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TIME AVERAGING FOR ORDINARY DIFFERENTIAL EQUATIONS AND RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We prove averaging theorems for non-autonomous ordinary differential equations and retarded functional differential equations in the case where the vector fields are continuous in the spatial variable uniformly with respect to the time and the solution of the averaged system exists on some given interval. Our assumptions are weaker than those required in the results of the existing literature. Usually, we require that the non-autonomous differential equation and the autonomous averaged equation are locally Lipschitz and that the solutions of both equations exist on some given interval. Our results are formulated in classical mathematics. Their proofs use the stroboscopic method which is a tool of the nonstandard asymptotic theory of differential equations.

1. Introduction

Averaging is an important method for analysis of nonlinear oscillation equations containing a small parameter. This method is well-known for ordinary differential equations (ODEs) and fundamental averaging results (see, for instance, [6, 13, 16, 38, 39] and references therein) assert that the solutions of a non-autonomous equation in *normal form*

$$x'(\tau) = \varepsilon f(\tau, x(\tau)), \tag{1.1}$$

where ε is a small positive parameter, are approximated by the solutions of the autonomous averaged equation

$$y'(\tau) = \varepsilon F(y(\tau)). \tag{1.2}$$

The approximation holds on time intervals of order $1/\varepsilon$ when ε is sufficiently small. In (1.2), the function F is the average of the function f in (1.1) defined by

$$F(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x)dt. \tag{1.3}$$

The method of averaging for ODEs is known also as the Krylov-Bogolyubov-Mitropolsky (KBM) method [6].

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The method of averaging was extended by Hale [15] (see also of [26, Section 2.1]) to the case of retarded functional differential equations (RFDEs) containing a small parameter when the equations are considered in normal form

$$x'(\tau) = \varepsilon f(\tau, x_\tau), \tag{1.4}$$

where, for $\theta \in [-r, 0]$, $x_\tau(\theta) = x(\tau + \theta)$. Equations of the form (1.4) cover a wide class of differential equations including those with point-wise delay for which a method of averaging was developed by Halanay [14], Medvedev [34] and Volosov et al. [44]. Note that the averaged equation corresponding to (1.4) is the ODE

$$y'(\tau) = \varepsilon F(\tilde{y}^\tau), \tag{1.5}$$

where, for τ fixed and $\theta \in [-r, 0]$, $\tilde{y}^\tau(\theta) = y(\tau)$ and the average function F is defined by (1.3). Recently, Lehman and Weibel [27] proposed to retain the delay in the averaged equation and proved that equation (1.4) is approximated by the averaged RFDE

$$y'(\tau) = \varepsilon F(y_\tau). \tag{1.6}$$

They observed, using numerical simulations, that equation (1.4) is better approximated by the RFDE (1.6) than by the ODE (1.5), see Remark 2.9.

The change from the fast time scale τ to the slow time scale $t = \varepsilon\tau$ transforms equations (1.1) and (1.2), respectively, into

$$\dot{x}(t) = f(t/\varepsilon, x(t)) \tag{1.7}$$

and

$$\dot{y}(t) = F(y(t)). \tag{1.8}$$

Thus a method of averaging can be developed for (1.7), that is, if ε is sufficiently small, the difference between the solution x of (1.7) and the solution y of (1.8), with the same initial condition, is small on finite time intervals.

The analog of equation (1.7) for RFDEs is

$$\dot{x}(t) = f\left(t/\varepsilon, x_t\right). \tag{1.9}$$

The averaged equation corresponding to (1.9) is the RFDE

$$\dot{y}(t) = F(y_t),$$

where the average function F is defined by (1.3).

Notice that the RFDEs (1.4) and (1.9) are not equivalent under the change of time $t = \varepsilon\tau$, as it was the case for the ODEs (1.1) and (1.7). Indeed, by rescaling τ as $t = \varepsilon\tau$ equation (1.4) becomes

$$\dot{x}(t) = f\left(t/\varepsilon, x_{t,\varepsilon}\right), \tag{1.10}$$

where, for $\theta \in [-r, 0]$, $x_{t,\varepsilon}(\theta) = x(t + \varepsilon\theta)$. Equation (1.10) is different from (1.9), so that the results obtained for (1.10) cannot be applied to (1.9). This last equation deserves a special attention. It was considered by Hale and Verduyn Lunel in [17] where a method of averaging is developed for infinite dimensional evolutionary equations which include RFDEs such (1.9) as a particular case (see also Section 12.8 of Hale and Verduyn Lunel's book [18] and Section 2.3 of [26]).

Following our previous works [19, 20, 21, 22, 24, 25, 39, 41, 42] we consider in this paper all equations (1.7), (1.9) and (1.10). Our aim is to give theorems of averaging under weaker conditions than those of the literature. We want to emphasize that our main contribution is the weakening of the regularity conditions on the equation under which the averaging method is justified in the existing literature. Indeed, usually classical averaging theorems require that the vector field f in (1.7), (1.9) and (1.10) is at least locally Lipschitz with respect to the second variable uniformly with respect to the first one (see Remarks 2.4, 2.8 and 2.13 below). In our results this condition is weakened and it is only assumed that f is continuous in the second variable uniformly with respect to the first one. Also, it is often assumed that the solutions x and y exist on the same finite interval of time. In this paper we assume only that the solution y of the averaged equation exists on some finite interval and we give conditions on the vector field f so that, for ε sufficiently small, the solution x of (1.7), (1.9) or (1.10) will be defined at least on the same interval. The *uniform quasi-boundedness* of the vector field f is thus introduced for this purpose. Recall that the property of quasi-boundedness is strongly related to results on continuation of solutions of RFDEs. It should be noticed that the existing literature [15, 17, 26] proposed also important results on the infinite time interval $[0, \infty)$, provided that more hypotheses are made on the non-autonomous system and its averaged system. For example, to a hyperbolic equilibrium point of the averaged system there corresponds a periodic solution of the original equation if ε is small. Of course, for such results, stronger assumptions on the regularity of the vector field f are required.

In this work our averaging results are formulated in classical mathematics. We prove them within *Internal Set Theory* (IST) [35] which is an axiomatic approach to *Nonstandard Analysis* (NSA) [37]. The idea to use NSA in perturbation theory of differential equations goes back to the 1970s with the Reebian school. Relative to this use, among many works we refer the interested reader, for instance, to [3, 5, 10, 28, 29, 30, 31, 32, 33, 43, 47] and the references therein. It has become today a well-established tool in asymptotic theory of differential equations. Among the famous discoveries of the nonstandard asymptotic theory of differential equations we can cite the *canards* which appear in slow-fast vector fields and are closely related to the stability loss delay phenomenon in dynamical bifurcations [1, 2, 4, 7, 8, 10, 11, 12, 45, 46].

The structure of the paper is as follows. In Section 2 we introduce the notations and present our main results : Theorems 2.2, 2.6 and 2.11. We discuss also both periodic and almost periodic special cases. In Section 3 we start with a short tutorial to NSA and then state our main (nonstandard) tool, the so-called *stroboscopic method*. In Section 4, we give the proofs of Theorems 2.2, 2.6 and 2.11. Some of the auxiliary results can be found, for instance, in [22, 39, 41]. They are included here to keep the paper self-contained for the benefit of the reader. In Section 5 we say exactly what are the results that are already proved in our previous articles and we discuss the differences with the previous works.

Let us notice that our proofs do not need to be translated into classical mathematics, because IST is a conservative extension of ZF, that is, any classical statement which is a theorem of IST is also a theorem of ZFC.

2. Notation and Main Results

In this section we will present our main results on averaging for fast oscillating ODEs (1.7), RFDEs in normal form (1.10) and fast oscillating RFDEs (1.9). First we introduce some necessary notations. We assume that $r \geq 0$ is a fixed real number and denote by $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^d)$ the Banach space of continuous functions from $[-r, 0]$ into $\mathbb{R}^d$ with the norm $\|\phi\| = \sup\{|\phi(\theta)| : \theta \in [-r, 0]\}$, where $|\cdot|$ is a norm of

$\mathbb{R}^d$. Let $L \geq 0$. If $x : [-r, L] \to \mathbb{R}^d$ is a continuous function then, for each $t \in [0, L]$, we define $x_t \in \mathcal{C}$ by setting $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-r, 0]$. Note that when $r = 0$ the Banach space $\mathcal{C}$ can be identified with $\mathbb{R}^d$ and x_t with $x(t)$ for each $t \in [0, L]$.

2.1. **Averaging for ODEs.** Let U be an open subset of $\mathbb{R}^d$ and let $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$, $(t, x) \mapsto f(t, x)$, be a continuous function. Let $x_0 \in U$ be an initial condition. We consider the initial value problem

$$\dot{x}(t) = f\left(t/\varepsilon, x(t)\right), \quad x(0) = x_0, \tag{2.1}$$

where $\varepsilon > 0$ is a small parameter. We state the precise assumptions on this problem in the following definition.

Definition 2.1. A vector field $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ is said to be a KBM-vector field if it is continuous and satisfies the following conditions

- (C1) The function f is continuous in the second variable uniformly with respect to the first one.
- (C2) For all $x \in U$, there exists a limit $F(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x)dt$.
- (C3) The initial value problem
$$\dot{y}(t) = F(y(t)), \quad y(0) = x_0 \tag{2.2}$$
has a unique solution.

Notice that from conditions (C1) and (C2) we deduce that the average of the function f, that is, the function $F : U \to \mathbb{R}^d$ in (C2), is continuous (see Lemma 4.1). So, the averaged initial value problem (2.2) is well defined.

The main theorem of this section is on averaging for fast oscillating ODEs. It establishes nearness of solutions of (2.1) and (2.2) on finite time intervals, and reads as follows.

Theorem 2.2. *Let $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ be a KBM-vector field. Let $x_0 \in U$. Let y be the solution of (2.2) and let $L \in J$, where J is the positive interval of definition of y. Then, for every $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, every solution x of (2.1) is defined at least on $[0, L]$ and satisfies $|x(t) - y(t)| < \delta$ for all $t \in [0, L]$.*

Let us discuss now the result above when the function f is periodic or more generally almost periodic in the first variable. We will see that some of the conditions in Theorem 2.2 can be removed. Indeed, in the case where f is periodic in t, from continuity plus periodicity properties one can easily deduce condition (C1). Periodicity also implies condition (C2) in an obvious way. The average of f is then given, for every $x \in U$, by

$$F(x) = \frac{1}{T} \int_0^T f(t, x)dt, \tag{2.3}$$

where T is the period. In the case where f is almost periodic in t it is well-known that for all $x \in U$, the limit

$$F(x) = \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} f(t, x)dt \tag{2.4}$$

exists uniformly with respect to $s \in \mathbb{R}$. So, condition (C2) is satisfied when $s = 0$. We point out also that in a number of cases encountered in applications the function f is a finite sum of periodic functions in t. As in the periodic case above, condition (C1) is satisfied. Hence we have the following result.

Corollary 2.3 (Periodic and Almost periodic cases)**.** *The conclusion of Theorem 2.2 holds when $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ is a continuous function which is periodic (or a sum of periodic functions) in the first variable and satisfies condition (C3). It holds also when f is continuous, almost periodic in the first variable and satisfies conditions (C1) and (C3).*

Remark 2.4. In the results of the classical literature, for instance [26, Theorem 1, p. 202], it is assumed that f has bounded partial derivatives with respect to the second variable.

2.2. **Averaging for RFDEs in normal form.** This section concerns the use of the method of averaging to approximate initial value problems of the form

$$\dot{x}(t) = f(t/\varepsilon, x_{t,\varepsilon}), \quad x(t) = \phi(t/\varepsilon), t \in [-\varepsilon r, 0]. \tag{2.5}$$

Here $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$, $(t, x) \mapsto f(t, x)$, is a continuous function, $\Omega = \mathcal{C}([-r, 0], U)$, $r > 0$, where U is an open subset of $\mathbb{R}^d$, $\phi \in \Omega$ is an initial condition and $\varepsilon > 0$ is a small parameter. For each $t \geq 0$, $x_{t,\varepsilon}$ denotes the element of $\mathcal{C}$ given by $x_{t,\varepsilon}(\theta) = x(t + \varepsilon\theta)$ for all $\theta \in [-r, 0]$.

We recall that the change of time scale $t = \varepsilon\tau$ transforms (2.5) into the following initial value problem, associated to a RFDE in normal form :

$$x'(\tau) = \varepsilon f(\tau, x_\tau), \quad x_0 = \phi.$$

Definition 2.5. A vector field $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is said to be a KBM-vector field if it is continuous and satisfies the following conditions.

- (H1) The function f is continuous in the second variable uniformly with respect to the first one.
- (H2) The function f is quasi-bounded in the second variable uniformly with respect to the first one, that is, for every compact subset $W \subset U$, f is bounded on $\mathbb{R}_+ \times \Lambda$, where $\Lambda = \mathcal{C}([-r, 0], W)$.
- (H3) For all $x \in \Omega$, the limit $F(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t, x)dt$ exists.
- (H4) The initial value problem
$$\dot{y}(t) = G(y(t)), \quad y(0) = \phi(0) \tag{2.6}$$
has a unique solution. Here $G : U \to \mathbb{R}^d$ is defined by $G(x) = F(\tilde{x})$ where, for each $x \in U$, $\tilde{x} \in \Omega$ is given by $\tilde{x}(\theta) = x$, $\theta \in [-r, 0]$.

As we will see later, condition (H2) is used essentially to prove continuability of solutions of (2.5) at least on every finite interval of time on which the solution of (2.6) is defined. For more details and a complete discussion about quasi-boundedness and its crucial role in the continuability of solutions of RFDEs, we refer the reader to Sections 2.3 and 3.1 of [18].

In assumption (H4) we anticipate the existence of solutions of (2.6). This will be justified a posteriori by Lemma 4.1 where we show that the function $F : \Omega \to \mathbb{R}^d$ in (H3), which is the average of the function f, is continuous. This implies the continuity of $G : U \to \mathbb{R}^d$ in (2.6) and then guaranties the existence of solutions.

The result below is our main theorem on averaging for RFDEs in normal form. It states closeness of solutions of (2.5) and (2.6) on finite time intervals.

Theorem 2.6. *Let $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a KBM-vector field. Let $\phi \in \Omega$. Let y be the solution of* (2.6) *and let $L \in J$, where J is the positive interval of definition of y. Then, for every $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$,* *every solution x of* (2.5) *is defined at least on* $[-\varepsilon r, L]$ *and satisfies* $|x(t)-y(t)| < \delta$ *for all* $t \in [0, L]$.

As in Section 2.1, we discuss now both periodic and almost periodic special cases. In each one, some of the conditions in Theorem 2.6 can be either removed or weakened. Let us consider the following (weak) condition which will be used hereafter instead of condition (H2):

(H5) The function f is quasi-bounded, that is, for every compact interval I of $\mathbb{R}_+$ and every compact subset $W \subset U$, f is bounded on $I \times \Lambda$, where $\Lambda = \mathcal{C}([-r, 0], W)$.

When f is periodic it is easy to see that condition (H1) derives from the continuity and the periodicity properties of f. On the other hand, by periodicity and condition (H5), condition (H2) is also satisfied. The average F in condition (H3) exists and is now given by formula (2.3) where T is the period. When f is almost periodic, condition (H5) imply condition (H2) and the average F is given by formula (2.4). Quite often the function f is a finite sum of periodic functions so that condition (H1) is satisfied. Hence we have the following result.

Corollary 2.7 (Periodic and Almost periodic cases)**.** *The conclusion of Theorem 2.6 holds when* $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ *is a continuous function which is periodic (or a sum of periodic functions) in the first variable and satisfies condition (H4) and (H5). It holds also when f is continuous, almost periodic in the first variable and satisfies conditions (H1), (H4) and (H5).*

Consider now the special case of equations with point-wise delay of the form

$$\dot{x}(t) = f(t/\varepsilon, x(t), x(t - \varepsilon r))$$

which is obtained, by letting $\tau = t/\varepsilon$, from equation

$$x'(\tau) = \varepsilon f(\tau, x(\tau), x(\tau - r)).$$

In this case, for both periodic and almost periodic functions, condition (H5) follows from the continuity property and then may be removed in Corollary 2.7.

Remark 2.8. In the results of the literature, for instance [26, Theorem 3, p. 206], f is assumed to be locally Lipschitz with respect to the second variable. Note that local Lipschitz condition with respect to the second variable implies condition (H1). It also assures the local existence for the solution of (2.5). But, in opposition to the case of ODEs, it is well known (see Sections 2.3 and 3.1 of [18] that without condition (H5) one cannot extend the solution x to finite time intervals where the solution y is defined in spite of the closeness of x and y. So, in the existing literature it is assumed that the solutions x and y are both defined at least on the same interval $[0, L]$.

Remark 2.9. In the introduction, we noticed that Lehman and Weibel [27] proposed to retain the delay in the averaged equation (1.6). At time scale $t = \varepsilon\tau$, their observation is that equation (2.5) is better approximated by the averaged RFDE

$$\dot{y}(t) = F(y_{t,\varepsilon}) \tag{2.7}$$

than by the averaged ODE (2.6). It should be noticed that the averaged RFDE (2.7) depends on the small parameter ε, which is not the case of the averaged equation (2.6).

2.3. **Averaging for fast oscillating RFDEs.** The aim here is to approximate the solutions of the initial value problem

$$\dot{x}(t) = f\left(t/\varepsilon, x_t\right), \quad x_0 = \phi, \tag{2.8}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$, $(t, x) \mapsto f(t, x)$, is a continuous function, $\Omega = \mathcal{C}([-r, 0], U)$, $r > 0$, where U is an open subset of $\mathbb{R}^d$, $\phi \in \Omega$ is an initial condition and $\varepsilon > 0$ is a small parameter.

Definition 2.10. A vector field $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is said to be a KBM-vector field if it is continuous and satisfies conditions (H1), (H2), (H3) in Definition 2.5 and the following condition

(H6) The initial value problem

$$\dot{y}(t) = F(y_t), \quad y_0 = \phi \tag{2.9}$$

has a unique solution.

The averaged initial value problem (2.9) associated to (2.8) is well defined since conditions (H1) and (H3) imply the continuity of the function $F : \Omega \to \mathbb{R}^d$ in (H3).

We may state our main result on averaging for fast oscillating RFDEs. It shows that the solution of (2.9) is an approximation of solutions of (2.8) on finite time intervals.

Theorem 2.11. *Let $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a KBM-vector field. Let $\phi \in \Omega$. Let y be the solution of (2.9) and let $L \in J$ be positive, where J is the interval of definition of y. Then, for every $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(L, \delta) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, every solution x of (2.8) is defined at least on $[-r, L]$ and satisfies $|x(t) - y(t)| < \delta$ for all $t \in [0, L]$.*

In the same manner as in Section 2.2 we have the following result corresponding to the periodic and almost periodic special cases.

Corollary 2.12 (Periodic and Almost periodic cases)**.** *The conclusion of Theorem 2.11 holds when $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ is a continuous function which is periodic (or a sum of periodic functions) in the first variable and satisfies condition (H5) and (H6). It holds also when f is continuous, almost periodic in the first variable and satisfies conditions (H1), (H5) and (H6).*

For fast oscillating equations with point-wise delay of the form

$$\dot{x}(t) = f(t/\varepsilon, x(t), x(t - r)),$$

in the periodic case as well as in the almost periodic one, condition (H5) derives from the continuity property and then can be removed in Corollary 2.12.

Remark 2.13. In the results of the classical literature, for instance, [26, Theorem 4, p. 210], it is assumed that f is locally Lipschitz with respect to the second variable and the existence of the solutions x and y on the same interval $[0, L]$ is required.

3. The Stroboscopic Method

3.1. **Internal Set Theory.** In this section we give a short tutorial of NSA. Additional information can be found in [5, 10, 35, 37]. *Internal Set Theory* (IST) is a theory extending ordinary mathematics, say ZFC (Zermelo-Fraenkel set theory with the axiom of choice), that axiomatizes (Robinson's) *nonstandard analysis* (NSA).

We adjoin a new undefined unary predicate *standard* (st) to ZFC. In addition to the usual axioms of ZFC, we introduce three others for handling the new predicate in a relatively *consistent* way. Hence *all theorems* of ZFC *remain valid* in IST. What is new in IST is an addition, not a change. In the external formulas, we use the following abbreviations [35] :

$$\forall^{\rm st} A \text{ for } \forall x(\mathrm{st}x \Rightarrow A) \quad \text{and} \quad \exists^{\rm st} A \text{ for } \exists x(\mathrm{st}x \,\&A).$$

A real number x is said to be *infinitesimal* if $|x| < a$ for all standard positive real numbers a and *limited* if $|x| \le a$ for some standard positive real number a. A limited real number which is not infinitesimal is said to be *appreciable*. A real number which is not limited is said to be *unlimited*. The notations $x \simeq 0$ and $x \simeq +\infty$ are used to denote, respectively, x is infinitesimal and x is unlimited positive.

Let D be a standard subset of some standard normed space E. A vector $x \in D$ is *infinitesimal* (resp. *limited, unlimited*) if its norm $\|x\|$ is infinitesimal (resp. limited, unlimited). Two elements $x, y \in D$ are said to be *infinitely close*, in symbols, $x \simeq y$, if $\|x-y\| \simeq 0$. An element $x \in D$ is said to be *near-standard* (resp. *near-standard in D*) if $x \simeq x_0$ for some standard $x_0 \in E$ (resp. for some standard $x_0 \in D$). The element x_0 is called the *standard part* or *shadow* of x. It is unique and is usually denoted by ${}^o x$. Note that when $E = \mathbb{R}^d$, each limited vector $x \in D$ is near-standard (but not necessary near-standard in D).

The *shadow* of a subset A of E, denoted by ${}^o A$, is the unique standard set whose standard elements are precisely those standard elements $x \in E$ for which there exists $y \in A$ such that $y \simeq x$. Note that ${}^o A$ is a closed subset of E and if $A \subset B$ then ${}^o A \subset {}^o B$. When A is standard, ${}^o A = \overline{A}$. We need the following result

Lemma 3.1. *Let U be a standard open subset of $\mathbb{R}^d$. Let A be near-standard in U (i.e. $\forall x \in A$, x is near-standard in U). Then, there exists a standard and compact set W such that $A \subset W \subset U$.*

Proof. For better readability we break the proof into three steps.

Step 1. We show that the shadow ${}^o A$ of A is compact in $\mathbb{R}^d$. ${}^o A$ is standard and closed. Let us prove that ${}^o A$ is bounded. Since A is near-standard, each element of A is limited. Hence $\forall x \in A\ \exists^{\rm st} a > 0\ |x| \le a$. By idealization, there exists a standard and finite set a' such that $\forall x \in A\ \exists a \in a'\ |x| \le a$. Let $a = \max(a')$. Then $\forall x \in A\ |x| \le a$. Hence $A \subset F = \{x \in \mathbb{R}^d : |x| \le a\}$, from where we deduce that ${}^o A \subset {}^o F = F$. This proves that ${}^o A$ is bounded. Finally, we conclude that ${}^o A$ is compact in $\mathbb{R}^d$ since it is closed and bounded.

Step 2. We show that ${}^o A \subset U$. Let x be standard in ${}^o A$. Let $y \in A$ such that $y \simeq x$. Since A is near-standard in U and x is the standard part of y, we have $x \in U$. By transfer we deduce that every $x \in {}^o A$ belongs to U. Thus ${}^o A \subset U$.

Step 3. We show that there exists a standard and compact set W such that $A \subset W \subset U$. Let W be the standard and compact neighborhood, around the standard and compact set ${}^o A$, given by $W = \{y \in \mathbb{R}^d / \ \exists x \in {}^o A : |x-y| \le \rho\}$, for some standard $\rho > 0$ chosen such that $W \subset U$. Let $x \in A$ and $x_0 \in U$, x_0 standard, such that $x \simeq x_0$. Thus $x_0 \in {}^o A$. Hence $x \in W$, since W is a standard neighborhood of x_0, which proves that $A \subset W$. □

Let $I \subset \mathbb{R}$ be some interval and $f : I \to \mathbb{R}^d$ be a function, with d standard. We say that f is *S-continuous at* a standard point $x \in I$ if, for all $y \in I$, $y \simeq x$ implies

$f(y) \simeq f(x)$, f is *S-continuous on* I if f is S-continuous at each standard point of I and f is *S-uniformly-continuous on* I if, for all $x, y \in I$, $x \simeq y$ implies $f(x) \simeq f(y)$. If I is standard and compact, S-continuity on I and S-uniform-continuity on I are the same. When f (and then I) is standard, the first definition is the same as saying that f is continuous at a standard point x, the second definition corresponds to the continuity of f on I and the last one to the uniform continuity of f on I.

We need the following result on S-uniformly-continuous functions on compact intervals of $\mathbb{R}$. This result is a particular case of the so-called "Continuous Shadow Theorem" [10].

Theorem 3.2. *Let I be a standard and compact interval of $\mathbb{R}$, D be a standard subset of $\mathbb{R}^d$ (with d standard) and $x : I \to D$ be a function. If x is S-uniformly-continuous on I and for each $t \in I$, $x(t)$ is near-standard in D then there exists a standard and continuous function $y : I \to D$ such that, for all $t \in I$, $x(t) \simeq y(t)$.*

The function y in Theorem 3.2 is unique. It is defined as the unique standard function y which, for t standard in I, is given by $y(t) = {}^o x(t)$. The function y is called the *standard part* or *shadow* of the function x and denoted by $y = {}^o x$.

3.2. **The Stroboscopic Method for ODEs.** Let U be a standard open subset of $\mathbb{R}^d$. Let $x_0 \in U$ be standard and let $F : \mathbb{R}_+ \times U \to \mathbb{R}^d$ be a standard and continuous function. Let I be some subset of $\mathbb{R}$ and let $x : I \to U$ be a function such that $0 \in I$ and $x(0) \simeq x_0$.

Definition 3.3 (F-Stroboscopic property)**.** A real number $t \geq 0$ is said to be an *instant of observation* if t is limited, $[0, t] \subset I$ and $x(s)$ is near standard in U for all $s \in [0, t]$. The function x is said to satisfy the F-Stroboscopic property on I if there exists $\mu > 0$ such that, for all instant of observation $t \in I$, there exists $t' \in I$ such that $\mu < t' - t \simeq 0$, $[t, t'] \subset I$, $x(s) \simeq x(t)$ for all $s \in [t, t']$ and $\frac{x(t')-x(t)}{t'-t} \simeq F(t, x(t))$.

Now, if a function satisfies the F-stroboscopic property on I, the result below asserts that it can be approximated by a solution of the ODE

$$\dot{y}(t) = F(t, y(t)), \quad y(0) = x_0. \tag{3.1}$$

Theorem 3.4 (Stroboscopic Lemma for ODEs)**.** *Suppose that*

(a) *The function x satisfies the F-stroboscopic property on I (Definition 3.3).*
(b) *The initial value problem* (3.1) *has a unique solution y. Let $J = [0, \omega)$, $0 < \omega \leq \infty$, be its maximal positive interval of definition.*

Then, for every standard $L \in J$, $[0, L] \subset I$ and the approximation $x(t) \simeq y(t)$ holds for all $t \in [0, L]$.

The proof of Stroboscopic Lemma for ODEs needs some results which are given in the section below.

3.2.1. *Preliminaries.*

Lemma 3.5. *Let $L > 0$ be limited such that $[0, L] \subset I$. Suppose that*

(i) *For all $t \in [0, L]$, $x(t)$ is near-standard in U.*
(ii) *There exist some positive integer N and some infinitesimal partition $\{t_n : n = 0, \dots, N+1\}$ of $[0, L]$ such that $t_0 = 0$, $t_N \leq L < t_{N+1}$ and, for $n = 0, \dots, N$, $t_{n+1} \simeq t_n$, $x(t) \simeq x(t_n)$ for all $t \in [t_n, t_{n+1}]$, and $\frac{x(t_{n+1})-x(t_n)}{t_{n+1}-t_n} \simeq F(t_n, x(t_n))$.*

Then the function x is S-uniformly-continuous on $[0, L]$.

Proof. Let $t, t' \in [0, L]$ and $p, q \in \{0, \dots, N\}$ be such that $t \le t'$, $t \simeq t'$, $t \in [t_p, t_{p+1}]$ and $t' \in [t_q, t_{q+1}]$. We write

$$x(t_q) - x(t_p) = \sum_{n=p}^{q-1} (x(t_{n+1}) - x(t_n)) = \sum_{n=p}^{q-1} (t_{n+1} - t_n)[F(t_n, x(t_n)) + \eta_n], \quad (3.2)$$

where

$$\eta_n = \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} - F(t_n, x_{t_n}) \simeq 0,$$

for all $n \in \{p, \dots, q-1\}$. Denote

$$\eta = \max_{p \le n \le q-1} |\eta_n| \quad \text{and} \quad m = \max_{p \le n \le q-1} |F(t_n, x(t_n))|.$$

We have $\eta \simeq 0$ and $m = |F(t_s, x(t_s))|$ for some $s \in \{p, \dots, q-1\}$. Since the function F is standard and continuous, and $(t_s, x(t_s))$ is near-standard in $\mathbb{R}_+ \times U$, $F(t_s, x(t_s))$ is near-standard. So is m. Hence (3.2) leads to the approximation

$$|x(t') - x(t)| \simeq |x(t_q) - x(t_p)| \le (m + \eta)(t_q - t_p) \simeq 0$$

which proves the S-uniform-continuity of x on $[0, L]$ and completes the proof. □

When we suppose L standard instead of limited, then more properties about the function x can be obtained and the following lemma can be written.

Lemma 3.6. *Let $L > 0$ be standard such that $[0, L] \subset I$. Suppose that conditions (i) and (ii) in Lemma 3.5 hold. Then the shadow $y = {}^o x$ of the function x is a solution of* (3.1). *Moreover, the approximation $x(t) \simeq y(t)$ holds for all $t \in [0, L]$.*

Proof. By Lemma 3.5 the function x is S-uniformly-continuous on $[0, L]$. From hypothesis (i) and Theorem 3.2 we deduce that y is continuous on $[0, L]$ and $y(t) \simeq x(t)$ for all $t \in [0, L]$. Let us show now that the function y is a solution of (3.1), that is, for all $t \in [0, L]$ it satisfies

$$y(t) = x_0 + \int_0^t F(s, y(s)) ds.$$

Let $t \in [0, L]$ be standard and let $n \in \{0, \dots, N\}$ be such that $t \in [t_n, t_{n+1}]$. Then

$$\begin{aligned} y(t) - x_0 \simeq x(t_n) - x(0) &= \sum_{k=0}^{n-1} (x(t_{k+1}) - x(t_k)) \\ &= \sum_{k=0}^{n-1} (t_{k+1} - t_k)[F(t_k, x(t_k)) + \eta_k], \end{aligned}$$

where $\eta_k \simeq 0$ for all $k \in \{0, \dots, n-1\}$. As F is standard and continuous, and $x(t_k) \simeq y(t_k)$ with $x(t_k)$ near-standard in U, we have $F(t_k, x(t_k)) = F(t_k, y(t_k)) + \beta_k$ where $\beta_k \simeq 0$ for all $k \in \{0, \dots, n-1\}$. Hence (3.3) gives

$$y(t) - x_0 \simeq \sum_{k=0}^{n-1} (t_{k+1} - t_k)[F(t_k, y(t_k)) + \beta_k + \eta_k] \simeq \int_0^t F(s, y(s)) ds.$$

Thus the approximation

$$y(t) \simeq x_0 + \int_0^t F(s, y(s)) ds \quad (3.3)$$

holds for all standard $t \in [0, L]$. Actually (3.3) is an equality since both sides of which are standard. We have thus, for all standard $t \in [0, L]$,

$$y(t) = x_0 + \int_0^t F(s, y(s))ds \tag{3.4}$$

and by transfer (3.4) holds for all $t \in [0, L]$. The proof is complete. □

The following statement is a consequence of Lemma 3.6.

Lemma 3.7. *Let $L > 0$ be standard such that $[0, L] \subset I$. Suppose that*

(i) *For all $t \in [0, L]$, $x(t)$ is near-standard in U.*
(ii) *The function x satisfies the F-stroboscopic property on $[0, L]$ (Definition 3.3).*

Then the function x is S-uniformly-continuous on $[0, L]$ and its shadow is a solution y of (3.1). So, we have $x(t) \simeq y(t)$ for all $t \in [0, L]$

Proof. First of all we have $\lambda \in A_\mu$ for all standard real number $\lambda > 0$, where A_μ is the subset of $\mathbb{R}$ defined by $A_\mu = \{\lambda \in \mathbb{R} \ / \ \forall t \in [0, L] \ \exists t' \in I : \mathcal{P}_\mu(t, t', \lambda)\}$ and $\mathcal{P}_\mu(t, t', \lambda)$ is the property

$$\mu < t'-t < \lambda,\ [t, t'] \subset I,\ \forall s \in [t, t']\ |x(s)-x(t)| < \lambda,\ \Big|\frac{x(t') - x(t)}{t' - t} - F(t, x(t))\Big| < \lambda.$$

By overspill there exists also $\lambda_0 \in A_\mu$ with $0 < \lambda_0 \simeq 0$. Thus, for all $t \in [0, L]$, there is $t' \in I$ such that $\mathcal{P}_\mu(t, t', \lambda_0)$ holds. Applying now the axiom of choice to obtain a function $c : [0, L] \to I$ such that $c(t) = t'$, that is, $\mathcal{P}_\mu(t, c(t), \lambda_0)$ holds for all $t \in [0, L]$. Since $c(t) - t > \mu$ for all $t \in [0, L]$, there are a positive integer N and an infinitesimal partition $\{t_n : n = 0, \dots, N + 1\}$ of $[0, L]$ such that $t_0 = 0$, $t_N \le L < t_{N+1}$ and $t_{n+1} = c(t_n)$. Finally, the conclusion follows from Lemma 3.6. □

3.2.2. *Proof of Theorem 3.4.* Let $L > 0$ be standard in J. Fix $\rho > 0$ to be standard such that the (standard) neighborhood around $\Gamma = \{y(t) : t \in [0, L]\}$ given by $W = \{z \in \mathbb{R}^d \ / \ \exists t \in [0, L] : |z - y(t)| \le \rho\}$ is included in U.

Let A be the subset of $[0, L]$ defined by

$$A = \{L_1 \in [0, L] \ / \ [0, L_1] \subset I \text{ and } \{x(t) : t \in [0, L_1]\} \subset W\}.$$

The set A is nonempty ($0 \in A$) and bounded above by L. Let L_0 be the upper bound of A and let $L_1 \in A$ be such that $L_0 - \mu < L_1 \le L_0$. Since $\{x(t) : t \in [0, L_1]\} \subset W$, the function x is near-standard in U on $[0, L_1]$. Hence, for any standard real number T such that $0 < T \le L_1$, hypotheses (i) and (ii) of Lemma 3.7 are satisfied. We have then

$$x(t) \simeq y(t), \quad \forall t \in [0, T], \tag{3.5}$$

where y is as in hypothesis (b). By overspill approximation (3.5) still holds for some $T \simeq L_1$. Next, by the S-uniform-continuity of x and the continuity of y on $[0, L_1]$ we have $x(t) \simeq x(T)$ and $y(t) \simeq y(T)$, for all $t \in [T, L_1]$. Combining this with (3.5) yields

$$x(t) \simeq y(t), \quad \forall t \in [0, L_1]. \tag{3.6}$$

Moreover, by hypothesis (a) there exists $L_1' \simeq L_1$ such that $L_1' > L_1+\mu$, $[L_1, L_1'] \subset I$ and $x(t) \simeq y(t)$ for all $t \in [L_1, L_1']$. By (3.6) we have $x(t) \simeq y(t)$ for all $t \in [0, L_1']$.

It remains to verify that $L \leq L_1'$. If this is not true, then $[0, L_1'] \subset I$ and $\{x(t) : t \in [0, L_1']\} \subset W$ imply $L_1' \in A$. This contradicts the fact that $L_1' > L_0$. So the proof is complete.

3.3. **The Stroboscopic Method for RFDEs.** Let $r \geq 0$ be standard. Let $\Omega = \mathcal{C}([-r, 0], U)$, where U is a standard open subset of $\mathbb{R}^d$ and $\phi \in \Omega$ be standard. Let $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a standard and continuous function. Let I be some subset of $\mathbb{R}$ and let $x : I \to U$ be a function such that $[-r, 0] \subset I$, $x_0 \simeq \phi$ and, for each $t \in I$, $t \geq 0$, $x_t \in \Omega$.

Definition 3.8 (F-Stroboscopic property)**.** A real number $t \geq 0$ is said to be an instant of observation if t is limited, $[0, t] \subset I$ and for all $s \in [0, t]$, $x(s)$ is near-standard in U and $F(s, x_s)$ is limited. The function x is said to satisfy the F-Stroboscopic property on I if there exists $\mu > 0$ such that, for all instant of observation t, there exists $t' \in I$ such that $\mu < t' - t \simeq 0$, $[t, t'] \subset I$, $x(s) \simeq x(t)$ for all $s \in [t, t']$ and $\frac{x(t')-x(t)}{t'-t} \simeq F(t, x_t)$.

In the same manner as in Section 2, for $r = 0$ we identify the Banach space $\mathcal{C}$ with $\mathbb{R}^d$ (and then Ω with U) and x_t with $x(t)$. By continuity property of F, if $x(s)$ is near-standard in U for all $s \in [0, t]$ then $F(s, x(s))$ is near-standard and then limited for all $s \in [0, t]$. So, Definition 3.3 is a particular case of Definition 3.8.

In the following result we assert that a function which satisfies the F-stroboscopic property on I can be approximated by a solution of the RFDE

$$\dot{y}(t) = F(t, y_t), \quad y_0 = \phi. \tag{3.7}$$

Theorem 3.9 (Stroboscopic Lemma for RFDEs)**.** *Suppose that*

(a) *The function x satisfies the F-stroboscopic property on I (Definition 3.8).*
(b) *The initial value problem* (3.7) *has a unique solution y. Let $J = [-r, \omega)$, $0 < \omega \leq \infty$, be its maximal interval of definition.*

Then, for every standard and positive $L \in J$, $[-r, L] \subset I$ and the approximation $x(t) \simeq y(t)$ holds for all $t \in [-r, L]$.

To prove Stroboscopic Lemma for RFDEs we need first to establish the following preliminary lemmas.

3.3.1. *Preliminaries.*

Lemma 3.10. *Let $L > 0$ be limited such that $[0, L] \subset I$. Suppose that*

(i) *For all $t \in [0, L]$, $x(t)$ is near-standard in U and $F(t, x_t)$ is limited.*
(ii) *There exist some positive integer N and some infinitesimal partition $\{t_n : n = 0, \dots, N+1\}$ of $[0, L]$ such that $t_0 = 0$, $t_N \leq L < t_{N+1}$ and, for $n = 0, \dots, N$, $t_{n+1} \simeq t_n$, $x(t) \simeq x(t_n)$ for all $t \in [t_n, t_{n+1}]$, and $\frac{x(t_{n+1})-x(t_n)}{t_{n+1}-t_n} \simeq F(t_n, x_{t_n})$.*

Then the function x is S-uniformly-continuous on $[0, L]$.

Proof. The proof is similar to the proof of Lemma 3.5. For $t, t' \in [0, L]$ with $t \leq t'$ and $t \simeq t'$ we have

$$x(t_q) - x(t_p) = \sum_{n=p}^{q-1} (x(t_{n+1}) - x(t_n)) = \sum_{n=p}^{q-1} (t_{n+1} - t_n)[F(t_n, x_{t_n}) + \eta_n], \tag{3.8}$$

where $p, q \in \{0, \dots, N\}$ are such that $t \in [t_p, t_{p+1}]$ and $t' \in [t_q, t_{q+1}]$ with $t_p \simeq t_q$. Let

$$\eta = \max_{p \le n \le q-1} |\eta_n| \quad \text{and} \quad m = \max_{p \le n \le q-1} |F(t_n, x_{t_n})|.$$

Since $\eta_n \simeq 0$ for $n = p, \dots, q-1$, we have $\eta \simeq 0$. Since $m = |F(t_s, x_{t_s})|$ for some $s \in \{p, \dots, q-1\}$, by hypothesis (i), m is limited. Hence (3.8) yields

$$|x(t') - x(t)| \simeq |x(t_q) - x(t_p)| \le (m + \eta)(t_q - t_p) \simeq 0$$

which shows the S-uniform-continuity of x on $[0, L]$. □

If the real number L in Lemma 3.10 is standard, instead of limited, one obtains more precise information about the function x.

Lemma 3.11. *Let $L > 0$ be standard such that $[0, L] \subset I$. Suppose that conditions (i) and (ii) in Lemma 3.10 are satisfied. Then the shadow $y = {}^o x$, of the function x is a solution of* (3.7) *and satisfies*

$$x(t) \simeq y(t), \quad \forall t \in [0, L]. \tag{3.9}$$

Proof. The proof is the same as the proof of Lemma 3.6. Notice that by (3.9) we obtain that for all $t \in [0, L]$, x_t is near-standard in Ω with $x_t \simeq y_t$. The details are omitted. □

From Lemma 3.11 we deduce the result below.

Lemma 3.12. *Let $L > 0$ be standard such that $[0, L] \subset I$. Suppose that*

- (i) *For all $t \in [0, L]$, $x(t)$ is near-standard in U and $F(t, x_t)$ is limited.*
- (ii) *The function x satisfies the F-stroboscopic property on $[0, L]$ (Definition 3.8).*

Then the function x is S-uniformly-continuous on $[0, L]$ and its shadow is a solution of (3.7) *and satisfies approximation* (3.9).

Proof. As in the proof of Lemma 3.7, we obtain a function $c : [0, L] \to I$ satisfying, for all $t \in [0, L]$,

$$\mu < c(t) - t \simeq 0, \ [t, c(t)] \subset I, \ \forall s \in [t, c(t)] \ x(s) \simeq x(t), \ \frac{x(c(t)) - x(t)}{c(t) - t} \simeq F(t, x_t).$$

If we let $t_0 = 0$ and $t_{n+1} = c(t_n)$ for $n = 0, \dots, N$, where the integer N is such that $t_N \le L < t_{N+1}$, the conclusion follows by applying Lemma 3.11. □

3.3.2. *Proof of Theorem 3.9.* Let $L > 0$ be standard in J and let $W_0 \subset U$ be the standard neighborhood around $\Gamma_0 = \{y(t) : t \in [0, L]\}$ defined by $W_0 = \{z \in \mathbb{R}^d \ / \ \exists t \in [0, L] : |z - y(t)| \le \rho_0\}$, where $\rho_0 > 0$ is a given standard real number.

Now, since F is standard and continuous, and $[0, L] \times \Gamma$ is a standard compact subset of $\mathbb{R}_+ \times \Omega$, where $\Gamma = \{y_t : t \in [0, L]\}$, there exists $\rho > 0$ and standard such that F is limited on $[0, L] \times W$, where $W \subset \Omega$ is the standard neighborhood around Γ given by $W = \{z \in \mathcal{C} \ / \ \exists t \in [0, L] : |z - y_t| \le \rho\}$. Consider the set

$$A = \{L_1 \in [0, L] \, / \, [0, L_1] \subset I, \, \{x(t) : t \in [0, L_1]\} \subset W_0 \text{ and } \{x_t : t \in [0, L_1]\} \subset W\}.$$

The set A is nonempty ($0 \in A$) and bounded above by L. Let $L_1 \in A$ such that $L_0 - \mu < L_1 \le L_0$, where $L_0 = \sup A$. Then

$$\{x(t) : t \in [0, L_1]\} \subset W_0 \quad \text{and} \quad [0, L_1] \times \{x_t : t \in [0, L_1]\} \subset [0, L] \times W.$$

Hence, for all $t \in [0, L_1]$, $x(t)$ is near-standard in U and $F(t, x_t)$ is limited.

Thus, for any standard real number T such that $0 < T \leq L_1$, hypotheses (i) and (ii) of Lemma 3.12 are satisfied. We have then

$$x(t) \simeq y(t), \quad \forall t \in [0,T],$$

where y is as in hypothesis (b). By overspill the property above holds for some $T \simeq L_1$. On the other hand, due to the S-uniform-continuity of x on $[0, L_1]$ and the continuity of y on the same interval, we have $x(t) \simeq x(T)$ and $y(t) \simeq y(T)$, for all $t \in [T, L_1]$, which achieves to prove that

$$x(t) \simeq y(t), \quad \forall t \in [0, L_1]. \tag{3.10}$$

By hypothesis (a), there exists some $L_1' \simeq L_1$ such that $L_1' > L_1 + \mu$, $[L_1, L_1'] \subset I$ and $x(t) \simeq y(t)$, for all $t \in [L_1, L_1']$. Combining with (3.10) yields

$$x(t) \simeq y(t), \quad \forall t \in [0, L_1']. \tag{3.11}$$

Now, taking into account that $x_0 \simeq \phi = y_0$, from (3.11) we deduce that $x_t \simeq y_t$ for all $t \in [0, L_1']$.
It remains to verify that $L \leq L_1'$. Assume that $L_1' \leq L$. Then $[0, L_1'] \subset I$, $\{x(t) : t \in [0, L_1']\} \subset W_0$ and $\{x_t : t \in [0, L_1']\} \subset W$. This implies $L_1' \in A$, which is absurd since $L_1' > L_0$. Thus $L_1' > L$. Finally, for any standard $L \in J$ we have shown that $x(t) \simeq y(t)$ for all $t \in [0, L] \subset [0, L_1']$. This completes the proof of the theorem.

4. Proofs of the Results

We prove Theorems 2.2, 2.6 and 2.11 within IST. By transfer it suffices to prove those results for *standard data* f, x_0 and ϕ. We will do this by applying Stroboscopic Lemma for ODEs (Theorem 3.4) in both cases of Theorems 2.2 and 2.6, and Stroboscopic Lemma for RFDEs (Theorem 3.9) in case of Theorem 2.11. For this purpose we need first to translate all conditions (C1) and (C2) in Section 2.1, and (H1), (H2) and (H3) in Section 2.2 into their external forms and then prove some technical lemmas.

Let U be a standard open subset of $\mathbb{R}^d$ and let $\Omega = \mathcal{C}([-r, 0], U)$, where $r \geq 0$ is standard. Let $f : \mathbb{R}_+ \times U \to \mathbb{R}^d$ or $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a standard and continuous function. The external formulations of conditions (C1) and (C2) are:

- (C1') $\forall^{\rm st} x \in U\ \forall x' \in U\ \forall t \in \mathbb{R}_+\ \big(x' \simeq x \Rightarrow f(t, x') \simeq f(t, x)\big)$.
- (C2') $\exists^{\rm st} F : U \to \mathbb{R}^d\ \forall^{\rm st} x \in U\ \forall R \simeq +\infty\ F(x) \simeq \frac{1}{R}\int_0^R f(t, x)dt$.

The external formulation of conditions (H1), (H2) and (H3) are, respectively:

- (H1') $\forall^{\rm st} x \in \Omega\ \forall x' \in \Omega\ \forall t \in \mathbb{R}_+\ \big(x' \simeq x \Rightarrow f(t, x') \simeq f(t, x)\big)$.
- (H2') $\forall^{\rm st} W$ compact, $W \subset U$, $\forall t \in \mathbb{R}_+$, $\forall x \in \Lambda = \mathcal{C}([-r, 0], W)$, $f(t, x)$ is limited.
- (H3') $\exists^{\rm st} F : \Omega \to \mathbb{R}^d\ \forall^{\rm st} x \in \Omega\ \forall R \simeq +\infty\ F(x) \simeq \frac{1}{R}\int_0^R f(t, x)dt$.

4.1. Technical Lemmas.

In Lemmas 4.1 and 4.2 below we formulate some properties of the average F of the function f defined in (C2) and (H3).

Lemma 4.1. *Suppose that the function f satisfies conditions (C1) and (C2) when $r = 0$ and conditions (H1) and (H3) when $r > 0$. Then the function F in (C2) or (H3) is continuous and satisfies*

$$F(x) \simeq \frac{1}{R}\int_0^R f(t, x)dt$$

for all $x \in U$ or $x \in \Omega$, x near-standard in U or in Ω and all $R \simeq +\infty$.

Proof. The proof is the same in both cases $r = 0$ and $r > 0$. So, there is no restriction to suppose that $r = 0$. Let $x, {}^o x \in U$ be such that ${}^o x$ is standard and $x \simeq {}^o x$. Fix $\delta > 0$ to be infinitesimal. By condition (C2) there exists $T_0 > 0$ such that

$$\Big|F(x) - \frac{1}{T}\int_0^T f(t,x)dt\Big| < \delta, \quad \forall T > T_0.$$

Hence there exists $T \simeq +\infty$ such that

$$F(x) \simeq \frac{1}{T}\int_0^T f(t,x)dt.$$

By condition (C1') we have $f(t,x) \simeq f(t, {}^o x)$ for all $t \in \mathbb{R}_+$. Therefore

$$F(x) \simeq \frac{1}{T}\int_0^T f(t,{}^o x)dt.$$

By condition (C2') we deduce that $F(x) \simeq F({}^o x)$. Thus F is continuous. Moreover, for all $T \simeq +\infty$, we have

$$F(x) \simeq F({}^o x) \simeq \frac{1}{T}\int_0^T f(t,{}^o x)dt \simeq \frac{1}{T}\int_0^T f(t,x)dt.$$

So, the proof is complete. □

Lemma 4.2. *Suppose that the function f satisfies conditions (C1) and (C2) when $r = 0$ and conditions (H1) and (H3) when $r > 0$. Let F be as in (C2) or (H3). Let $\varepsilon > 0$ be infinitesimal. Then, for all limited $t \in \mathbb{R}_+$ and all $x \in U$ or $x \in \Omega$, x near-standard in U or in Ω, there exists $\alpha = \alpha(\varepsilon, t, x)$ such that $0 < \alpha \simeq 0$, $\varepsilon/\alpha \simeq 0$ and*

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} f(\tau,x)d\tau \simeq TF(x), \quad \forall T \in [0,1].$$

Proof. The proof is the same in both cases $r = 0$ and $r > 0$. Let t be limited in $\mathbb{R}_+$ and let x be near-standard in Ω. We denote for short $g(r) = f(r,x)$. Let $T \in [0,1]$. We consider the following two cases.

Case 1: t/ε is limited. Let $\alpha > 0$ be such that $\varepsilon/\alpha \simeq 0$. If $T\alpha/\varepsilon$ is limited then we have $T \simeq 0$ and

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r)dr \simeq 0 \simeq TF(x).$$

If $T\alpha/\varepsilon \simeq +\infty$ we write

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r)dr = \big(T + \frac{t}{\alpha}\big)\frac{1}{t/\varepsilon + T\alpha/\varepsilon}\int_0^{t/\varepsilon + T\alpha/\varepsilon} g(r)dr - \frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon} g(r)dr.$$

By Lemma 4.1 we have

$$\frac{1}{t/\varepsilon + T\alpha/\varepsilon}\int_0^{t/\varepsilon + T\alpha/\varepsilon} g(r)dr \simeq F(x).$$

Since $\frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon} g(r)dr \simeq 0$ and $t/\alpha \simeq 0$, we have

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon + T\alpha/\varepsilon} g(r)dr \simeq TF(x).$$

This approximation is satisfied for all $\alpha > 0$ such that $\varepsilon/\alpha \simeq 0$. Choosing then α such that $0 < \alpha \simeq 0$ and $\varepsilon/\alpha \simeq 0$ gives the desired result.

Case 2: t/ε is unlimited. Let $\alpha > 0$. We have

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+T\alpha/\varepsilon} g(r)dr = T\eta(\alpha) + \frac{t}{\alpha}\left[\eta(\alpha) - \eta(0)\right], \tag{4.1}$$

where

$$\eta(\alpha) = \frac{1}{t/\varepsilon + T\alpha/\varepsilon}\int_0^{t/\varepsilon+T\alpha/\varepsilon} g(r)dr.$$

By Lemma 4.1 we have $\eta(\alpha) \simeq F(x)$ for all $\alpha \geq 0$. Return to (4.1) and assume that α is not infinitesimal. Then

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+T\alpha/\varepsilon} g(r)dr \simeq TF(x), \tag{4.2}$$

By overspill (4.2) holds for some $\alpha \simeq 0$ which can be chosen such that $\varepsilon/\alpha \simeq 0$. $\square$

4.2. **Proof of Theorem 2.2.** We need first to prove the following result which discuss some properties of solutions of a certain ODE needed in the sequel.

Lemma 4.3. *Let $\omega \in \mathbb{R}_+$, $\omega \simeq +\infty$. Let $g : \mathbb{R}_+ \times B(0,\omega) \to \mathbb{R}^d$ and $h : \mathbb{R}_+ \to \mathbb{R}^d$ be continuous functions, where $B(0,\omega) \subset \mathbb{R}^d$ is the ball of center 0 and radius ω. Let x_0 be limited in $\mathbb{R}^d$. Suppose that*

(i) *$g(t,x) \simeq h(t)$ holds for all $t \in [0,1]$ and all $x \in B(0,\omega)$.*
(ii) *$\int_0^t h(s)ds$ is limited for all $t \in [0,1]$.*

Then any solution x of the initial value problem

$$\dot{x} = g(t,x),\ t \in [0,1]; \quad x(0) = x_0$$

is defined and limited on $[0,1]$ and satisfies

$$x(t) \simeq x_0 + \int_0^t h(s)ds, \quad \forall t \in [0,1].$$

Proof. Assume that there exists $t \in [0,1]$ such that $x(t) \in B(0,\omega)$ and $x(t) \simeq \infty$. Then we have

$$x(t) = x_0 + \int_0^t g(s,x(s))ds \simeq x_0 + \int_0^t h(s)ds$$

whence, in view of hypothesis (ii), $x(t)$ is limited; this is a contradiction. Therefore $x(t)$ is defined and limited for all $t \in [0,1]$. $\square$

Let us now prove Theorem 2.2. Assume that x_0 and L are standard. To prove Theorem 2.2 is equivalent to show that, for every infinitesimal $\varepsilon > 0$, every solution x of (2.1) is defined at least on $[0,L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0,L]$. Fix $\varepsilon > 0$ to be infinitesimal and let $x : I \to U$ be a maximal solution of (2.1). We claim that x satisfies the F-stroboscopic property on I (Definition 3.3). To see this, let $t_0 \geq 0$ be an instant of observation of the stroboscopic method for ODEs; that is t_0 is limited, $t_0 \in I$ and $x(t)$ is near-standard in U for all $t \in [0,t_0]$. By Lemma 4.2 there exists $\alpha = \alpha(\varepsilon, t_0, x(t_0))$ such that $0 < \alpha \simeq 0$, $\varepsilon/\alpha \simeq 0$ and

$$\frac{\varepsilon}{\alpha}\int_{t_0/\varepsilon}^{t_0/\varepsilon+T\alpha/\varepsilon} f(t,x(t_0))dt \simeq TF(x(t_0)), \quad \forall T \in [0,1]. \tag{4.3}$$

Introduce the function

$$X(T) = \frac{x(t_0+\alpha T) - x(t_0)}{\alpha}, \quad T \in [0,1].$$

Differentiating and substituting the above into (2.1) gives, for $T \in [0,1]$,

$$\frac{dX}{dT}(T) = f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, x(t_0) + \alpha X(T)\Big). \tag{4.4}$$

By (C1') and Lemma 4.3 the function X, as a solution of (4.4), is defined and limited on $[0,1]$ and, for $T \in [0,1]$,

$$X(T) \simeq \int_0^T f\big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, x(t_0)\big)dt = \frac{\varepsilon}{\alpha}\int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x(t_0))dt.$$

Using now (4.3) this leads to the approximation

$$X(T) \simeq TF(x(t_0)), \quad \forall T \in [0,1].$$

Define $t_1 = t_0 + \alpha$ and set $\mu = \varepsilon$. Then $\mu < \alpha = t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$ and $x(t_0 + \alpha T) = x(t_0) + \alpha X(T) \simeq x(t_0)$ for all $T \in [0,1]$, that is, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, whereas

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(1) \simeq F(x(t_0)),$$

which shows the claim. Finally, by (C3) and Theorem 3.4, the solution x is defined at least on $[0, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$. The proof of Theorem 2.2 is complete.

4.3. **Proof of Theorem 2.6.** We start by showing the following auxiliary lemma, which is needed to prove Lemmas 4.5 and 4.6. Lemma 4.5 is used in the proof of Theorem 2.6 and Lemma 4.6 is used in the proof of Theorem 2.7.

Lemma 4.4. *Let U be a standard open subset of $\mathbb{R}^d$ and $\Omega = \mathcal{C}([-r,0],U)$, where $r \geq 0$ is standard. Let $g : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a continuous function. Suppose that*

- (A) *for all standard and compact subset $W \subset U$, all $t \in \mathbb{R}_+$ and all $x \in \Lambda = \mathcal{C}([-r,0]), W)$, $g(t,x)$ is limited.*

Let $\phi \in \Omega$ be standard. Let $x : I = [-\varepsilon r, b) \to U$, with $b > 0$, be a maximal solution of the initial value problem

$$\dot{x}(t) = g(t, x_{t,\varepsilon}), \quad x(t) = \phi(t/\varepsilon),\ t \in [-\varepsilon r, 0], \tag{4.5}$$

or $x : I = [-r, b) \to U$, with $b > 0$, be a maximal solution of the initial value problem

$$\dot{x}(t) = g(t, x_t), \quad x_0 = \phi. \tag{4.6}$$

Let $t_0 \in [0,b)$ be limited such that $x(t)$ is near-standard in U for all $t \in [0, t_0]$.

If $b \simeq t_0$ then $x(t')$ is not near-standard in U for some $t' \in [t_0, b)$.

Proof. The proof is the same for the solution x of the initial value problem (4.5) or the solution x of the initial value problem (4.6). We give the details in the first case. Assume by contradiction that $x(t)$ is near-standard in U for all $t \in [t_0, b)$.

Claim 1: $\sup_{t \in [t_0,b)} |g(t, x_{t,\varepsilon})|$ *is limited.* Since $x([-\varepsilon r, 0]) = \phi([-r, 0])$, $x(\theta)$ is near-standard in U for all $\theta \in [-\varepsilon r, 0]$. Thus $x(t)$ is near-standard in U for all $t \in [-\varepsilon r, b)$. By Lemma 3.1, there exists a standard and compact set W such that $x([-\varepsilon r, b)) \subset W \subset U$. We have $x_{t,\varepsilon} \in \Lambda = \mathcal{C}([-r,0], W)$ for all $t \in [t_0, b)$. By assumption (A), $\sup_{t \in [t_0,b)} |g(t, x_{t,\varepsilon})|$ is limited.

Claim 2: $\lim_{t\to b} x(t)$ *exists and is in* U. Let $(\tau_n)_n$ be a sequence in $[t_0, b)$ which converges to b. For $n, m \in \mathbb{N}$, we have

$$|x(\tau_m) - x(\tau_n)| = \Big|\int_{\tau_n}^{\tau_m} g(t, x_{t,\varepsilon})dt\Big| \leq |\tau_m - \tau_n| \sup_{t\in[t_0,b)} |g(t, x_{t,\varepsilon})|.$$

By Claim 1, the sequence $(x(\tau_n))_n$ is a Cauchy sequence, and hence, it converges to some $\xi \in \mathbb{R}^n$. Let $t \in [t_0, b)$ and $n \in \mathbb{N}$ such that $\tau_n \geq t$. By

$$|x(\tau_n) - x(t)| \leq \int_t^{\tau_n} |g(s, x_{s,\varepsilon})|ds \leq (\tau_n - t) \sup_{s\in[t_0,b)} |g(s, x_{s,\varepsilon})|,$$

we conclude that $\lim_{t\to b} x(t) = \xi$. By Claim 1, for each $t \in [t_0, b)$, we have

$$|x(t) - x(t_0)| \leq \int_{t_0}^{t} |g(s, x_{s,\varepsilon})|ds \leq (t - t_0) \sup_{s\in[t_0,t]} |g(s, x_{s,\varepsilon})| \simeq 0.$$

Since $x(t_0)$ is near-standard in U and $x(t) \simeq x(t_0)$, we have $\xi \in U$.

Now, one can extend x to a continuous function on $[-\varepsilon r, b]$ by setting $x(b) = \xi$. Consequently, $x_{b,\varepsilon} \in \Omega$ and then one can find a solution of (4.5) through the point $(b, x_{b,\varepsilon})$ to the right of b, which contradicts the noncontinuability hypothesis on x. So the proof is complete. □

Lemma 4.5. *Let* U *be a standard open subset of* $\mathbb{R}^d$ *and* $\Omega = \mathcal{C}([-r, 0], U)$, *where* $r \geq 0$ *is standard. Let* $g : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ *be a continuous function. Suppose that condition (A) in Lemma 4.4 holds. Let* $\phi \in \Omega$ *be standard and let* $x : I = [-\varepsilon r, b) \to U$, *with* $0 < b \leq \infty$, *be a maximal solution of the initial value problem* (4.5). *Let* $t_0 \in [0, b)$ *be limited such that* $x(t)$ *is near-standard in* U *for all* $t \in [0, t_0]$. *Then the solution* x *is such that*

(i) *the restriction of* x *to interval* $[0, t_0]$ *is S-uniformly-continuous.*

(ii) $x(t)$ *is defined and near-standard in* U *for all* $t \geq t_0$ *such that* $t \simeq t_0$.

Proof. (i) Let $t, t' \in [0, t_0]$ such that $t \leq t'$ and $t \simeq t'$. We have

$$|x(t') - x(t)| \leq \int_t^{t'} |g(s, x_{s,\varepsilon})|ds \leq (t' - t) \sup_{s\in[t,t']} |g(s, x_{s,\varepsilon})|.$$

By Lemma 3.1, there exists a standard and compact set W such that $x([-\varepsilon r, t_0]) \subset W \subset U$. We have $x_{s,\varepsilon} \in \Lambda = \mathcal{C}([-r, 0], W)$ for all $s \in [t, t']$. By assumption (A), $\sup_{s\in[t,t']} |g(s, x_{s,\varepsilon})|$ is limited so that $x(t') \simeq x(t)$. Thus x is S-uniformly-continuous on $[0, t_0]$.

(ii) Assume, by contradiction, that $x(t)$ is not defined or not near-standard in U for all $t \geq t_0$ such that $t \geq t_0$. If $x(t)$ is not defined for some $t \geq t_0$ such that $t \geq t_0$, then $b \simeq t_0$. By Lemma 4.4, we have $x(t')$ is not near-standard in U for some $t' \in [t_0, b)$. If $x(t)$ is not near-standard in U for some $t \geq t_0$ such that $t \geq t_0$, then obviously, we have $x(t')$ is not near-standard in U for some $t' \in [t_0, b)$. Hence, in both cases there exists $t' > t_0$, $t' \simeq t_0$ such that $x(t')$ is not near-standard in U. Now, by the continuity of x, there exists $t_1 \in [t_0, t']$ such that $x(t)$ is near-standard in U for all $t \in [t_0, t_1]$ and $x(t_1) \not\simeq x(t_0)$. By property (i) of the lemma, x is S-uniformly-continuous on $[0, t_1]$. Thus $x(t_1) \simeq x(t_0)$, which is a contradiction. □

The proof of Theorem 2.6 is as follows. Assume that ϕ and L are standard. To prove Theorem 2.6 is equivalent to prove that, when $\varepsilon > 0$ is infinitesimal, every solution x of (2.5) is defined at least on $[-\varepsilon r, L]$ and satisfies $x(t) \simeq y(t)$ for all

$t \in [0, L]$. Let $\varepsilon > 0$ be infinitesimal. Let x be a maximal solution of (2.5) defined on I, an interval of $\mathbb{R}$. Let $t_0 \geq 0$ be an instant of observation of the stroboscopic method for ODEs; that is t_0 is limited, $t_0 \in I$ and $x(t)$ is near-standard in U for all $t \in [0, t_0]$. Since $x(t_0)$ is near-standard so is $\tilde{x}^{t_0}$ where $\tilde{x}^{t_0} \in \Omega$ is defined by $\tilde{x}^{t_0}(\theta) = x(t_0)$ for all $\theta \in [-r, 0]$. Now we apply Lemma 4.2 to obtain some constant $\alpha = \alpha(\varepsilon, t_0, \tilde{x}^{t_0})$ such that $0 < \alpha \simeq 0$, $\varepsilon/\alpha \simeq 0$ and

$$\frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, \tilde{x}^{t_0})dt \simeq TF(\tilde{x}^{t_0}) = TG(x(t_0)), \quad \forall T \in [0, 1]. \tag{4.7}$$

By Lemma 4.5 $x(t)$ is defined and near-standard in U for all $t \geq t_0$ and $t \simeq t_0$. Hence one can consider the function

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \varepsilon\theta) - x(t_0)}{\alpha}, \quad \theta \in [-r, 0],\ T \in [0, 1].$$

We have, for $T \in [0, 1]$,

$$X(0, T) = \frac{x(t_0 + \alpha T) - x(t_0)}{\alpha}, \quad x_{t_0 + \alpha T, \varepsilon} = \tilde{x}^{t_0} + \alpha X(\cdot, T).$$

Note that, since $\tilde{x}^{t_0}(\theta) + \alpha X(\theta, T)$ is near-standard in U for all $\theta \in [-r, 0]$ and all $T \in [0, T]$, by Lemma 3.1 there exists a standard and compact set W such that $\{\tilde{x}^{t_0}(\theta) + \alpha X(\theta, T) : \theta \in [-r, 0],\ T \in [0, T]\} \subset W \subset U$. From this we deduce that $\tilde{x}^{t_0} + \alpha X(\cdot, T) \in \Lambda = \mathcal{C}([-r, 0], W)$ for all $T \in [0, 1]$. Differentiate now $X(0, \cdot)$ to obtain

$$\frac{\partial X}{\partial T}(0, T) = f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}T, \tilde{x}^{t_0} + \alpha X(\cdot, T)\Big), \quad T \in [0, T].$$

Integration between 0 and T, for $T \in [0, 1]$, yields

$$X(0, T) = \int_0^T f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0} + \alpha X(\cdot, t)\Big)dt. \tag{4.8}$$

Here after we will consider the following two cases:

Case 1: $T \in [0, \varepsilon r/\alpha]$. Using (H2') and taking into account that $\varepsilon r/\alpha \simeq 0$, (4.8) leads to the approximation

$$X(0, T) \simeq 0. \tag{4.9}$$

Case 2: $T \in [\varepsilon r/\alpha, 1]$. By Lemma 4.5 the restriction of x to interval $[0, t_0 + \alpha]$ is S-uniformly-continuous so that, for $\theta \in [-r, 0]$,

$$\alpha X(\theta, T) = x(t_0 + \alpha T + \varepsilon\theta) - x(t_0) \simeq 0,$$

since $t_0 + \alpha T + \varepsilon\theta \in [t_0, t_0 + \alpha] \subset [0, t_0 + \alpha]$ and $t_0 + \alpha T + \varepsilon\theta \simeq t_0$.

Return now to (4.8). For $T \in [0, 1]$, we write

$$X(0, T) = \Big(\int_0^{\varepsilon r/\alpha} + \int_{\varepsilon r/\alpha}^T\Big) f\big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0} + \alpha X(\cdot, t)\big)dt.$$

Using (4.9), (H1'), (H2') and (4.7), we thus get, for $T \in [0, 1]$,

$$\begin{aligned} X(0, T) &\simeq \int_{\varepsilon r/\alpha}^T f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0}\Big)dt \simeq \int_0^T f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon}t, \tilde{x}^{t_0}\Big)dt \\ &= \frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, \tilde{x}^{t_0})dt \simeq TG(x(t_0)). \end{aligned}$$

Defining $t_1 = t_0 + \alpha$ and setting $\mu = \varepsilon$, the following properties are true: $\mu < \alpha = t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t_0 + \alpha T) = x(t_0) + \alpha X(0, T) \simeq x(t_0)$ for all $T \in [0, 1]$, that is, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$, and

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0, 1) \simeq G(x(t_0)).$$

This proves that x satisfies the F-stroboscopic property on I (Definition 3.3). Taking (H4) into account, we finally apply Theorem 3.4 (Stroboscopic Lemma for ODEs) to obtain the desired result, that is, the solution x is defined at least on $[-\varepsilon r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$. The theorem is proved.

4.4. **Proof of Theorem 2.11.** We first prove the following result.

Lemma 4.6. *Let U be a standard open subset of $\mathbb{R}^d$ and $\Omega = \mathcal{C}([-r, 0], U)$, where $r \geq 0$ is standard. Let $g : \mathbb{R}_+ \times \Omega \to \mathbb{R}^d$ be a continuous function. Suppose that condition (A) in Lemma 4.4 holds. Let $\phi \in \Omega$ be standard and let $x : I = [-r, b) \to U$, with $0 < b \leq \infty$, be a maximal solution of the initial value problem* (4.6)*. Let $t_0 \in [0, b)$ be limited such that $x(t)$ is near-standard in U for all $t \in [0, t_0]$. Then*

- (i) *x is S-uniformly-continuous on $[-r, t_0]$ and x_t is near-standard in Ω for all $t \in [0, t_0]$.*
- (ii) *$x(t)$ is defined and near-standard in U for all $t \simeq t_0$, $t \geq t_0$.*

Proof. (i) We first note that x is S-uniformly-continuous on $[-r, 0]$, since it coincides with the standard and continuous function ϕ on the (standard) interval $[-r, 0]$. Now consider the interval $[0, t_0]$. Let $t, t' \in [0, t_0]$ such that $t \leq t'$ and $t \simeq t'$. Then

$$|x(t') - x(t)| \leq \int_t^{t'} |g(s, x_s)| ds \leq (t' - t) \sup_{s \in [t, t']} |g(s, x_s)|.$$

By Lemma 3.1, there exists a standard and compact set W such that $x([-r, t_0]) \subset W \subset U$. We have $x_{s,\varepsilon} \in \Lambda = \mathcal{C}([-r, 0], W)$ for all $s \in [t, t']$. By assumption (A), $\sup_{s \in [t, t']} |g(s, x_{s,\varepsilon})|$ is limited so that $x(t') \simeq x(t)$. Thus x is S-uniformly-continuous on $[0, t_0]$.

It remains to prove that x_t is near-standard in Ω for all $t \in [0, t_0]$. Since $x(t)$ is near-standard in U for all $t \in [-r, t_0]$ and S-uniformly-continuous on $[-r, t_0]$ then, for any fixed $t \in [0, t_0]$, $x_t(\theta)$ is near-standard in U for all $\theta \in [-r, 0]$ and x_t is S-uniformly-continuous on $[-r, 0]$. So, the result follows from Theorem 3.2.

(ii) Assume, by contradiction, that $x(t)$ is not defined or not near-standard in U for all $t \geq t_0$ such that $t \geq t_0$. If $x(t)$ is not defined for some $t \geq t_0$ such that $t \geq t_0$, then $b \simeq t_0$. By Lemma 4.4, we have $x(t')$ is not near-standard in U for some $t' \in [t_0, b)$. If $x(t)$ is not near-standard in U for some $t \geq t_0$ such that $t \geq t_0$, then obviously, we have $x(t')$ is not near-standard in U for some $t' \in [t_0, b)$. Hence, in both cases there exists $t' > t_0$, $t' \simeq t_0$ such that $x(t')$ is not near-standard in U.

By the continuity of x there exists $t_1 \in [t_0, t']$ such that $x(t)$ is near-standard in U for all $t \in [t_0, t_1]$ and $x(t_1) \not\simeq x(t_0)$. By property (i) of the lemma x is S-uniformly-continuous on $[-r, t]$. Since $t \simeq t_0$, it follows that $x(t) \simeq x(t_0)$, which is absurd. This proves that $x(t)$ is defined and near-standard in U for all $t \simeq t_0$. □

Let us prove Theorem 2.11. Let ϕ and L be standard. To prove Theorem 2.11 is equivalent to show that for every infinitesimal $\varepsilon > 0$, every solution x of (2.8) is defined at least on $[-r, L]$ and $x(t) \simeq y(t)$ holds for all $t \in [0, L]$. We fix $\varepsilon > 0$ to be infinitesimal and we let $x : I \to U$ to be a maximal solution of (2.8). We show that x

satisfies the F-stroboscopic property on I (Definition 3.8). Let $t_0 \geq 0$ be an instant of observation of the stroboscopic method for RFDEs; that is t_0 is limited, $t_0 \in I$ and $x(t)$ is near-standard in U and $F(x_t)$ is limited for all $t \in [0, t_0]$. According to (H2') Lemma 4.6 applies. Thus x_t is near-standard in Ω for all $t \in [0, t_0]$.

Now, applied to t_0 and x_{t_0}, Lemma 4.2 gives

$$\frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x_{t_0}) dt \simeq TF(x_{t_0}), \quad \forall T \in [0,1] \tag{4.10}$$

for some $\alpha = \alpha(\varepsilon, t_0, x_{t_0})$ such that $0 < \alpha \simeq 0$ and $\varepsilon/\alpha \simeq 0$.

Let $X : [-r, 0] \times [0, 1] \to \mathbb{R}^d$ be the function given by

$$X(\theta, T) = \frac{x(t_0 + \alpha T + \theta) - x(t_0 + \theta)}{\alpha}, \quad \theta \in [-r, 0], \ T \in [0, 1].$$

By Lemma 4.6 the function X is well defined. It satisfies, for $T \in [0, 1]$,

$$X(0, T) = \frac{x(t_0 + \alpha T) - x(t_0)}{\alpha}, \quad x_{t_0 + \alpha T} = x_{t_0} + \alpha X(\cdot, T).$$

Hence, for $T \in [0, 1]$,

$$\frac{\partial X}{\partial T}(0, T) = f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon} T, x_{t_0} + \alpha X(\cdot, T)\Big).$$

Solving this equation gives, for $T \in [0, 1]$,

$$X(0, T) = \int_0^T f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon} t, x_{t_0} + \alpha X(\cdot, t)\Big) dt. \tag{4.11}$$

According to Lemma 4.6 the solution x is S-uniformly-continuous on $[-r, t_0 + \alpha]$. Therefore, for $\theta \in [-r, 0]$ and $T \in [0, 1]$, $X(\theta, T)$ satisfies, since $t_0 + \alpha T + \theta \simeq t_0 + \theta$,

$$\alpha X(\theta, T) = x(t_0 + \alpha T + \theta) - x(t_0 + \theta) \simeq 0. \tag{4.12}$$

By (H1'), (4.12), (4.11) and (4.10), we have for all $T \in [0, 1]$ the approximation

$$X(0, T) \simeq \int_0^T f\Big(\frac{t_0}{\varepsilon} + \frac{\alpha}{\varepsilon} t, x_{t_0}\Big) dt = \frac{\varepsilon}{\alpha} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T\alpha/\varepsilon} f(t, x_{t_0}) dt \simeq TF(x_{t_0}).$$

Let $t_1 = t_0 + \alpha$ and set $\mu = \varepsilon$. The instant t_1 and the constant μ are such that: $\mu < \alpha = t_1 - t_0 \simeq 0$, $[t_0, t_1] \subset I$, $x(t_0 + \alpha T) = x(t_0) + \alpha X(0, T) \simeq x(t_0)$ for all $T \in [0, 1]$, that is, $x(t) \simeq x(t_0)$ for all $t \in [t_0, t_1]$ and

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = X(0, 1) \simeq F(x_{t_0}),$$

which is the F-stroboscopic property on I. Finally, using (H6) we get, by means of Theorem 3.9 (Stroboscopic Lemma for RFDEs), that the solution x is defined at least on $[-r, L]$ and satisfies $x(t) \simeq y(t)$ for all $t \in [0, L]$. So the proof is complete.

5. Discussion

In this paper we presented averaging results for ODEs and RFDEs. The results are proved in an unified manner for both ODEs and RFDEs, by using the Stroboscopic method which is a nonstandard tool in the asymptotic theory of differential equations. It should be noticed that the usual approaches for averaging make use of different tools for ODEs [6, 13, 38] and for RFDEs [14, 15, 16, 17, 18, 26, 27, 34, 44].

The results on RFDEs presented in this paper were obtained in [22], in which the Stroboscopic method for RFDEs was stated for the first time (see also [25, 42]). We recall that the stroboscopic method was initially proposed for ODEs. In this paper, we presented a slightly modified version of this method (see Theorem 3.4) and then extended it (see Theorem 3.9) in the context of RFDEs. Here, the stroboscopic method is slightly extended since the time t' in Definition 3.3 is assumed to exist only for those limited values of t for which $x(s)$ is near-standard in U for all $s \in [0, t]$ (see also Definition 6 in [42]). In the previous papers the time t' was assumed to exist for those limited values of t for which $x(t)$ is near-standard in U, without any assumption on $x(s)$ for $s \in [0, t]$ (see Theorem 1 in [41] or Definition 5 in [42]). In the stroboscopic method for RFDEs, the main assumption is that the time t' in Definition 3.8 is assumed to exist for those limited values of t for which $x(s)$ is near-standard in U and $F(s, x_s)$ is limited for all $s \in [0, t]$.

The Stroboscopic method for ODEs was first obtained by Callot and Reeb [9, 36]. For more information on the discovery of the stroboscopic method, and its use in averaging and asymptotic analysis the reader can consult [32, 40, 41]. Lemma 3.6 of the present paper is simply the Stroboscopic Theorem of Callot (see Theorem 1 in [9] or Lemma 1 in [41]). Theorem 3.4 is similar to Theorem 1 in [41]. Lemma 3.7 is similar to Lemma 2 in [41]. In the case $r = 0$, Lemma 4.1 is Lemma 4 in [41]; in the case $r > 0$ it is Lemma 4.3.6 in [22]. In the case $r = 0$, Lemma 4.2 is Lemma 2 in [39] or Lemma 5 in [41]; in the case $r > 0$ it is Lemma 4.3.7 in [22]. Lemma 4.3 is Lemma 1 in [39] or Theorem 2 in [41]. The results of Sections 3.3, 4.3 and 4.4 are extensions of some of the results in [22]. In our previous papers [19, 20, 21, 24], the stroboscopic method for RFDEs was not yet established and the results of averaging were obtained directly through evaluations of integrals.

The KBM theorem of averaging on ODEs obtained previously (see Theorems 1 in [39] or Theorem 6 in [41]) concerned nonstandard differential equations of the form $\dot{x} = g(t/\varepsilon, x)$ where $g(t, x)$ is a perturbation of a standard KBM vector field $f(t, x)$ satisfying the conditions in Definition 2.1. From a classical point of view this theorem includes the case of deformations of the form

$$\dot{x}(t) = f(t/\varepsilon, x(t), \varepsilon),$$

where the vector field $f(t, x, \varepsilon)$ depends also on ε. More precisely the theorem concerns all initial value problems in a neighborhood of a KBM vector field in a suitable topology (see Theorem 7 in [41]). The KBM theorems of averaging on ODEs and RFDEs obtained in this paper does not concern deformations of the above form in the case of ODEs or of the form

$$\dot{x}(t) = f(t/\varepsilon, x_t, \varepsilon) \quad \text{or} \quad \dot{x}(t) = f(t/\varepsilon, x_{t,\varepsilon}, \varepsilon),$$

in the case of RFDEs. However we think that our approach will permit also to consider these deformations. We leave this problem for future investigations. An other adapted version of the stroboscopic method for the case of ordinary differential inclusions has been given in [23] and used there to prove an averaging result.

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