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EXISTENCE OF RADIAL POSITIVE SOLUTIONS VANISHING AT INFINITY FOR ASYMPTOTICALLY HOMOGENEOUS SYSTEMS

ALI DJELLIT, MOHAND MOUSSAOUI, SAADIA TAS

Abstract. In this article we study elliptic systems called asymptotically homogeneous because their nonlinearities may not have polynomial growth. Using the Gidas-Spruck Blow-up method, we obtain a priori estimates, and then using Leray-Schauder topological degree theory, we obtain radial positive solutions vanishing at infinity.

1. Introduction

We study asymptotically homogeneous systems involving nonlinearities which may not have polynomial growth. More precisely, we establish the existence of radial positive solutions vanishing at infinity, the so-called fundamental states, for systems of the form

$$\begin{aligned} -\Delta_p u &= a_{11}(|x|)f_{11}(u) + a_{12}(|x|)f_{12}(v) \quad \text{in } \mathbb{R}^N \\ -\Delta_q v &= a_{21}(|x|)f_{21}(u) + a_{22}(|x|)f_{22}(v) \quad \text{in } \mathbb{R}^N \end{aligned} \tag{1.1}$$

Here $1 < p$, $q < N$, the coefficients a_{ij} $(i, j = 1, 2)$ are positive continuous real-valued functions and f_{ij} $(i, j = 1, 2)$ belong to asymptotically homogeneous class of functions. Such functions have been introduced later by Garcia-Huidobro, Guerra, Manasevich, Schmitt and Ubilla [3, 4, 5] to deal with existence problems of quasilinear elliptic partial differential equations. Briefly this corresponds to a class of nonhomogeneous functions which are not asymptotically equivalent to any strength nevertheless they possess a suitable homogeneous behavior at the infinity or at the origin. The system (1.1) being nonvariational , a first step consists in establishing a priori estimates via Gidas-Spruck "Blow-up" method (see [6]). We use Leray-Schauder topological degree to guarantee the existence of fundamental states. We can refer the reader to the works of Clément, Manasevich and Mitidieri [1] on hamiltonian systems defined in a ball, as well as works on nonvariational system occurring sublinear growth conditions.

The main result established in this paper is expressed in the next section, namely the system (1.1) possesses at last a fundamental state.

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2. Existence of ground states

First, we introduce definitions and notation utilized in this note. Let the Banach space

$$X = \{(u, v) \in C^0([0, +\infty[) \times C^0([0, +\infty[), \lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0\}$$

be equipped with the norm $\|(u, v)\|_X = \|u\|_\infty + \|v\|_\infty$, $\|u\|_\infty = \sup_{r\in[0,+\infty[} |u(r)|$.

Let $K = \{(u, v) \in X, u \geq 0, v \geq 0\}$ a positive cone of X. For $h \geq 0$ and $\lambda \in [0, 1]$, we define two families of operators T_h and S_λ from X to itself by $T_h(u, v) = (w, z)$ such that (w, z) satisfies the system

$$\begin{gathered} -\big(r^{N-1}|w'(r)|^{p-2}w'(r)\big)' = r^{N-1}a_{11}(r)f_{11}(|u(r)|) + r^{N-1}a_{12}(r)[f_{12}(|v(r)|) + h] \\ \text{in } [0, +\infty[\,, \\ -\big(r^{N-1}|z'(r)|^{q-2}z'(r)\big)' = r^{N-1}a_{21}(r)f_{21}(|u(r)|) + r^{N-1}a_{22}(r)f_{22}(|v(r)|) \\ \text{in } [0, +\infty[\,, \\ w'(0) = z'(0) = 0, \quad \lim_{r\to+\infty} w(r) = \lim_{r\to+\infty} z(r) = 0, \end{gathered} \tag{2.1}$$

and $S_\lambda(u, v) = (w, z)$ such that (w, z) satisfies the system

$$\begin{gathered} -\big(r^{N-1}|w'(r)|^{p-2}w'(r)\big)' = \lambda r^{N-1}a_{11}(r)f_{11}(|u(r)|) + \lambda r^{N-1}a_{12}(r)f_{12}(|v(r)|) \\ \text{in } [0, +\infty[\,, \\ -\big(r^{N-1}|z'(r)|^{q-2}z'(r)\big)' = \lambda r^{N-1}a_{21}(r)f_{21}(|u(r)|) + \lambda r^{N-1}a_{22}(r)f_{22}(|v(r)|) \\ \text{in } [0, +\infty[\,, \\ w'(0) = z'(0) = 0, \quad \lim_{r\to+\infty} w(r) = \lim_{r\to+\infty} z(r) = 0 \end{gathered} \tag{2.2}$$

Let us recall the notion of "asymptotically homogeneous" functions and some of their properties.

A function $\varphi : \mathbb{R} \to \mathbb{R}$ defined in a neighborhood at the infinity (respect. at the origin) is said asymptotically homogeneous at the infinity (respect. at the origin) of order $\rho > 0$ if for all $\sigma > 0$, we have $\lim_{s\to+\infty} \frac{\varphi(\sigma s)}{\varphi(s)} = \sigma^\rho$ (respect. $\lim_{s\to 0} \frac{\varphi(\sigma s)}{\varphi(s)} = \sigma^\rho$).

As an example, we have the function $\varphi(s) = |s|^{\alpha-2}s(\ln(1 + |s|))^\beta$ with $\alpha > 1$ and $\beta > 1 - \alpha$. It is asymptotically homogeneous at infinity of order $\alpha - 1$ and at the origin of order $\alpha + \beta - 1$.

Proposition 2.1 ([3]). *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous, odd, asymptotically homogeneous at infinity (respect. at the origin) of order ρ such that $t\varphi(t) > 0$ for all $t \neq 0$ and $\varphi(t) \to infty$ as $t \to \infty$, then*

- (i) *For all $\varepsilon \in]0, \rho[$, there exists $t_0 > 0$ such that $\forall t \geq t_0$ (respect. $0 \leq t \leq t_0$), $c_1t^{\rho-\varepsilon} \leq \varphi(t) \leq c_2t^{\rho+\varepsilon}$; c_1, c_2 are positive constants. Moreover $\forall s \in [t_0, t]$: $(\rho + 1 - \varepsilon)\varphi(s) \leq (\rho + 1 + \varepsilon)\varphi(t)$.*
- (ii) *If $(w_n), (t_n)$ are real sequences such that $w_n \to w$ and $t_n \to +\infty$ (respect. $t_n \to 0$) then $\lim_{n\to+\infty} \frac{\varphi(t_nw_n)}{\varphi(t_n)} = w^\rho$.*

We assume that both the coefficients a_{ij} and the functions f_{ij} verify smooth conditions; explicitly:

(H1) For all $i, j = 1, 2$, the coefficient $a_{ij} : [0, +\infty[\to]0, +\infty[$ is continuous and satisfies $\exists \theta_{11}, \theta_{12} > p$; $\exists \theta_{21}, \theta_{22} > q$; there exists $R > 0$ such that $a_{ij}(\xi) = O(\xi^{-\theta_{ij}})$ for all $\xi > R$ and $\tilde{a}_i = \min_{r\in[0,R]} a_{ij}(r) > 0$; $i, j = 1, 2; i \neq j$.

(H2) For all $i, j = 1, 2$, the function $f_{ij} : \mathbb{R} \to \mathbb{R}$ is continuous, odd such that $sf_{ij}(s) > 0$ for all $s \neq 0$ and $\lim_{s\to+\infty} f_{ij}(s) = +\infty$.

(H3) For all $i, j = 1, 2$, f_{ij} is asymptotically homogeneous at the infinity of order δ_{ij} satisfying $\frac{\delta_{12}\delta_{21}}{(p-1)(q-1)} > 1$, $\alpha_1\delta_{11} - \alpha_1(p-1) - p < 0$, $\alpha_2\delta_{22} - \alpha_2(q-1) - q < 0$ and $\max(\beta_1, \beta_2) \geq 0$ where $\alpha_1 = \frac{p(q-1)+\delta_{12}q}{\delta_{12}\delta_{21}-(p-1)(q-1)}$, $\alpha_2 = \frac{q(p-1)+\delta_{21}p}{\delta_{12}\delta_{21}-(p-1)(q-1)}$, $\beta_1 = \alpha_1 - \frac{N-p}{p-1}$, $\beta_2 = \alpha_2 - \frac{N-q}{q-1}$.

(H4) For all $i, j = 1, 2$, f_{ij} is asymptotically homogeneous at the origin of order $\bar{\delta}_{ij}$ with $\bar{\delta}_{11}$, $\bar{\delta}_{12} > p - 1$, $\bar{\delta}_{21}$, $\bar{\delta}_{22} > q - 1$.

To show the existence result, it is necessary to state some lemmas.

Lemma 2.2. *Let $u \in C^1([0, +\infty[) \cap C^2(]0, +\infty[)$ be a positive solution of the problem*

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq 0$$

in $[0, +\infty[$ such that $u(0) > 0$ and $u'(0) \leq 0$, then

(i) *$u(r) > 0$ and $u'(r) \leq 0$ for all $r \geq 0$. Moreover, if $u'(s) = 0$ for all $s > 0$ then $u'(r) = 0$ for all $r \in [0, s]$.*

(ii) *The function M_p defined by $M_p(r) = ru'(r) + \frac{N-p}{p-1}u(r)$, $r \geq 0$, is nonnegative and nonincreasing. In particular, the function $r \mapsto r^{\frac{N-p}{p-1}}u(r)$ is nondecreasing in $[0, +\infty[$.*

Proof. To show (i), let us consider a nontrivial positive solution u of problem

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq 0 \quad \text{in } [0, +\infty[.$$

Integrating from s to r, we obtain $r^{N-1}|u'(r)|^{p-2}u'(r) \leq s^{N-1}|u'(s)|^{p-2}u'(s)$, for $0 < s < r$. Letting $s \to 0$, $u'(r) \leq 0$. If $u'(r) = 0$ then $u'(s) = 0$ for all $0 \leq s \leq r$. This means either u is a constant in $[0, +\infty[$ or there exists $r_0 \geq 0$ such that $u'(r) < 0$ for $r > r_0$ and $u'(r) = 0$, $u(r) = u(0)$ for $0 \leq r \leq r_0$. So u is non increasing and $u(0) > 0$.

Let us show (ii). Since u is a positive solution of the problem

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq 0 \quad \text{in } [0, +\infty[,$$

we have $-r^{N-1}(p-1)|u'(r)|^{p-2}u''(r) - (N-1)r^{N-2}|u'(r)|^{p-2}u'(r) \geq 0$. In other words $ru''(r) + \frac{N-1}{p-1}u'(r) \leq 0$, or $(ru'(r))' + \frac{N-p}{p-1}u'(r) \leq 0$. This yields that M_p is non increasing.

To show that $M_p(r) \geq 0$ for all $r \geq 0$, we use a contradiction argument. Indeed, assume that there exists $r_1 > 0$ such that $M_p(r_1) < 0$. Since M_p is non increasing, for all $r > r_1$, $M_p(r) \leq M_p(r_1)$ or $u'(r) + \frac{N-p}{p-1}\frac{u(r)}{r} \leq \frac{M_p(r_1)}{r}$.

On the other hand $u(r) > 0$, $\frac{N-p}{p-1} > 0$, hence $u'(r) \leq \frac{M_p(r_1)}{r}$. Consequently $u(r) - u(r_1) \leq M_p(r_1)\ln\frac{r}{r_1}$, $r > r_1$. It follows immediately that $\lim_{r\to+\infty} u(r) = -\infty$. This contradicts u begin positive. In particular

$$\frac{M_p(r)}{ru(r)} \geq 0 \quad \forall r > 0.$$

Finally, we obtain $\frac{u'(r)}{u(r)} + \frac{N-p}{p-1}\frac{1}{r} \geq 0$. In other words,

$$(\ln r^{\frac{N-p}{p-1}} u(r))' \geq 0.$$

This implies that the function $r \mapsto r^{\frac{N-p}{p-1}} u(r)$ is non decreasing. □

The study of the function M_p is essential and help us to estimate $u(r)$.

Lemma 2.3. *If* (H1) *is satisfied, then the operators T_h and S_λ are compact.*

The proof of the above lemma follows the same argument as in [2, Lemma 6], and is omitted.

We remark that the ground states of (1.1) are precisely the fixed points of the operator T_0. Now, we show a nonexistence result related to a "limit" system.

Theorem 2.4. *Under hypotheses* (H1)-(H3)*, the system*

$$\begin{aligned} -\Delta_p u &= a_{12}(|x|)|v|^{\delta_{12}-1}v \quad \text{in } \mathbb{R}^N \\ -\Delta_q v &= a_{21}(|x|)|u|^{\delta_{21}-1}u \quad \text{in } \mathbb{R}^N \end{aligned} \tag{2.3}$$

has no non-trivial radial positive solutions; in particular (2.3) *has no ground state.*

Proof. Let us argue by contradiction. Let (u, v) be a radial positive solution of System (2.3). Then (u, v) satisfies the differential system

$$\begin{gathered} -(r^{N-1}|u'(r)|^{p-2}u'(r))' = r^{N-1}a_{12}(r)(v(r))^{\delta_{12}} \quad \text{in } [0, +\infty[\,, \\ -(r^{N-1}|v'(r)|^{q-2}v'(r))' = r^{N-1}a_{21}(r)(u(r))^{\delta_{21}} \text{ in } [0, +\infty[\,, \\ u'(0) = v'(0) = 0 \end{gathered} \tag{2.4}$$

Hence,

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' \geq r^{N-1}\widetilde{a}_1 v^{\delta_{12}}\,, \tag{2.5}$$

$$-(r^{N-1}|v'(r)|^{q-2}v'(r))' \geq r^{N-1}\widetilde{a}_2 u^{\delta_{21}}\,. \tag{2.6}$$

First, consider the case $\beta_1 > 0$ or $\beta_2 > 0$. Integrating both (2.5) and (2.5) from 0 to r and taking into account that $u'(r) < 0$, $v'(r) < 0$ for all $r > 0$, we obtain

$$\begin{aligned} -u'(r) &\geq (\frac{\tilde{a}_1}{N})^{\frac{1}{p-1}} r^{\frac{1}{p-1}} v^{\frac{\delta_{12}}{p-1}}, \\ -v'(r) &\geq (\frac{\tilde{a}_2}{N})^{\frac{1}{q-1}} r^{\frac{1}{q-1}} u^{\frac{\delta_{21}}{q-1}}. \end{aligned}$$

By Lemma 2.2, we have $M_p \geq 0$, $M_q \geq 0$, thus

$$\begin{aligned} 0 &\geq -ru'(r) - \frac{N-p}{p-1}u(r) \geq (\frac{\tilde{a}_1}{N})^{\frac{1}{p-1}} r^{\frac{p}{p-1}} v^{\frac{\delta_{12}}{p-1}} - \frac{N-p}{p-1}u(r), \\ 0 &\geq -rv'(r) - \frac{N-q}{q-1}v(r) \geq (\frac{\tilde{a}_2}{N})^{\frac{1}{q-1}} r^{\frac{q}{q-1}} u^{\frac{\delta_{21}}{q-1}} - \frac{N-q}{q-1}v(r). \end{aligned}$$

This yields

$$u(r) \geq Cr^{\frac{p}{p-1}} v^{\frac{\delta_{12}}{p-1}}, \tag{2.7}$$

$$v(r) \geq Cr^{\frac{q}{q-1}} u^{\frac{\delta_{21}}{q-1}}. \tag{2.8}$$

Combining these two inequalities, we have

$$u(r) \le Cr^{-\alpha_1}, \tag{2.9}$$
$$v(r) \le Cr^{-\alpha_2}. \tag{2.10}$$

Since $r^{\frac{N-p}{p-1}}u(r)$ and $r^{\frac{N-q}{q-1}}v(r)$ are nondecreasing, for all $r > r_0 > 0$,

$$u(r) \ge Cr^{-\frac{N-p}{p-1}}, \tag{2.11}$$
$$v(r) \ge Cr^{-\frac{N-q}{q-1}}. \tag{2.12}$$

Inequalities (2.9)-(2.12) imply either $r^{\beta_1} \le C$ or $r^{\beta_2} \le C$. This yields a contradiction. Suppose now that $\beta_1 = 0$ (we may prove in a similar manner for $\beta_2 = 0$). Integrating with respect to r the first equation of System (2.4) from $r_0 > 0$ to r, we obtain

$$r^{N-1}|u'(r)|^{p-1} - r_0^{N-1}|u'(r_0)|^{p-1} \ge \tilde{a}_1 \int_{r_0}^{r} s^{N-1} v^{\delta_{12}}(s) ds.$$

Then (2.8) yields

$$v^{\delta_{12}}(s) \ge Cs^{\frac{\delta_{12}q}{q-1}} u^{\frac{\delta_{12}\delta_{21}}{q-1}}(s).$$

Consequently,

$$r^{N-1}|u'(r)|^{p-1} \ge C \int_{r_0}^{r} s^{N-1+\frac{\delta_{12}q}{q-1}} u^{\frac{\delta_{12}\delta_{21}}{q-1}}(s) ds.$$

Taking into account inequality (2.11) and the fact that $\beta_1 = 0$, we have

$$r^{N-1}|u'(r)|^{p-1} \ge C \int_{r_0}^{r} s^{N-1+\frac{\delta_{12}q}{q-1} - \frac{N-p}{p-1}\frac{\delta_{12}\delta_{21}}{q-1}} \, ds = C \int_{r_0}^{r} s^{-1} \, ds = C \ln \frac{r}{r_0}.$$

On the other hand, $M_p(r) \ge 0$ for $r > 0$ implies $(\frac{N-p}{p-1})^{p-1} u^{p-1}(r) \ge r^{p-1}|u'(r)|^{p-1}$. Hence

$$u^{p-1}(r) \ge Cr^{p-1}|u'(r)|^{p-1} \ge Cr^{p-N} \ln \frac{r}{r_0}.$$

Then we write

$$r^{\frac{N-p}{p-1}} u(r) \ge C\big(\ln \frac{r}{r_0}\big)^{\frac{1}{p-1}}.$$

This together with (2.9) yields a contradiction. □

We now show that the eventual radial positive solutions of System (2.1) are bounded.

Theorem 2.5. *Assume* (H1)-(H4). *If* (u, v) *is a ground state of* (2.1). *then there exists a constant* $C > 0$ *(independent of* u *and* v*) such that* $\|(u, v)\|_X \le C$.

Proof. Let (u, v) be a ground state of (2.1) for $h = 0$, then (u, v) satisfies the system

$$\begin{gathered}
-(r^{N-1}|u'(r)|^{p-2}u'(r))' = r^{N-1}a_{11}(r)f_{11}(u(r)) + r^{N-1}a_{12}(r)f_{12}(v(r)) \\
\text{in } [0,+\infty[, \\
-(r^{N-1}|v'(r)|^{q-2}v'(r))' = r^{N-1}a_{21}(r)f_{21}(u(r)) + r^{N-1}a_{22}(r)f_{22}(v(r)) \\
textin[0,+\infty[\,, \\
u'(0) = v'(0) = 0, \quad \lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0
\end{gathered} \tag{2.13}$$

Assume now that there exists a sequence (u_n, v_n) of positive solutions of (2.13) such that $\|u_n\|_\infty \to \infty$ as $n \to +\infty$ or $\|v_n\|_\infty \to \infty$ as $n \to +\infty$. Taking $\gamma_n = \|u_n\|_\infty^{\frac{1}{\alpha_1}} + \|v_n\|_\infty^{\frac{1}{\alpha_2}}$, and using (H3), we have $\alpha_1 > 0$ and $\alpha_2 > 0$. So $\gamma_n \to +\infty$ as $n \to +\infty$.

Now we introduce the transformations

$$y = \gamma_n r, \quad w_n(y) = \frac{u_n(r)}{\gamma_n^{\alpha_1}}, \quad z_n(y) = \frac{v_n(r)}{\gamma_n^{\alpha_2}}.$$

Observe that for all $y \in [0, +\infty[$, $0 \le w_n(y) \le 1$, $0 \le z_n(y) \le 1$. Furthermore it is easy to see that for any n the pair (w_n, z_n) is a solution of the system

$$\begin{aligned}
&-(y^{N-1}|w_n'(y)|^{p-2}w_n'(y))' \\
&= y^{N-1}a_{11}(\frac{y}{\gamma_n})\frac{f_{11}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} + y^{N-1}a_{12}(\frac{y}{\gamma_n})\frac{f_{12}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} \quad \text{in } [0,+\infty[\,, \\
&-(y^{N-1}|z_n'(y)|^{q-2}z_n'(y))' \\
&= y^{N-1}a_{21}(\frac{y}{\gamma_n})\frac{f_{21}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_2(q-1)+q}} + y^{N-1}a_{22}(\frac{y}{\gamma_n})\frac{f_{22}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_2(q-1)+q}} \quad \text{in } [0,+\infty[\,, \\
&\qquad w_n'(0) = z_n'(0) = 0, \quad \lim_{r\to+\infty} w_n(r) = \lim_{r\to+\infty} z_n(r) = 0\,.
\end{aligned} \tag{2.14}$$

Let $R > 0$ be fixed. We claim that (w_n') and (z_n') are bounded in $C([0, R])$. Indeed passing to a subsequence of (w_n') (denoted again (w_n')) assume that $\|w_n'\|_{C([0,R])} \to +\infty$ as $n \to +\infty$. Hence there exists a sequence (y_n) in $[0, R]$ such that for all $A > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $|w_n'(y_n)| > A$.

Integrating with respect to y the first equation of System (2.14), we obtain

$$\begin{aligned}
&|w_n'(y_n)|^{p-1} \\
&= \frac{1}{y_n^{N-1}} \int_0^{y_n} \Big(y^{N-1}a_{11}(\frac{y}{\gamma_n})\frac{f_{11}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} + y^{N-1}a_{12}(\frac{y}{\gamma_n})\frac{f_{12}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} \Big) dy.
\end{aligned}$$

From the fact that f_{1j}, $j = 1, 2$, are asymptotically homogeneous at the infinity together with part (i) of Proposition 2.1, we arrive to the statement: for all $\varepsilon \in [0, \delta_{1j}[$, there exists $c_{1j}^1, c_{1j}^2 > 0$, $s_0 > 0$ such that for all $s \ge s_0$

$$c_{1j}^1 s^{\delta_{1j}-\varepsilon} \le f_{1j}(s) \le c_{1j}^2 s^{\delta_{1j}+\varepsilon}.$$

Since (w_n) and (z_n) are bounded, we conclude that

$$c_{11}^1\gamma_n^{\alpha_1(\delta_{11}-\varepsilon)-\alpha_1(p-1)-p} \le \frac{f_{11}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} \le c_{11}^2\gamma_n^{\alpha_1(\delta_{11}+\varepsilon)-\alpha_1(p-1)-p},$$
$$c_{12}^1\gamma_n^{\alpha_2(\delta_{12}-\varepsilon)-\alpha_1(p-1)-p} \le \frac{f_{12}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} \le c_{12}^2\gamma_n^{\alpha_2(\delta_{12}+\varepsilon)-\alpha_1(p-1)-p}.$$

By choosing ε sufficiently small, the assumption (H3) yields

$$\frac{f_{11}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} \to 0 \quad \text{and} \quad \frac{f_{12}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_1(p-1)+p}} \to c_1 \quad \text{as } n \to +\infty$$

where c_1 is positive constant. So there exists $n_1 \in \mathbb{N}$ such that for any $n \ge n_1$, we have

$$|w_n'(y_n)|^{p-1} \le \frac{a_{12}(0)}{y_n^{N-1}}c_1\int_0^{y_n} y^{N-1}dy = \frac{c_1}{N}a_{12}(0)y_n \le \frac{Rc_1}{N}a_{12}(0) \equiv c.$$

Setting $n \ge \max(n_0, n_1)$, we have $A < |w_n'(y_n)| \le c$. This contradicts the fact that A may be infinitely large. Similarly we prove that (z_n') is bounded in $C([0,R])$. Consequently (w_n) and (z_n) are equicontinuous in $C([0,R])$. By Arzéla-Ascoli theorem, there exists a subsequence of (w_n) denoted again (w_n) (respect. (z_n)) such that $w_n \to w$ (respect. $z_n \to z$) in $C([0,R])$.

On the other hand,

$$\|w_n\|_\infty^{\frac{1}{\alpha_1}} + \|z_n\|_\infty^{\frac{1}{\alpha_2}} = 1,$$

this implies that the real-valued sequences $(\|w_n\|_\infty)$ and $(\|z_n\|_\infty)$ are bounded. Hence there exist subsequences denoted again $(\|w_n\|_\infty)$ and $(\|z_n\|_\infty)$ such that $\|w_n\|_\infty \to w_0$, $\|z_n\|_\infty \to z_0$ and $w_0^{\frac{1}{\alpha_1}} + z_0^{\frac{1}{\alpha_2}} = 1$. In view of the uniqueness of the limit in $C([0,R])$, we get $\|w\|_\infty^{\frac{1}{\alpha_1}} + \|z\|_\infty^{\frac{1}{\alpha_2}} = 1$. This implies that (w,z) is not identically null. Integrating from 0 to $y \in [0,R]$, the first and the second equation of System (2.14), we obtain

$$w_n(0) - w_n(y) = \int_0^y (g_n(s))^{\frac{1}{p-1}}ds, \tag{2.15}$$
$$z_n(0) - z_n(y) = \int_0^y (h_n(s))^{\frac{1}{q-1}}ds\,. \tag{2.16}$$

Clearly $g_n(y)$ and $h_n(y)$ are defined by

$$g_n(y) = \frac{1}{y^{N-1}}\int_0^y \Big(s^{N-1}a_{11}(\frac{s}{\gamma_n})\frac{f_{11}(\gamma_n^{\alpha_1}w_n(s))}{\gamma_n^{\alpha_1(p-1)+p}} + s^{N-1}a_{12}(\frac{s}{\gamma_n})\frac{f_{12}(\gamma_n^{\alpha_2}z_n(s))}{\gamma_n^{\alpha_1(p-1)+p}}\Big)ds$$
$$h_n(y) = \frac{1}{y^{N-1}}\int_0^y \Big(s^{N-1}a_{21}(\frac{s}{\gamma_n})\frac{f_{21}(\gamma_n^{\alpha_1}w_n(s))}{\gamma_n^{\alpha_2(q-1)+q}} + s^{N-1}a_{22}(\frac{s}{\gamma_n})\frac{f_{22}(\gamma_n^{\alpha_2}z_n(s))}{\gamma_n^{\alpha_2(q-1)+q}}\Big)ds.$$

Compiling Proposition 2.1 and (H3), we obtain

$$\frac{f_{11}(\gamma_n^{\alpha_1}w_n(s))}{\gamma_n^{\alpha_1(p-1)+p}} \to 0, \quad \frac{f_{22}(\gamma_n^{\alpha_2}z_n(s))}{\gamma_n^{\alpha_2(q-1)+q}} \to 0,$$
$$\frac{f_{12}(\gamma_n^{\alpha_2}z_n(s))}{\gamma_n^{\alpha_1(p-1)+p}} = \frac{f_{12}(\gamma_n^{\alpha_2})}{\gamma_n^{\alpha_1(p-1)+p}}\frac{f_{12}(\gamma_n^{\alpha_2}z_n(s))}{f_{12}(\gamma_n^{\alpha_2})} \to cz^{\delta_{12}}(s),$$
$$\frac{f_{21}(\gamma_n^{\alpha_1}w_n(s))}{\gamma_n^{\alpha_2(q-1)+q}} = \frac{f_{21}(\gamma_n^{\alpha_1})}{\gamma_n^{\alpha_2(q-1)+q}}\frac{f_{21}(\gamma_n^{\alpha_1}w_n(s))}{f_{21}(\gamma_n^{\alpha_1})} \to cw^{\delta_{21}}(s),$$

as $n \to \infty$. By the Lebesgue theorem on dominated convergence, it follows that

$$g_n(y) \to \frac{c}{y^{N-1}} \int_0^y s^{N-1} a_{12}(0) z^{\delta_{12}}(s) ds,$$
$$h_n(y) \to \frac{c}{y^{N-1}} \int_0^y s^{N-1} a_{21}(0) w^{\delta_{21}}(s) ds,$$

as $n \to \infty$. Passing to the limit in (2.15) and (2.16), we arrive to

$$w(0) - w(y) = c \int_0^y \frac{1}{\tau^{N-1}} \Big(\int_0^\tau s^{N-1} a_{12}(0) z^{\delta_{12}}(s) ds \Big)^{\frac{1}{p-1}} d\tau,$$
$$z(0) - z(y) = c \int_0^y \frac{1}{\tau^{N-1}} \Big(\int_0^\tau s^{N-1} a_{21}(0) w^{\delta_{21}}(s) ds \Big)^{\frac{1}{q-1}} d\tau.$$

In this way $w \geq 0$, $z \geq 0$, $w, z \in C^1([0, R]) \cap C^2(]0, R])$ and satisfy the system

$$\begin{gathered} -(y^{N-1} |w'(y)|^{p-2} w'(y))' = c a_{12}(0) y^{N-1} (z(y))^{\delta_{12}} \quad \text{in } [0, R] \\ -(y^{N-1} |z'(y)|^{q-2} z'(y))' = c a_{21}(0) y^{N-1} (w(y))^{\delta_{21}} \quad \text{in } [0, R] \\ w'(0) = z'(0) = 0 \end{gathered} \tag{2.17}$$

If we use the same arguments on $[0, R^*]$ where $R^* > R$, we obtain a solution (w^*, z^*) of System (2.17) with R^* in stead of R, which coincide with (w, z) in $[0, R]$. To this end, we indefinitely extend (w, z) to $[0, +\infty[$. By Lemma 2.2 we have $w(y) > 0$, $z(y) > 0$, for all $y \geq 0$. The pair (w, z) also satisfies System (2.17). In other words (w, z) is a radial positive solution of (2.4). This contradicts Theorem 2.4. □

Lemma 2.6. *Under assumptions* (H1)-(H4)*, there exists $h_0 > 0$ such that the problem $(u, v) = T_h(u, v)$ has no solution for $h \geq h_0$.*

Proof. Suppose by contradiction that there is a solution $(u, v) \in X$ of the above problem. Then (u, v) satisfies system

$$\begin{gathered} -(r^{N-1} |u'(r)|^{p-2} u'(r))' = r^{N-1} a_{11}(r) f_{11}(|u(r)|) + r^{N-1} a_{12}(r) [f_{12}(|v(r)|) + h] \\ \text{in } [0, +\infty[\,, \\ -(r^{N-1} |v'(r)|^{q-2} v'(r))' = r^{N-1} a_{21}(r) f_{21}(|u(r)|) + r^{N-1} a_{22}(r) f_{22}(|v(r)|) \\ \text{in } [0, +\infty[\,, \\ u'(0) = v'(0) = 0, \quad \lim_{r \to +\infty} u(r) = \lim_{r \to +\infty} v(r) = 0 \end{gathered} \tag{2.18}$$

Assume that there exists a sequence (h_n) $h_n \to +\infty$ as $n \to +\infty$, such that (2.18) admits a pair of solutions (u_n, v_n). In accordance with Lemma 2.2, we have $u_n(r) > 0$, $v_n(r) > 0$, $u_n'(r) \leq 0$ and $v_n'(r) \leq 0$, for all $n \in \mathbb{N}$. Integrating the first equation of System (2.18), from R to $2R$, $R > 0$, we obtain

$$u_n(R) \geq \int_R^{2R} \Big(\eta^{1-N} \int_0^\eta \xi^{N-1} a_{12}(\xi) h_n d\xi \Big)^{\frac{1}{p-1}} d\eta \geq c R h_n^{\frac{1}{p-1}}$$

Here

$$c = \Big(\frac{1}{(2R)^{N-1}} \int_0^R \xi^{N-1} a_{12}(\xi) d\xi \Big)^{\frac{1}{p-1}}.$$

Consequently $u_n(R) \geq cRh_n^{\frac{1}{p-1}}$. Passing to the limit we get $u_n(R) \to +\infty$. On the other hand, integrating the second equation of (2.18), from R to $2R$, we obtain

$$v_n(R) \geq \int_R^{2R} (\eta^{1-N} \int_0^{\eta} \xi^{N-1} a_{21}(\xi) f_{21}(u_n(\xi)) d\xi)^{\frac{1}{q-1}} d\eta \geq cR(f_{21}(u_n(R)))^{\frac{1}{q-1}}$$

By hypothesis (H3) and Proposition 2.1, we have $v_n(R) \geq c(u_n(R))^{\frac{\delta_{21}-\varepsilon}{q-1}}$ Operating similarly, we obtain $u_n(R) \geq c(v_n(R))^{\frac{\delta_{12}-\varepsilon}{p-1}}$. It follows from the last two inequalities, that

$$(u_n(R))^{\frac{(\delta_{12}-\varepsilon)(\delta_{21}-\varepsilon)-(p-1)(q-1)}{(p-1)(q-1)}} \leq \frac{1}{c}.$$

This is the desired contradiction since $u_n(R)$ increases to infinitely. $\square$

Lemma 2.7. *There exists $\bar{\rho} > 0$ such that for all $\rho \in]0, \bar{\rho}[$ and all $(u, v) \in X$ satisfying $\|(u, v)\| = \rho$, the equation $(u, v) = S_\lambda(u, v)$ has no solution.*

Proof. Assume that there exist $(\rho_n) \in \mathbb{R}_+$, $\rho_n \to 0$; $(\lambda_n) \subset [0, 1]$ and $(u_n, v_n) \in X$ such that $(u_n, v_n) = S_{\lambda_n}(u_n, v_n)$ with $\|(u_n, v_n)\| = \rho_n$. Taking (H4) into account,

$$\|u_n\|_\infty \leq c\lambda_n^{\frac{1}{p-1}} \Big(\|u_n\|_\infty^{\frac{\bar{\delta}_{11}-\varepsilon}{p-1}} + \|v_n\|_\infty^{\frac{\bar{\delta}_{12}-\varepsilon}{p-1}} \Big)$$
$$\|v_n\|_\infty \leq c\lambda_n^{\frac{1}{q-1}} \Big(\|u_n\|_\infty^{\frac{\bar{\delta}_{21}-\varepsilon}{q-1}} + \|v_n\|_\infty^{\frac{\bar{\delta}_{22}-\varepsilon}{q-1}} \Big)$$

Adding term by term, we obtain

$$\|(u_n, v_n)\| \leq C\Big(\|(u_n, v_n)\|^{\frac{\bar{\delta}_{11}-\varepsilon}{p-1}} + \|(u_n, v_n)\|^{\frac{\bar{\delta}_{12}-\varepsilon}{p-1}}, + \|(u_n, v_n)\|^{\frac{\bar{\delta}_{21}-\varepsilon}{q-1}} + \|(u_n, v_n)\|^{\frac{\bar{\delta}_{22}-\varepsilon}{q-1}} \Big).$$

This implies

$$1 \leq C\Big(\|(u_n, v_n)\|^{\frac{\bar{\delta}_{11}-\varepsilon}{p-1}-1} + \|(u_n, v_n)\|^{\frac{\bar{\delta}_{12}-\varepsilon}{p-1}-1} + \|(u_n, v_n)\|^{\frac{\bar{\delta}_{21}-\varepsilon}{q-1}-1} + \|(u_n, v_n)\|^{\frac{\bar{\delta}_{22}-\varepsilon}{q-1}-1} \Big).$$

The above inequality contradicts the fact that $\|(u_n, v_n)\| = \rho_n \to$ as $n \to +\infty$. $\square$

Theorem 2.8. *Under hypotheses (H1)-(H4), System* (1.1) *has positive radial solution.*

Proof. To show the existence of ground states for (1.1) (or (2.1) with $h = 0$), it is sufficient to prove that the compact operator T_0 admits a fixed point. In view of Theorem 2.5, the eventual fixed point (u, v) of T_0 are bounded; explicitly there exists $C > 0$ such that $\|(u, v)\|_X \leq C$. Let us chose $R_1 > C$ and let us designate by B_{R_1} the ball of X, centered at the origin with radius R_1. To this end, the Leray-Schauder degree $\deg_{LS}(I - T_h, B_{R_1}, 0)$ is well defined. It being understood that I denote the identical operator in X. Moreover, by Lemma 2.6, we have $\deg_{LS}(I - T_h, B_{R_1}, 0) = 0$ for all $h \geq h_0$. It follows from the homotopy invariance of the Leray-Schauder degree that

$$\deg_{LS}(I - T_0, B_{R_1}, 0) = \deg_{LS}(I - T_h, B_{R_1}, 0) = 0.$$

On the other hand, by Lemma 2.7, there exists $0 < \rho < \bar{\rho} < R_1$ such that $\deg_{LS}(I - S_\lambda, B_\rho, 0)$ is well defined. Once again, the homotopy invariance of the Leray-Schauder degree yields

$$\begin{aligned} 1 &= \deg_{LS}(I, B_\rho, 0) \\ &= \deg_{LS}(I - S_\lambda, B_\rho, 0) \\ &= \deg_{LS}(I - S_1, B_\rho, 0) \\ &= \deg_{LS}(I - T_0, B_\rho, 0). \end{aligned}$$

Using the additivity of the Leray-Schauder degree,

$$\deg_{LS}(I - T_0, B_{R_1} \backslash B_\rho, 0) = \deg_{LS}(I - T_0, B_{R_1}, 0) - \deg_{LS}(I - T_0, B_\rho, 0) = -1.$$

This implies that T_0 has fixed point in $B_{R_1} \backslash B_\rho$. Consequently, there exists a nontrivial ground state. □

References

[1] Ph. Clément, R. Manasevich, E. Mitidieri; *Positive solutions for a quasilinear system via Blow up*, Comm. in Partial Diff. Eq. 18(12), 2071-2106 (1993).
[2] A. Djellit, S. Tas; *On some nonlinear elliptic systems*, Nonlinear Analysis, 59 (2004) 695-706
[3] M. Garcia-Huidobro, I. Guerra, R. Manasevich: *Existence of positive radial solutions for a weakly coupled system via Blow up*, Abstract Appl. Anal. 3 (1998), 105-131.
[4] M. Garcia-Huidobro, R. Manasevich, K. Schmitt; *Some bifurcation results for a class of p-Laplacian like operators*, Differential and Integral Equations 10 (1997), 51-66.
[5] M. Garcia-Huidobro, R. Manasevich, P. Ubilla; *Existence of positive solutions for some Dirichlet problems with an asymptotically homogeneous operator*, Electron. J. Diff. Equ., Vol 1995(1995), No. 10, 1-22.
[6] B. Gidas, J. Spruck; *A priori Bounds for Positive Solutions of Nonlinear Elliptic Equations*, Comm. in PDE, 6(8), 1981, 883-901.

Ali Djellit
University of Annaba, Faculty of Sciences, Department of Mathematics, BP 12, 23000 Annaba, Algeria
E-mail address: a_djellit@hotmail.com

Mohand Moussaoui
Ecole Centrale de Lyon, Department of Mathematics, 36 Rue de Collongue, 69130 Ecully, France
E-mail address: mohand.moussaoui@ec-lyon.fr

Saadia Tas
University of Bejaia, Faculty of Sciences and Engineering Sciences, Department of Mathematics, 06000 Bejaia, Algeria
E-mail address: tas_saadia@yahoo.fr