) \\
&\quad + L_g(K_bk + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b) + \|g(t_k, 0, 0)\| \\
&\quad + b\|A\|\widetilde{N}L_g(K_bk + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b) + \|A\|\widetilde{N}\int_0^b \|g(s, 0, 0)\|ds \\
&\quad + N\int_0^b \alpha(s)ds\Omega(K_bk + K_b\|y\|_b + M_b\|\varphi\|_{\mathcal{B}} + k + \|y'\|_b),
\end{aligned}$$

which implies

$$\begin{aligned}
1 &< L_g(K_b+1)(Nb+1+\|A\|\widetilde{Nb}) \\
&\quad + (N+\widetilde{N})\int_0^b \alpha(s)ds \limsup_{k\to\infty} \frac{\Omega(K_bk+K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}+k+\|y'\|_b)}{k}) \\
&\le (K_b+1)(L_g(Nb+1+\|A\|\widetilde{N}b)+(N+\widetilde{N})\int_0^b \alpha(s)ds \limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau}) < 1,
\end{aligned}$$

which is a contradiction.

Step 2. Next we show that Γ is $\chi - contraction$. To clarify this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, for $t \ge 0$, where

$$\begin{gathered}
\Gamma_1 x(t) = -\int_0^t C(t-s)g(s, x_s + y_s, x'(s) + y'(s))ds, \\
\Gamma_2 x(t) = \int_0^t S(t-s)f(s, x_s + y_s, x'(s) + y'(s))ds.
\end{gathered}$$

First, we show the Γ_1 is Lipschitz continuous. For arbitrary $x_1, x_2 \in B_k$, from Definition 2.1 and hypotheses conditions, we obtain

$$\begin{aligned}
&\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\|_1 \\
&\le \|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\| + \|(\Gamma_1 x_1)'(t) - (\Gamma_1 x_2)'(t)\| \\
&\le \|\int_0^t C(t-s)(g(s, x_{1s}+y_s, x_1'(s)+y'(s)) - g(s, x_{2s}+y_s, x_2'(s)+y'(s)))ds\| \\
&\quad + \|g(t, x_{1t}+y_t, x_1'(t)+y'(t)) - g(t, x_{2t}+y_t, x_2'(t)+y'(t))\| \\
&\quad + \|\int_0^t AS(t-s)(g(s, x_{1s}+y_s, x_1'(s)+y'(s)) - g(s, x_{2s}+y_s, x_2'(s)+y'(s)))ds\| \\
&\le N\int_0^t L_g(\|x_{1s}-x_{2s}\|_{\mathcal{B}} + \|x_1'(s)-x_2'(s)\|)ds \\
&\quad + L_g(\|x_{1t}-x_{2t}\|_{\mathcal{B}} + \|x_1'(t)-x_2'(t)\|) \\
&\quad + \|A\|\widetilde{N}\int_0^t L_g(\|x_{1s}-x_{2s}\|_{\mathcal{B}} + \|x_1'(s)-x_2'(s)\|)ds \\
&\le N\int_0^t L_g(K(s)\sup_{0\le\tau\le s}\|x_1(\tau)-x_2(\tau)\| + \|x_1'(s)-x_2'(s)\|)ds \\
&\quad + L_g(K(t)\sup_{0\le\tau\le t}\|x_1(\tau)-x_2(\tau)\| + \|x_1'(t)-x_2'(t)\|) \\
&\quad + \|A\|\widetilde{N}\int_0^t L_g(K(s)\sup_{0\le\tau\le s}\|x_1(\tau)-x_2(\tau)\| + \|x_1'(s)-x_2'(s)\|)ds \\
&\le L_g(Nb+1+\|A\|\widetilde{N}b)\sup_{0\le\tau\le t}(K(t)\|x_1(\tau)-x_2(\tau)\| + \|x_1'(t)-x_2'(t)\|) \\
&\le L_g(Nb+1+\|A\|\widetilde{N}b)(K_b+1)\|x_1-x_2\|_1;
\end{aligned}$$

that is,

$$\|\Gamma_1 x_1(t) - \Gamma_1 x_2(t)\|_{1b} \le L_g(Nb+1+\|A\|\widetilde{N}b)(K_b+1)\|x_1-x_2\|_{1b}.$$

Hence, Γ_1 is Lipschitz continuous with Lipschitz constant $L' = L_g(Nb + 1 + \|A\|\widetilde{N}b)(K_b + 1)$.

Next, taking $W \subset \Gamma(B_k)$. Obviously, $S(t)$ is equicontinuous. From Lemma 2.7, W is equicontinuous and $\chi_C(W) = \sup\{\chi(W(t)), t \in [a,b]\}$, we have

$$\begin{aligned}
&\chi_{C^1}(\Gamma_2 W(t)) \\
&= \chi_{C^1}\Big(\int_0^t S(t-s)f(s, W_s + y_s, W'(s) + y'(s))ds\Big) \\
&= \max\Big\{\max_{t\in J}\chi_C\Big(\int_0^t S(t-s)f(s, W_s + y_s, W'(s) + y'(s))\Big)\Big\}ds\Big), \\
&\quad \max_{t\in J}\chi_C\Big(\int_0^t C(t-s)f(s, W_s + y_s, W'(s) + y'(s))ds\Big)\Big\} \\
&\le \max\Big\{\max_{t\in J}\int_0^t \eta_1(s)\sup_{-\infty<\theta\le 0}\chi_C(W(s+\theta) + y(s+\theta))ds\Big), \\
&\quad \max_{t\in J}\int_0^t \eta_2(s)\sup_{-\infty<\theta\le 0}\chi_C(W'(s+\theta) + y'(s+\theta))ds\Big)\Big\} \\
&\le \max\Big\{\int_0^t \eta_1(s)\sup_{0\le\tau\le s}\chi_C(W(\tau))ds, \int_0^t \eta_2(s)\sup_{0\le\tau\le s}\chi_C(W'(\tau))ds\Big\} \\
&\le \max\Big\{\int_0^b \eta_1(s)ds, \int_0^b \eta_2(s)ds\Big\}\max\Big\{\sup_{0\le\tau\le b}\chi_C(W(\tau)), \sup_{0\le\tau\le b}\chi_C(W'(\tau))\Big\} \\
&\le \max\Big\{\int_0^b \eta_1(s)ds, \int_0^b \eta_2(s)ds\Big\}\max\{\chi_C(W), \chi_C(W')\} \\
&\le \max\Big\{\int_0^b \eta_1(s)ds, \int_0^b \eta_2(s)ds\Big\}\chi_{C^1}(W),
\end{aligned}$$

for each bounded set $W \in C^1(J;X)$. Since

$$\begin{aligned}
\chi_{C^1}(\Gamma W) &= \chi_{C^1}(\Gamma_1 W + \Gamma_2 W) \le \chi_{C^1}(\Gamma_1 W) + \chi_{C^1}(\Gamma_2 W) \\
&\le \Big(L_g(Nb + 1 + \|A\|\widetilde{N}b) + \max\Big\{\int_0^b \eta_1(s)ds, \int_0^b \eta_2(s)ds\Big\}\Big)\chi_{C^1}(W).
\end{aligned}$$

The function Γ is χ-contraction. In view of Lemma 2.5, we conclude that Γ has at least one fixed point in W. Let x be a fixed of Γ on $S^1(b)$, then $z = x + y$ is a mild solution of (1.3)-(1.4). So we deduce the existence of a mild solution of (1.3)-(1.4). □

4. Examples

4.1. **The phase space $C_r \times L^2(h, X)$.** Let $h(\cdot) : (-\infty, -r] \to R$ be a positive Lebesgue integrable function and $\mathcal{B} := C_r \times L^2(h; X)$, $r \ge 0$, be the space formed of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi|_{[-r,0]} \in C([-r, 0], X)$, $\phi(\cdot)$ is Lebesgue-measurable on $(-\infty, -r]$ and $h|\varphi|^2$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup_{\theta\in[-r,0]}|\varphi(\theta)| + \Big(\int_{-\infty}^{-r} h(\theta)|\varphi(\theta)|^2 d\theta\Big)^{1/2}.$$

Assume that $h(\cdot)$ satisfies [17, conditions (g-6) and (g-7)], function G is locally bounded on $(-\infty, 0]$. Proceeding as in the proof of [17, Theorem 1.3.8] it follows that $\mathcal{B}$ is a phase space which satisfies the axioms (A) and (B). Moreover, when

$r = 0$ this space coincides with $X \times L^2(h, X)$ and the parameter $H = 1$, as in [17, Theorem 1.3.8]; $M(t) = G(-t)^{1/2}$ and $K(t) = 1 + (\int_{-t}^{0} h(\tau)d\tau)^{1/2}$, for $t \geq 0$ (see [17]).

Let $X = L^2([0, \pi])$ and let A be the operator $Af = f''$ with domain

$$D(A) := \{f \in L^2([0, \pi]) : f'' \in L^2([0, \pi]), f(0) = f(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a C_0-semigroup and of a strongly continuous cosine function on X, which will be denoted by $(C(t))_{t\in R}$. Moreover, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with corresponding normalized eigenvectors $z_n(\xi) := (\frac{2}{\pi})^{1/2} \sin(n\xi)$ and the following properties hold:

(a) $\{z_n : n \in \mathbf{N}\}$ is an orthonormal basis of X.
(b) For $f \in X$, $(-A)^{-1/2}f = \sum_{n=1}^{\infty} \frac{1}{n}\langle f, z_n\rangle z_n$ and $\|(-A)^{-1/2}\| = 1$.
(c) For $f \in X$, $C(t)f = \sum_{n=1}^{\infty} \cos(nt)\langle f, z_n\rangle z_n$. Moreover, it follow from this expression that $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle \varphi, z_n\rangle z_n$, that $S(t)$ is compact for $t > 0$ and that $\|C(t)\| = 1$ and $\|S(t)\| = 1$ for every $t \in R$.
(d) If Φ denotes the group of translations on X defined by $\Phi(t)x(\xi) = \widetilde{x}(\xi + t)$, where $\widetilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t))$; $A = B^2$ where B is the infinitesimal generator of the group Φ and $E = \{x \in H^1(0, \pi) : x(0) = x(\pi) = 0$, see [6] for details.

In the next applications, $\mathcal{B}$ will be the phase space $X \times L^2(h, X)$.

4.2. **A second order neutral equation.** Now we discuss the existence of solutions for the second order neutral differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}(\frac{\partial u(t,\xi)}{\partial t} + \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)u(s,\eta)d\eta ds) \\ &= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_{-\infty}^{t} F(t, t-s, \xi, u(s,\xi))ds, \quad t \in [0,a], \xi \in [0,\pi], \end{aligned} \tag{4.1}$$

$$u(t, 0) = u(t, \pi) = 0, \quad t \in [0, a], \tag{4.2}$$

$$u(\tau, \xi) = \varphi(\tau, \xi), \quad \tau \leq 0,\ 0 \leq \xi \leq \pi, \tag{4.3}$$

where $\varphi \in X \times L^2(h; X)$, and

(a) The functions $b(s, \eta, \xi)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s, \eta, \pi) = b(s, \eta, 0) = 0$ and
$$L_g := \max\{(\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{h(s)}(\frac{\partial^i b(s,\eta,\xi)}{\partial \xi^i})^2 d\eta ds d\xi)^{1/2} : i = 0, 1\} < \infty;$$
(b) The function $F : \mathbb{R}^4 \to \mathbb{R}$ is continuous and there is continuous function $\mu : \mathbb{R}^2 \to \mathbb{R}$ such that
$$\int_{-\infty}^{0} \frac{\mu(t,s)^2}{h(s)} ds < \infty$$
and $|F(t, s, \xi, x)| \leq \mu(t, s)|x|$, $(t, s, \xi, x) \in \mathbb{R}^4$;

Assuming that conditions (a),(b) are satisfied, problem (4.1)-(4.3) can be modelled as the abstract Cauchy problem (1.1)-(1.2) by defining

$$g(t, \psi)(\xi) := \int_{-\infty}^{0}\int_{0}^{\pi} b(s, \nu, \xi)\psi(s, \nu)d\nu ds, \tag{4.4}$$

$$f(t,\psi)(\xi) := \int_{-\infty}^{0} F(t,s,\xi,\psi(s,\xi))ds. \tag{4.5}$$

Moreover, $\|f(t,\psi)\| \leq d(t)\|\psi\|_{\mathcal{B}}$ for every $t \in [0,a]$, where $d(t) := (\int_{-\infty}^{0} \frac{\mu(t,s)^2}{h(s)} ds)^{1/2}$ is a Lebesgue integrable function.

The next result is a consequence of Theorem 3.2.

Proposition 4.1. *Let the previous conditions be satisfied. If*

$$(1 + (\int_{-a}^{0} h(\tau)d\tau)^{1/2})(aL_g + \int_{0}^{a} d(t)dt) < 1,$$

then there exists a mild solution of (4.1)-(4.3).

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