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TWO MODIFICATIONS OF THE LEGGETT-WILLIAMS FIXED POINT THEOREM AND THEIR APPLICATIONS

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Abstract. This article presents two modifications of the Leggett-Williams fixed point theorem, and two applications of these results to a terminal and to a boundary value problem for ordinary differential equations.

1. Introduction

This article presents two modifications of the fixed point theorem named after Leggett and Williams [9], published in 1979, as well as two applications of these results to a terminal and to a boundary value problem for ordinary differential equations. The widely used version of the Leggett-Williams fixed point theorem provides conditions which ensure the existence of at least three fixed points. However this version of the theorem is only an extension of the original result, presented in the same paper by the authors, which is in turn a modification, but not a true extension, of the well-known Krasnoselskii fixed point theorem. The original result presents conditions which guarantee the existence of at least one fixed point, just like the Krasnoselskii fixed point theorem does. One of the differences between the two theorems, lies in the sets chosen to replace the order intervals, which are present in the class of fixed point theorems based on the classical Schauder theorem. Namely, at the Krasnoselskii fixed point theorem that set is of the form $\{x : a \leq \|x\| \leq b\}$, where $a, b \in (0, +\infty)$, whilst at the Leggett-Williams fixed point theorem that set is of the form $\{x : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$, where $a, b \in (0, +\infty)$ and α is a properly chosen functional. Although the two approaches are not easily comparable, using the functional α, which, by its definition, cannot coincide with the norm, allows for easier calculations and more versatile results. In this context, although the essence of the two theorems, and indeed lots of others based on them, is more or less the same, it is preferable to use the Leggett-Williams approach.

Here, following the ideas demonstrated in [2, 4, 5, 6], see also [1, 8, 12], the set $\{x : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$ is replaced by the set $\{x : a \leq \alpha(x) \text{ and } \beta(x) \leq b\}$, where β is another properly chosen functional. This modification is an extension of the original Leggett-Williams fixed point theorem, since the functional β can coincide with the norm. A closely related result can be found in [3]. Additionally,

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going a step further, the set $\{x : a \leq \alpha(x) \text{ and } \|x\| \leq b\}$ is replaced by the set $\{x : u \preceq A(x) \text{ and } B(x) \preceq v\}$, where A, B are operators and u, v are functions. Since we are not aware of any known proofs for these results, we use the fixed point index to prove them.

A specific example regarding a terminal value problem and another one regarding a boundary value problem, both for second order differential equations, are provided to demonstrate the applicability of the results and to pinpoint the advantages of their use. These problems are well-known in the literature, for example the terminal value problem is studied in [10, 11, 13] and the boundary value problem is studied in [7]. It is worth mentioning that, to the best of our knowledge, the results we obtain here are new.

2. Existence Theorems

Let $\mathbb{R}$ be the set of real numbers. For any interval $I \subseteq \mathbb{R}$ and any set $S \subseteq \mathbb{R}$, by $C(I, S)$ we denote the set of all continuous functions defined on I, which have values in S.

Lemma 2.1 (Fixed Point Index)**.** *Let Q be a retract of a Banach space E. For every open subset U of Q and every completely continuous map $A : \overline{U} \to Q$ which has no fixed points on ∂U (i.e. the boundary of U), there exists an integer $i(A, U, Q)$ satisfying the following*

(i) *if $A : \overline{U} \to U$ is a constant map, then $i(A, U, Q) = 1$.*
(ii) *if U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} \backslash (U_1 \cup U_2)$, then $i(A, U, Q) = i(A, U_1, Q) + i(A, U_2, Q)$, where $i(A, U_k, Q) = i(A|\overline{U_k}, U_k, Q)$, $k = 1, 2$.*
(iii) *if I is a compact interval in $\mathbb{R}$ and $h : I \times \overline{U} \to Q$ is a continuous map with relatively compact range such that $h(\lambda, x) \neq x$ for $(\lambda, x) \in I \times \partial U$, then $i(h(\lambda, \cdot), U, Q)$ is well-defined and independent of λ.*
(iv) *if $i(A, U, Q) \neq 0$, then A has at least one fixed point in U.*
(v) *if Q_1 is a retract of Q and $A(\overline{U}) \subset Q_1$, then $i(A, U, Q) = i(A, U \cap Q_1, Q_1)$, where $i(A, U \cap Q_1, Q_1) = i(A|\overline{U \cap Q_1}, U \cap Q_1, Q_1)$.*
(vi) *if V is open in U and A has no fixed points in $\overline{U} \backslash V$, then $i(A, U, Q) = i(A, V, Q)$.*

Definition 2.2. Let E be a real Banach space. A nonempty closed convex set $K \subseteq E$ is called a cone if it satisfies the following two conditions

(i) for every $x \in K$ and $\lambda \geq 0$ it holds that $\lambda x \in K$,
(ii) if $x \in K$ and $-x \in K$ then $x = 0$.

Every cone induces an ordering in E given by
$$x \leq y \quad \text{if and only if} \quad y - x \in K.$$

Definition 2.3. A map α is said to be a concave positive functional on a cone K of a real Banach space E if $\alpha : K \to [0, +\infty)$ is continuous and
$$\alpha(\lambda x + (1 - \lambda)y) \geq \lambda\alpha(x) + (1 - \lambda)\alpha(y)$$
for all $x, y \in K$ and $\lambda \in [0, 1]$. Similarly, we say that the map β is a convex positive functional on a cone K of a real Banach space E if $\beta : P \to [0, +\infty)$ is continuous and
$$\beta(\lambda x + (1 - \lambda)y) \leq \lambda\alpha(x) + (1 - \lambda)\alpha(y)$$

for all $x, y \in K$ and $\lambda \in [0,1]$.

Theorem 2.4 (Leggett-Williams [9]). *Let K be a cone in a Banach space E and define the sets*
$$K_{\epsilon_1} := \{x \in K : \|x\| \leq \epsilon_1\}, \text{ for } \epsilon_1 > 0$$
and
$$S(\beta, \epsilon_2, \epsilon_3) := \{x \in K : \epsilon_2 \leq \beta(x) \text{ and } \|x\| \leq \epsilon_3\},$$
for $\epsilon_3 > \epsilon_2 > 0$ and any concave positive functional β defined on the cone K, with $\beta(x) \leq \|x\|$.

Suppose that $c \geq b > a > 0$, α is a concave positive functional with $\alpha(x) \leq \|x\|$ and $A : K_c \to K$ is a completely continuous operator, such that

(i) *$\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$, and $\alpha(Ax) > a$ if $x \in S(\alpha, a, b)$,*
(ii) *$Ax \in K_c$ if $x \in S(\alpha, a, c)$,*
(iii) *$\alpha(Ax) > a$ for all $x \in S(\alpha, a, c)$ with $\|Ax\| > b$.*

Then A has a fixed point in $S(\alpha, a, c)$.

Theorem 2.5. *Let $I \subseteq \mathbb{R}$ and E be the Banach space of all bounded functions $x \in C(I, \mathbb{R})$ endowed with the norm*
$$\|x\| := \sup\{|x(t)| : t \in I\}, \ x \in C(I, \mathbb{R}).$$
Suppose that

- *K is a cone in E and for any $\epsilon > 0$*
$$K_\epsilon := \{x \in K : \|x\| \leq \epsilon\}$$
- *$0 < a < b < c < d$ are real numbers*
- *$T : K_d \to K$ is completely continuous*
- *α is a concave positive functional and β is a convex positive functional such that $\alpha(x) \leq \beta(x)$, $x \in K$*

and set
$$K_{\alpha,\beta}(a, b) := \{x \in K : \alpha(x) \geq a \text{ and } \beta(x) \leq b\}.$$
If

(i) *$\mathrm{int}_{K_d}(K_{\alpha,\beta}(a, b) \cap K_c) \neq \emptyset$ (i.e. the internal of $K_{\alpha,\beta}(a, b) \cap K_c$ with respect to K_d is non-empty) and for $x \in K_{\alpha,\beta}(a, b) \cap K_c$ it holds that*
$$\beta(Tx) < b, \quad \alpha(Tx) > a$$
(ii) *$Tx \in K_d$ for $x \in \overline{K_{\alpha,\beta}(a, b) \cap K_d}$*
(iii) *$\beta(Tx) < b$ and $\alpha(Tx) > a$ for $x \in K_{\alpha,\beta}(a, b) \cap K_d$ with $\|Tx\| > c$*

then the operator T has at least one fixed point
$$y \in \mathrm{int}_{K_d}(K_{\alpha,\beta}(a, b) \cap K_d)$$
i.e. $\alpha(y) > a$, $\beta(y) < b$, $\|y\| \leq d$.

Proof. Suppose that $x \in \partial(\mathrm{int}_{K_d}(K_{\alpha,\beta}(a, b) \cap K_d))$ is a fixed point of operator T. Then
$$\alpha(x) = a \text{ or } \beta(x) = b. \tag{2.1}$$
Also, obviously either
$$x \in K_{\alpha,\beta}(a, b) \cap K_c$$
or $\|x\| > c$.

- If $x \in K_{\alpha,\beta}(a,b) \cap K_c$ then according to assumption (i), we have
$$\beta(x) = \beta(Tx) < b \text{ and } \alpha(x) = \alpha(Tx) > a,$$
which contradicts (2.1).
- If $\|x\| > c$ then
$$\|Tx\| = \|x\| > c,$$
so according to assumption (iii), we have
$$\beta(Tx) < b \text{ and } \alpha(Tx) > a,$$
which contradicts (2.1).

So T has no fixed points on $\partial\big(\mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d)\big)$.

Since $\mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_c) \neq \emptyset$, we choose $x_0 \in \mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_c)$ and define the map
$$h : [0,1] \times \overline{\mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d)} \to K_d$$
by $h(t,x) = (1-t)Tx + tx_0$. It is easy to see that h is continuous and
$$h\left([0,1] \times \overline{\mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d)}\right)$$
is relatively compact.

Suppose there exists
$$(t,x) \in [0,1] \times \partial\big(\mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d)\big)$$
such that $h(t,x) = x$. Then
$$\alpha(x) = a \quad \text{or} \quad \beta(x) = b.$$

- If $\|Tx\| > c$ then by assumption (iii) we have
$$\beta(Tx) < b \quad \text{and} \quad \alpha(Tx) > a,$$
so
  - if $\alpha(x) = a$ then
$$\begin{aligned} \alpha(x) &= \alpha(h(t,x)) = \alpha((1-t)Tx + tx_0) \\ &\geq (1-t)\alpha(Tx) + t\alpha(x_0) > (1-t)a + ta \\ &= a \end{aligned}$$
which contradicts $\alpha(x) = a$.
  - if $\beta(x) = b$ then
$$\begin{aligned} \beta(x) &= \beta(h(t,x)) = \beta((1-t)Tx + tx_0) \\ &\leq (1-t)\beta(Tx) + t\beta(x_0) < (1-t)b + tb \\ &= b \end{aligned}$$
which contradicts $\beta(x) = b$.
- If $\|Tx\| \leq c$ then
$$\begin{aligned} \|x\| &= \|h(t,x)\| = \|(1-t)Tx + tx_0\| \\ &\leq (1-t)\|Tx\| + t\|x_0\| < (1-t)c + tc \\ &= c, \end{aligned}$$
therefore by assumption (i) we have
$$\beta(Tx) < b \quad \text{and} \quad \alpha(Tx) > a,$$
so

– if $\alpha(x) = a$ then
$$\begin{aligned}\alpha(x) &= \alpha(h(t,x)) = \alpha((1-t)Tx + tx_0) \\ &\geq (1-t)\alpha(Tx) + t\alpha(x_0) > (1-t)a + ta \\ &= a\end{aligned}$$
which contradicts $\alpha(x) = a$.

– if $\beta(x) = b$ then
$$\begin{aligned}\beta(x) &= \beta(h(t,x)) = \beta((1-t)Tx + tx_0) \\ &\leq (1-t)\beta(Tx) + t\beta(x_0) < (1-t)b + tb \\ &= b\end{aligned}$$
which contradicts $\beta(x) = b$.

Consequently, for each
$$(t,x) \in [0,1] \times \partial \left(\mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d)\right)$$
it holds that $h(t,x) \neq x$. So, by Lemma 2.1, we have
$$i\left(T, \mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d), K_d\right) = i\left(x_0, \mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d), K_d\right) = 1.$$
Hence, operator T has at least one fixed point $y \in \mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_d)$, i.e.
$$\alpha(y) > a, \ \beta(y) < b, \quad \|y\| \leq d.$$
□

Definition 2.6. Let $I \subseteq \mathbb{R}$ be bounded and $f, g \in C(I,\mathbb{R})$. We define the relation $\preceq$ by
$$f \preceq g \quad \text{if and only if} \quad f(t) \leq g(t), \ \forall t \in I$$
and the relation $\prec$ by
$$f \prec g \quad \text{if and only if} \quad f(t) < g(t), \ \forall t \in I.$$

Definition 2.7. Let $I \subseteq \mathbb{R}$ be bounded and $A \in C(C(I,\mathbb{R}), C(I,\mathbb{R}))$. We say that operator A satisfies

- property P1 if and only if
$$(1-t)A(x) + tA(y) \preceq A((1-t)x + ty), \quad \forall x, y \in C(I,\mathbb{R}), \ \forall t \in [0,1].$$
- property P2 if and only if
$$A((1-t)x + ty) \preceq (1-t)A(x) + tA(y), \quad \forall x, y \in C(I,\mathbb{R}), \ \forall t \in [0,1].$$
- property P3 if and only if
$$Ax(t) \geq 0, \ \forall x \in C(I,\mathbb{R}), \quad \forall t \in [0,1].$$

Theorem 2.8. *Let $I \subseteq \mathbb{R}$ be bounded and E be the Banach space of all bounded functions $x \in C(I,\mathbb{R})$ endowed with the norm*
$$\|x\| := \sup\{|x(t)| : t \in I\}, \quad x \in C(I,\mathbb{R}).$$
Suppose that

- *K is a cone in E and for any $\epsilon > 0$*
$$K_\epsilon := \{x \in K : \|x\| \leq \epsilon\}$$
- *$0 < c < d$ are real numbers*
- *$T : K_d \to K$ is completely continuous*

- *A is an operator satisfying properties P1 and P3, and B is an operator satisfying properties P2 and P3, such that $A(x) \preceq B(x)$, $x \in K$*
- *$u, v \in C(I, [0, +\infty))$ with $u \prec v$*

and set

$$K_{A,B}(u, v) := \{x \in K : u \preceq A(x) \text{ and } B(x) \preceq v\}.$$

If

(i) *$\operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_c) \neq \emptyset$ (i.e. the internal of $K_{A,B}(u, v) \cap K_c$ with respect to K_d is non-empty) and for $x \in K_{A,B}(u, v) \cap K_c$ it holds that*

$$B(Tx) \prec v \quad \text{and} \quad u \prec A(Tx)$$

(i) *$Tx \in K_d$ for $x \in \overline{K_{A,B}(u, v) \cap K_d}$*

(i) *$B(Tx) \prec v$ and $u \prec A(Tx)$ for $x \in K_{A,B}(u, v) \cap K_d$ with $\|Tx\| > c$*

then the operator T has at least one fixed point

$$y \in \operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_d)$$

i.e.

$$u \prec A(y), \quad B(y) \prec v, \quad \|y\| \leq d.$$

Proof. Suppose that $x \in \partial\left(\operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_d)\right)$ is a fixed point of operator T. Then

$$Ax(t_0) = u(t_0) \text{ or } Bx(t_0) = v(t_0), \text{ for some } t_0 \in I. \tag{2.2}$$

Also, obviously either

$$x \in K_{A,B}(u, v) \cap K_c$$

or $\|x\| > c$.

- If $x \in K_{A,B}(u, v) \cap K_c$ then according to assumption (i), we have
$$B(x) = B(Tx) \prec v \text{ and } u \prec A(Tx) = A(x),$$
which contradicts (2.2).
- If $\|x\| > c$ then
$$\|Tx\| = \|x\| > c,$$
so according to assumption (iii), we have
$$B(Tx) \prec v \text{ and } u \prec A(Tx),$$
which contradicts (2.2).

So T has no fixed points on $\partial\left(\operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_d)\right)$.

Since $\operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_c) \neq \emptyset$, we choose $x_0 \in \operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_c)$ and define the map

$$h : [0, 1] \times \overline{\operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_d)} \to K_d$$

by

$$h(t, x) = (1 - t)Tx + tx_0.$$

It is easy to see that h is continuous and

$$h\left([0, 1] \times \overline{\operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_d)}\right)$$

is relatively compact.

Suppose there exists

$$(t, x) \in [0, 1] \times \partial\left(\operatorname{int}_{K_d}(K_{A,B}(u, v) \cap K_d)\right)$$

such that $h(t,x)=x$. Then

$$Ax(t_0)=u(t_0) \quad \text{or} \quad Bx(t_0)=v(t_0), \text{ for some } t_0 \in I.$$

- If $\|Tx\|>c$ then by assumption (iii) we have

  $$B(Tx) \prec v \quad \text{and} \quad u \prec A(Tx),$$

  so
  - if $Ax(t_0)=u(t_0)$ then

    $$\begin{aligned} A(x) &= A(h(t,x)) = A((1-t)Tx+tx_0) \\ &\succeq (1-t)A(Tx)+tA(x_0) \succ (1-t)u+tu \\ &= u \end{aligned}$$

    which contradicts $Ax(t_0)=u(t_0)$.
  - if $Bx(t_0)=v(t_0)$ then

    $$\begin{aligned} B(x) &= B(h(t,x)) = B((1-t)Tx+tx_0) \\ &\preceq (1-t)B(Tx)+tB(x_0) \prec (1-t)v+tv \\ &= v \end{aligned}$$

    which contradicts $Bx(t_0)=v(t_0)$.
- If $\|Tx\| \le c$ then

  $$\begin{aligned} \|x\| &= \|h(t,x)\| = \|(1-t)Tx+tx_0\| \\ &\le (1-t)\|Tx\|+t\|x_0\| < (1-t)c+tc \\ &= c, \end{aligned}$$

  therefore by assumption (i) we have

  $$B(Tx) \prec v \quad \text{and} \quad u \prec A(Tx),$$

  so
  - if $Ax(t_0)=u(t_0)$ then

    $$\begin{aligned} A(x) &= A(h(t,x)) = A((1-t)Tx+tx_0) \\ &\succeq (1-t)A(Tx)+tA(x_0) \succ (1-t)u+tu \\ &= u \end{aligned}$$

    which contradicts $Ax(t_0)=u(t_0)$.
  - if $Bx(t_0)=v(t_0)$ then

    $$\begin{aligned} B(x) &= B(h(t,x)) = B((1-t)Tx+tx_0) \\ &\preceq (1-t)B(Tx)+tB(x_0) \prec (1-t)v+tv \\ &= v \end{aligned}$$

    which contradicts $Bx(t_0)=v(t_0)$.

Consequently, for each

$$(t,x) \in [0,1] \times \partial\left(\operatorname{int}_{K_d}(K_{A,B}(u,v)\cap K_d)\right)$$

it holds that $h(t,x) \neq x$. So, by Lemma 2.1, we have

$$i\left(T, \operatorname{int}_{K_d}(K_{A,B}(u,v)\cap K_d), K_d\right) = i\left(x_0, \operatorname{int}_{K_d}(K_{A,B}(u,v)\cap K_d), K_d\right) = 1.$$

Hence, operator T has at least one fixed point $y \in \operatorname{int}_{K_d}(K_{A,B}(u,v)\cap K_d)$, i.e.

$$u \prec A(y), \quad B(y) \prec v, \quad \|y\| \le d.$$

□

3. An Application of Theorem 2.5 to a Terminal Value Problem

For any interval $I \subseteq \mathbb{R}$ and any set $S \subseteq \mathbb{R}$, by $C^2(I,S)$ we denote the set of all twice continuously differentiable functions defined on I, which have values in S. Let J be an unbounded interval in $\mathbb{R}^+$. It is easy to see that the set

$$BC^2(J,\mathbb{R}^+) := \{x \in C^2(J,\mathbb{R}^+) : x \text{ is bounded }\}$$

endowed with the norm

$$\|x\| := \sup_{t\in J} |x(t)|, \quad x \in BC^2(J,\mathbb{R}^+),$$

is a Banach space. We are looking for functions $x \in BC^2(J,\mathbb{R}^+)$ which satisfy the second order differential equation

$$x''(t) + f(t,x(t)) = 0, \quad t \in J \tag{3.1}$$

as well as the terminal condition

$$\lim_{t\to+\infty} x(t) = \xi, \tag{3.2}$$

where $\xi \in \mathbb{R}^+$, $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and

$$\int_t^{+\infty}\int_s^{+\infty} f(\sigma,y(\sigma))\,d\sigma\,ds \le \xi, \quad \text{for every } t \in J \text{ and every } y \in BC^2(J,\mathbb{R}^+).$$

Define the following set K, which is a cone in $BC^2(J,\mathbb{R}^+)$

$$K := \{x \in BC^2(J,\mathbb{R}^+) : x(t) \ge 0,\ x'(t) \ge 0 \text{ and } x''(t) \le 0,\ \text{for all } t \in J\}.$$

Lemma 3.1. *Let $\epsilon > 0$. A function $x \in K_\epsilon$ is a solution of the terminal value problem* (3.1)–(3.2) *if and only if x is a fixed point of the operator $T : K_d \to C(J,\mathbb{R}^+)$ defined by the formula*

$$Ty(t) := \xi - \int_t^{+\infty}\int_s^{+\infty} f(\sigma,y(\sigma))\,d\sigma\,ds, \quad t \in J.$$

Definition 3.2. A set U of real valued functions defined on the interval J is called equiconvergent at ∞ if all functions in U are convergent in $\mathbb{R}$ at the point ∞ and, in addition, for each $\epsilon > 0$, there exists $T \equiv T(\epsilon) > 0$ such that, for all functions $u \in U$, it holds

$$|u(t) - \lim_{s\to\infty} u(s)| < \epsilon, \quad \text{for every } t \ge T.$$

Lemma 3.3. *Let U be an equicontinuous and uniformly bounded subset of the Banach space $BC^2(J,\mathbb{R})$. If U is equiconvergent at ∞, it is also relatively compact.*

Lemma 3.4. *Let $\epsilon > 0$. Operator T is completely continuous and maps K_ϵ into K.*

Theorem 3.5. *Let $r_1, r_2 \in J$, with $r_1 < r_2$, and $0 < a < b < c < d$, with $d \ge \xi$. Also, define the functionals $\alpha(x) = x'(r_2)$, $x \in K$, and $\beta(x) = x'(r_1)$, $x \in K$. Suppose that for any $x \in K_{\alpha,\beta}(a,b) \cap K_c$ as well as for any $x \in K_{\alpha,\beta}(a,b) \cap K_d$ with $\|Tx\| > c$, it holds that*

$$\int_{r_1}^{+\infty} f(s,x(s))ds < b \quad \textit{and} \quad \int_{r_2}^{+\infty} f(s,x(s))ds > a.$$

Then the terminal value problem (3.1)–(3.2) *has at least one solution* y *such that*

$$y'(r_2) > a, \quad y'(r_1) < b, \quad \|y\| \le d.$$

Proof. Obviously α is a concave positive functional and β is a convex positive functional such that

$$\alpha(x) \le \beta(x), \quad x \in K.$$

It is easy to see that any function $x \in K$ with

$$x(t) = \lambda t, \quad t \in [r_1, r_2],$$

where $\lambda \in (a, b)$, and such that $\|x\| < c$, belongs to $\mathrm{int}_{K_d}(K_{\alpha,\beta}(a,b) \cap K_c)$. Also, since $d \ge \xi$, it is obvious that

$$Tx \in K_d, \quad \forall x \in \overline{K_{\alpha,\beta}(a,b) \cap K_d}.$$

The rest of the proof is easy. □

Corollary 3.6. *The terminal value problem*

$$x''(t) + \frac{1}{t^3 + x(t)} = 0, \quad t \in [1, +\infty), \tag{3.3}$$

$$\lim_{t \to +\infty} x(t) = 2. \tag{3.4}$$

has at least one non-negative solution y *such that*

$$y'(3) > \frac{1}{20}, \quad y'(2) < \frac{1}{6}, \quad \sup_{t \in [1,+\infty)} |y(t)| \le 2.$$

Proof. The result follows from Theorem 3.5, for $r_1 = 2$, $r_2 = 3$, $a = \frac{1}{20}$, $b = \frac{1}{6}$, $c = 1$, $d = 2$ and $\xi = 2$. We notice that, for any $t \in [1, +\infty)$ and $x \in K_d$, it holds

$$\int_t^{+\infty} \int_s^{+\infty} \frac{1}{\sigma^3 + x(\sigma)} \, d\sigma \, ds \le \int_t^{+\infty} \int_s^{+\infty} \frac{1}{\sigma^3} \, d\sigma \, ds = \frac{1}{2t} \le 2 = \xi,$$

and, for any $x \in K_d$, we have

$$\int_2^{+\infty} \frac{1}{s^3 + x(s)} ds \le \int_2^{+\infty} \frac{1}{s^3} ds < \frac{1}{6}$$

and

$$\int_3^{+\infty} \frac{1}{s^3 + x(s)} ds \ge \int_3^{+\infty} \frac{1}{s^3 + 2} ds > \frac{1}{20}.$$

□

4. An Application of Theorem 2.8 to a Boundary Value Problem

Consider the second order boundary value problem

$$x''(t) - f(t, x(t)) = 0, \quad t \in [0, 1], \tag{4.1}$$

$$x(0) = 0, \quad x'(1) = a x'(0) \tag{4.2}$$

where $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $a > 1$. Define the following set K, which is a cone in $C([0,1], \mathbb{R}^+)$,

$$K := \{x \in C([0,1], \mathbb{R}^+) : x(t) \ge 0, \ \forall t \in [0,1] \text{ and } x'(t) \ge 0, \ \forall t \in [0,1]\}.$$

Lemma 4.1. *Let $\epsilon > 0$. A function $x \in K_\epsilon$ is a solution of the boundary value problem (4.1)–(4.2) if and only if x is a fixed point of the operator $T : K_\epsilon \to C([0,1], \mathbb{R}^+)$ defined by the formula*

$$Ty(t) := \frac{t}{a-1}\int_0^1 f(s, y(s))ds + \int_0^t \int_0^s f(\sigma, y(\sigma))\, d\sigma\, ds, \quad t \in [0,1].$$

Lemma 4.2. *Let $\epsilon > 0$. Operator T is completely continuous and maps K_ϵ into K.*

Theorem 4.3. *Let $u, v \in C([0,1], \mathbb{R}^+)$ with $u \prec v$, $u'(t) \geq 0$, $\forall t \in [0,1]$, and $v'(t) \geq 0$, $\forall t \in [0,1]$. Also, define the operators A, B by*

$$A(x) = B(x) = x', \quad x \in K.$$

and let $0 < c < d$, with

$$\sup_{t\in[0,1]} Tx(t) < d, \quad \forall x \in \overline{K_{A,B}(u,v) \cap K_d}.$$

Suppose that for any $x \in K_{A,B}(u,v) \cap K_c$ as well as for any $x \in K_{A,B}(u,v) \cap K_d$ with $\|Tx\| > c$, it holds that

$$u(t) < \frac{1}{a-1}\int_0^1 f(s, x(s))ds + \int_0^t f(s, x(s))ds < v(t), \quad t \in [0,1].$$

Then the boundary value problem (4.1)–(4.2) has at least one solution y such that

$$u \prec y', \ y' \prec v \text{ and } \|y\| \leq d.$$

Proof. It is easy to see that $\operatorname{int}_{K_d}(K_{A,B}(u,v) \cap K_c) \neq \emptyset$ and

$$Tx \in K_d, \quad \forall x \in \overline{K_{A,B}(u,v) \cap K_d}.$$

The rest of the proof is easy. □

Corollary 4.4. The boundary value problem

$$x''(t) - (1 + \sin^2(x(t))) = 0, \quad t \in [0,1], \tag{4.3}$$

$$x(0) = 0, \quad x'(1) = 2x'(0). \tag{4.4}$$

has at least one non-negative solution y such that

$$t < y(t) < t^2 + 2t, \ \forall t \in [0,1], \quad \text{and} \quad \sup_{t\in[0,1]} |y(t)| \leq 3.$$

Proof. The result follows from Theorem 4.3, for $c = 1$, $d = 3$, $u(t) = 1$, $t \in [0,1]$, and $v(t) = 2(t+1)$, $t \in [0,1]$. We notice that, for any $t \in [0,1]$, it holds

$$1 < \int_0^1 (1 + \sin^2(x(s)))ds + \int_0^t (1 + \sin^2(x(s)))ds$$

and

$$\int_0^1 (1 + \sin^2(x(s)))ds + \int_0^t (1 + \sin^2(x(s)))ds < 2(t+1).$$

Also, for any $x \in K_d$, we have

$$\int_0^t \int_0^s (1 + \sin^2(x(\sigma)))\, d\sigma\, ds \leq 2\int_0^t \int_0^s d\sigma\, ds = t^2 \leq 1, \quad \forall t \in [0,1],$$

and
$$\frac{t}{2}\int_0^1 \left(1+\sin^2(x(s))\right) ds \le t\int_0^1 ds = t \le 1, \quad \forall t \in [0,1],$$
so
$$\sup_{t\in[0,1]} Tx(t) \le 2 < d.$$
$\square$

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