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GROWTH OF SOLUTIONS OF HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

KARIMA HAMANI

Abstract. In this article, we study the growth of solutions of the linear differential equation

$$f^{(k)}+(A_{k-1}(z)e^{P_{k-1}(z)}+B_{k-1}(z))f^{(k-1)}+\dots+(A_0(z)e^{P_0(z)}+B_0(z))f=0,$$

where $k\geq 2$ is an integer, $P_j(z)$ are nonconstant polynomials and $A_j(z), B_j(z)$ are entire functions, not identically zero. We determine the hyper-order of these solutions, under certain conditions.

1. Introduction and statement of results

In this article, we assume that the reader is familiar with the fundamental results and standard notation of the Nevanlinna value distribution theory of meromorphic functions [7]. Let $\sigma(f)$ denote the order of growth of an entire function $f(z)$ and $\sigma_2(f)$ the hyper-order of $f(z)$, which as in [8, 11] is defined by

$$\sigma_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}=\limsup_{r\to+\infty}\frac{\log\log\log M(r,f)}{\log r}, \tag{1.1}$$

where $M(r,f)=\max_{|z|=r}|f(z)|$.

We define the linear measure of a set $E\subset[0,+\infty)$ by $m(E)=\int_0^{+\infty}\chi_E(t)dt$ and the logarithmic measure of a set $H\subset[1,+\infty)$ by $lm(H)=\int_1^{+\infty}\frac{\chi_H(t)}{t}dt$, where χ_E is the characteristic function of a set E .

Several authors [2, 6, 8] have studied the second-order linear differential equation

$$f''+h_1(z)e^{P(z)}f'+h_0(z)e^{Q(z)}f=0, \tag{1.2}$$

where $P(z)$ and $Q(z)$ are nonconstant polynomials, $h_1(z)$ and $h_0(z)\not\equiv 0$ are entire functions satisfying $\sigma(h_1)<\deg P$ and $\sigma(h_0)<\deg Q$. Gundersen showed in [6, p. 419] that if $\deg P\neq\deg Q$, then every nonconstant solution of (1.2) is of infinite order. If $\deg P=\deg Q$, then (1.2) can have nonconstant solutions of finite order. Indeed, $f(z)=z$ satisfies $f''-z^3e^zf'+z^2e^zf=0$. Kwon [8] studied the case where $\deg P=\deg Q$ and proved the following result:

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Theorem 1.1 ([8]). *Let $P(z)$ and $Q(z)$ be nonconstant polynomials such that*
$$P(z) = a_n z^n + \cdots + a_1 z + a_0, \tag{1.3}$$
$$Q(z) = b_n z^n + \cdots + b_1 z + b_0, \tag{1.4}$$
where a_i, b_i $(i = 0, 1, \dots, n)$ are complex numbers, $a_n \neq 0$ and $b_n \neq 0$. Let $h_j(z)$ $(j = 0, 1)$ be entire functions with $\sigma(h_j) < n$. Suppose that $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$. Then every nonconstant solution f of (1.2) is of infinite order and satisfies $\sigma_2(f) \geq n$.

Chen [3] also studied the growth of solutions of second-order linear differential equations and obtained the following result:

Theorem 1.2 ([3]). *Let $A_j(z)$ $(\not\equiv 0)$, $D_j(z)$ $(j = 0, 1)$ be entire functions with $\sigma(A_j) < 1$, $\sigma(D_j) < 1$, a, b be complex constants such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every solution f $(\not\equiv 0)$ of the equation*
$$f'' + (A_1(z)e^{az} + D_1(z))f' + (A_0(z)e^{bz} + D_0(z))f = 0 \tag{1.5}$$
is of infinite order.

Belaidi [1] extended Theorem 1.2 for higher-order linear differential equations as follows.

Theorem 1.3 ([1]). *Let $k \geq 2$ and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, \dots, a_{n,j}$ $(j = 0, \dots, k-1)$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, 1, \dots, k-1)$. Let $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \dots, k-1)$ be entire functions. Suppose that $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = c_j a_{n,0}$ $(0 < c_j < 1)$ $(j = 1, \dots, k-1)$ and $\sigma(A_j) < n$, $\sigma(B_j) < n$ $(j = 0, 1, \dots, k-1)$. Then every solution f $(\not\equiv 0)$ of the differential equation*
$$f^{(k)} + (A_{k-1}(z)e^{P_{k-1}(z)} + B_{k-1}(z))f^{(k-1)} + \cdots + (A_0(z)e^{P_0(z)} + B_0(z))f = 0 \tag{1.6}$$
is of infinite order and satisfies $\sigma_2(f) = n$.

Chen [4] also considered the growth of solutions of higher-order linear differential equations and proved the following result:

Theorem 1.4 ([4]). *Let $h_j(z)$ $(j = 0, 1, \dots, k-1)$ $(k \geq 2)$ be entire functions with $\sigma(h_j) < 1$, and $H_j(z) = h_j(z)e^{a_j z}$, where a_j $(j = 0, 1, \dots, k-1)$ are complex numbers. Suppose that there exists a_s such that $h_s \not\equiv 0$, for $j \neq s$, if $H_j \not\equiv 0$, $a_j = c_j a_s$ $(0 < c_j < 1)$; if $H_j \equiv 0$, we define $c_j = 0$. Then every transcendental solution f of the equation*
$$f^{(k)} + H_{k-1}(z)f^{(k-1)} + \cdots + H_s(z)f^{(s)} + \cdots + H_0(z)f = 0 \tag{1.7}$$
is of infinite order.

Furthermore, if $\max\{c_1, \dots c_{s-1}\} < c_0$, then every solution of (1.7) is of infinite order.

Recently, Tu and Yi [10] obtained the following result which is an extension of Theorem 1.1.

Theorem 1.5 ([10]). *Let $h_j(z)$ $(j = 0, 1, \dots, k-1)$ $(k \geq 2)$ be entire functions with $\sigma(h_j) < n$ $(n \geq 1)$, and let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be polynomials with degree n , where $a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,0} = |a_{n,0}|e^{i\theta_0}$, $a_{n,s} = |a_{n,s}|e^{i\theta_s}$, $a_{n,0}a_{n,s} \neq 0$ $(0 < s \leq k-1)$, θ_0, $\theta_s \in [0, 2\pi)$,*

$\theta_0 \neq \theta_s$, $h_0 h_s \not\equiv 0$; for $j \neq 0, s$, $a_{n,j}$ satisfies either $a_{n,j} = c_j a_{n,0}$ $(c_j < 1)$ or $\arg a_{n,j} = \theta_s$. Then every solution f $(\not\equiv 0)$ of the equation

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \dots + h_1(z)e^{P_1(z)} f' + h_0(z)e^{P_0(z)} f = 0 \tag{1.8}$$

is of infinite order and satisfies $\sigma_2(f) = n$.

The main purpose of this article is to investigate the growth of solutions of (1.6), and determine the hyper-order of these solutions. We shall prove the following results:

Theorem 1.6. *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, 1, \dots, k-1)$. Let $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \dots, k-1)$ be entire functions with $\sigma(A_j) < n$ and $\sigma(B_j) < n$. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $\arg a_{n,j} \neq \arg a_{n,s}$ $(j \neq s)$. Then every transcendental solution f of (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.*

Theorem 1.7. *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, 1, \dots, k-1)$. Let $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \dots, k-1)$ be entire functions with $\sigma(A_j) < n$ and $\sigma(B_j) < n$. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$. Then every transcendental solution f of (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.*

Furthermore, if $\max\{c_1, \dots c_{s-1}\} < c_0$, then every solution of (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.

Theorem 1.8. *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, 1, \dots, k-1)$ and $a_{n,0} = |a_{n,0}| e^{i\theta_0}$, $\theta_0 \in [0, 2\pi)$. Let $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \dots, k-1)$ be entire functions with $\sigma(A_j) < n$ and $\sigma(B_j) < n$ $(j = 0, 1, \dots, k-1)$. Suppose that there exists $s \in \{1, \dots, k-1\}$ such that $a_{n,s} = |a_{n,s}| e^{i\theta_s}$, $\theta_s \in [0, 2\pi)$, $\theta_s \neq \theta_0$ and for $j \in \{1, \dots, s-1, s+1, \dots, k-1\}$, $a_{n,j}$ satisfies either $a_{n,j} = c_j a_{n,0}$ $(c_j < 1)$ or $\arg a_{n,j} = \theta_s$. Then every solution f $(\not\equiv 0)$ of (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.*

Theorem 1.9. *Let $k \geq 2$ be an integer and $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \dots, k-1)$ be nonconstant polynomials, where $a_{0,j}, a_{1,j}, \dots, a_{n,j}$ $(j = 0, 1, \dots, k-1)$ are complex numbers such that $a_{n,j} \neq 0$ $(j = 0, 1, \dots, k-1)$. Let $A_j(z)$ $(\not\equiv 0)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \dots, k-1)$ be entire functions with $\sigma(A_j) < n$ and $\sigma(B_j) < n$ $(j = 0, 1, \dots, k-1)$. Suppose that there exist d, $s \in \{1, \dots, k-1\}$ such that $a_{n,d} = |a_{n,d}| e^{i\theta_d}$, $a_{n,s} = |a_{n,s}| e^{i\theta_s}$, θ_d, $\theta_s \in [0, 2\pi)$, $\theta_d \neq \theta_s$ and for $j \in \{0, \dots, k-1\} \setminus \{d, s\}$, $a_{n,j}$ satisfies either $a_{n,j} = c_j a_{n,d}$ $(c_j < 1)$ or $\arg a_{n,j} = \theta_s$. Then every transcendental solution f of (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.*

2. Preliminaries

Lemma 2.1 ([5]). *Let $f(z)$ be a transcendental meromorphic function and let $\alpha > 1$ and $\epsilon > 0$ be given constants. Then there exist a set $E_1 \subset [1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ that depends only on α and (i, j) $(i, j$*

positive integers with $i > j$) such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\Big|\frac{f^{(i)}(z)}{f^{(j)}(z)}\Big| \leq B\Big[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\Big]^{i-j}. \tag{2.1}$$

Lemma 2.2 ([4]). *Let $f(z)$ be a transcendental entire function Then there exists a set $E_2 \subset [1,+\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r,f)$, we have*

$$\Big|\frac{f(z)}{f^{(s)}(z)}\Big| \leq 2r^{s}, \tag{2.2}$$

where $s \geq 1$ is an integer.

Lemma 2.3 ([9]). *Let $P(z) = (\alpha+i\beta)z^n + \dots$ (α, β are real numbers, $|\alpha|+|\beta| \neq 0$) be a polynomial with degree $n \geq 1$ and $A(z)$ be an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P,\theta) = \alpha\cos n\theta - \beta\sin n\theta$. Then for any given $\epsilon > 0$, there exists a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for any $\theta \in [0,2\pi)\setminus H$ ($H = \{\theta \in [0,2\pi) : \delta(P,\theta) = 0\}$) and for $|z| = r \notin [0,1] \cup E_3$, we have*

(i) *if $\delta(P,\theta) > 0$, then*

$$\exp\{(1-\epsilon)\delta(P,\theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1+\epsilon)\delta(P,\theta)r^n\}, \tag{2.3}$$

(ii) *if $\delta(P,\theta) < 0$, then*

$$\exp\{(1+\epsilon)\delta(P,\theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1-\epsilon)\delta(P,\theta)r^n\}. \tag{2.4}$$

Lemma 2.4 ([4]). *Let $k \geq 2$ be an integer and let $A_j(z)$ ($j = 0,1,\dots,k-1$) be entire functions of finite order. If f is a solution of the differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0, \tag{2.5}$$

then $\sigma_2(f) \leq \max\{\sigma(A_j)\ (j = 0,1,\dots,k-1)\}$.

3. Proof of main results

3.1. **Proof of Theorem 1.6.** Assume f is a transcendental solution of (1.6). By Lemma 2.1, there exist a constant $B > 0$ and a set $E_1 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\Big|\frac{f^{(j)}(z)}{f^{(s)}(z)}\Big| \leq Br[T(2r,f)]^{j-s+1} \quad (j = s+1,\dots,k), \tag{3.1}$$

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leq Br[T(2r,f)]^{j+1} \quad (j = 1,2,\dots,s-1). \tag{3.2}$$

By Lemma 2.2, there exists a set $E_2 \subset [1,+\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r,f)$, we have

$$\Big|\frac{f(z)}{f^{(s)}(z)}\Big| \leq 2r^{s}. \tag{3.3}$$

Since $\arg a_{n,j} \neq \arg a_{n,s}$ ($j \neq s$), there is a ray $\arg z = \theta \in [0,2\pi) \setminus H$, where $H = \{\theta \in [0,2\pi) : \delta(P_0,\theta) = 0$ or $\dots$ or $\delta(P_{k-1},\theta) = 0\}$, such that $\delta(P_s,\theta) > 0$, $\delta(P_j,\theta) < 0$ ($j \neq s$). Set $\beta = \max\{\sigma(B_j)\ (j = 0,\dots,k-1)\}$. By Lemma 2.3, for any given ϵ ($0 < 2\epsilon < \min\{1, n-\beta\}$), there exists a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r, we have

$$|A_s(z)e^{P_s(z)} + B_s(z)| \geq (1 - o(1)) \exp\{(1-\epsilon)\delta(P_s, \theta)r^n\} \tag{3.4}$$

and

$$\begin{aligned} |A_j(z)e^{P_j(z)} + B_j(z)| &\leq \exp\{(1-\epsilon)\delta(P_j, \theta)r^n\} + \exp\{r^{\sigma(B_j)+\frac{\epsilon}{2}}\} \\ &\leq \exp\{r^{\sigma(B_j)+\epsilon}\} \\ &\leq \exp\{r^{\beta+\epsilon}\} \quad (j \neq s). \end{aligned} \tag{3.5}$$

We can rewrite (1.6) as

$$\begin{aligned} &A_s(z)e^{P_s(z)} + B_s(z) \\ &= \frac{f^{(k)}}{f^{(s)}} + (A_{k-1}(z)e^{P_{k-1}(z)} + B_{k-1}(z))\frac{f^{(k-1)}}{f^{(s)}} + \dots \\ &\quad + (A_{s+1}(z)e^{P_{s+1}(z)} + B_{s+1}(z))\frac{f^{(s+1)}}{f^{(s)}} + (A_{s-1}(z)e^{P_{s-1}(z)} + B_{s-1}(z))\frac{f^{(s-1)}}{f}\frac{f}{f^{(s)}} \\ &\quad + \dots + (A_1(z)e^{P_1(z)} + B_1(z))\frac{f'}{f}\frac{f}{f^{(s)}} + (A_0(z)e^{P_0(z)} + B_0(z))\frac{f}{f^{(s)}}. \end{aligned} \tag{3.6}$$

Hence from (3.1)-(3.6), for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_3$, $|f(z)| = M(r, f)$ and a sufficiently large r, we have

$$(1 - o(1)) \exp\{(1-\epsilon)\delta(P_s, \theta)r^n\} \leq M_1 r^{s+1} \exp\{r^{\beta+\epsilon}\}[T(2r, f)]^k, \tag{3.7}$$

where M_1 is some positive constant. Thus $0 < 2\epsilon < \min\{1, n - \beta\}$ implies $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. By Lemma 2.4, we have $\sigma_2(f) = n$.

3.2. **Proof of Theorem 1.7.** Assume f is a transcendental solution of (1.6). Since $a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$, it follows that $\delta(P_j, \theta) = c_j \delta(P_s, \theta)$ $(j \neq s)$. Put $c = \max\{c_j (j \neq s)\}$. Then $0 < c < 1$.We take a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where $H = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0\}$, such that $\delta(P_s, \theta) > 0$. By Lemma 2.3, for any given ϵ $(0 < 2\epsilon < \frac{1-c}{1+c})$, there exists a set $E_3 \subset [1, +\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r, we have

$$|A_s(z)e^{P_s(z)} + B_s(z)| \geq (1 - o(1)) \exp\{(1-\epsilon)\delta(P_s, \theta)r^n\} \tag{3.8}$$

and

$$|A_j(z)e^{P_j(z)} + B_j(z)| \leq (1 + o(1)) \exp\{(1+\epsilon)c\delta(P_s, \theta)r^n\} \quad (j \neq s). \tag{3.9}$$

Thus by (3.1)-(3.3), (3.6), (3.8) and (3.9), we obtain that for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_3$, $|f(z)| = M(r, f)$ and a sufficiently large r,

$$\begin{aligned} &(1 - o(1)) \exp\{(1-\epsilon)\delta(P_s, \theta)r^n\} \\ &\leq M_2 r^{s+1}(1 + o(1)) \exp\{(1+\epsilon)c\delta(P_s, \theta)r^n\}[T(2r, f)]^k, \end{aligned} \tag{3.10}$$

where M_2 is a positive constant. By $0 < 2\epsilon < \frac{1-c}{1+c}$ and (3.10), we have

$$\exp\{\frac{(1-c)}{2}\delta(P_s, \theta)r^n\} \leq M_3 r^{s+1}[T(2r, f)]^k, \tag{3.11}$$

where M_3 is a positive constant. Hence (3.11) implies $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. By Lemma 2.4, we have $\sigma_2(f) = n$.

Now we prove that if $\max\{c_1,\dots c_{s-1}\} < c_0$, then equation (1.6) cannot have a nonzero polynomial solution. Suppose that $c' = \max\{c_1,\dots c_{s-1}\} < c_0$ and let $f(z)$ be a nonzero polynomial solution of equation (1.6) with $\deg f(z) = m$. We take a ray $\arg z = \theta \in [0,2\pi) \setminus H$, where H is defined as above, such that $\delta(P_s,\theta) > 0$. By Lemma 2.3, for any given ϵ $(0 < 2\epsilon < \min\{\frac{1-c}{1+c}, \frac{c_0-c'}{c_0+c'}\})$, there exists a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r, we have (3.8) and (3.9).

If $m \geq s$, by (1.6), (3.8) and (3.9), we obtain for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r,

$$\begin{aligned}
&d_1 r^{m-s}(1-o(1))\exp\{(1-\epsilon)\delta(P_s,\theta)r^n\}\\
&\leq |A_s(z)e^{P_s(z)} + B_s(z)||f^{(s)}(z)|\\
&\leq d_2 r^m(1+o(1))\exp\{(1+\epsilon)c\delta(P_s,\theta)r^n\},
\end{aligned} \tag{3.12}$$

where d_1, d_2 are positive constants. By (3.12),

$$\exp\{\frac{(1-c)}{2}\delta(P_s,\theta)r^n\} \leq d_3 r^s, \tag{3.13}$$

where d_3 is a positive constant. Hence (3.13) is not possible.

If $m < s$, by (1.6), (3.8) and (3.9), we obtain for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r,

$$\begin{aligned}
&d_4 r^{s-1}(1-o(1))\exp\{(1-\epsilon)c_0\delta(P_s,\theta)r^n\}\\
&\leq |A_0(z)e^{P_0(z)} + B_0(z)||f(z)|\\
&\leq \sum_{j=1}^{s-1}|A_j(z)e^{P_j(z)} + B_j(z)||f^{(j)}(z)|\\
&\leq d_5 r^{s-2}(1+o(1))\exp\{(1+\epsilon)c\delta(P_s,\theta)r^n\},
\end{aligned} \tag{3.14}$$

where d_4, d_5 are positive constants. By (3.14),

$$\exp\{\frac{(c_0-c')}{2}\delta(P_s,\theta)r^n\} \leq \frac{d_6}{r}, \tag{3.15}$$

where d_6 is a positive constant. This contradiction implies that if $\max\{c_1,\dots c_{s-1}\}<c_0$, then every solution of (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.

3.3. **Proof of Theorem 1.8.** Assume f is a transcendental solution of (1.6). By Lemma 2.1, there exist a constant $B > 0$ and a set $E_1 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leq Br[T(2r,f)]^{k+1} \quad (j = 1,2,\dots,k). \tag{3.16}$$

Set $\beta = \max\{\sigma(B_j)\ (j = 0,\dots,k-1)\}$. Suppose that $a_{n,j_1},\dots,a_{n,j_m}$ satisfy $a_{n,j_\alpha} = c_{j_\alpha}a_{n,0}$, $j_\alpha \in \{1,\dots,s-1,s+1,\dots k-1\}$, $\alpha \in \{1,\dots,m\}$, $1 \leq m \leq k-2$ and $\arg a_{n,j} = \theta_s$ for $j \in \{1,\dots,s-1,s+1,\dots,k-1\} \setminus \{j_1,\dots,j_m\}$. Choose a constant c satisfying $\max\{c_{j_1},\dots,c_{j_m}\} = c < 1$. We divide the proof into two cases: $c < 0$ and $0 \leq c < 1$.

Case (a): $c < 0$. Since $\theta_0 \neq \theta_s$, there is a ray $\arg z = \theta \in [0,2\pi) \setminus H$, where $H = \{\theta \in [0,2\pi) : \delta(P_0,\theta) = 0 \text{ or } \delta(P_s,\theta) = 0\}$ such that $\delta(P_0,\theta) > 0$ and

$\delta(P_s,\theta)<0$. Hence

$$\delta(P_{j_\alpha},\theta)=c_{j_\alpha}\delta(P_0,\theta)<0(\alpha=1,\dots,m), \tag{3.17}$$
$$\delta(P_j,\theta)=|a_{n,j}|\cos(\theta_s+n\theta)<0, \tag{3.18}$$

where $j\in\{1,\dots,s-1,s+1,\dots,k-1\}\setminus\{j_1,\dots,j_m\}$. By Lemma 2.3, for any given ϵ $(0<2\epsilon<\min\{1,n-\beta\})$, there exists a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z=\theta$, $|z|=r\notin[0,1]\cup E_3$ and a sufficiently large r, we have

$$|A_0(z)e^{P_0(z)}+B_0(z)|\geq(1-o(1))\exp\{(1-\epsilon)\delta(P_0,\theta)r^n\} \tag{3.19}$$

and

$$\begin{aligned}|A_j(z)e^{P_j(z)}+B_j(z)|&\leq\exp\{(1-\epsilon)\delta(P_j,\theta)r^n\}+\exp\{r^{\sigma(B_j)+\frac{\epsilon}{2}}\}\\&\leq\exp\{r^{\sigma(B_j)+\epsilon}\}\\&\leq\exp\{r^{\beta+\epsilon}\}(j=1,\dots,k-1).\end{aligned} \tag{3.20}$$

We rewrite (1.6) as

$$\begin{aligned}&A_0(z)e^{P_0(z)}+B_0(z)\\&=\frac{f^{(k)}}{f}+(A_{k-1}(z)e^{P_{k-1}(z)}+B_{k-1}(z))\frac{f^{(k-1)}}{f}+\dots\\&\quad+(A_s(z)e^{P_s(z)}+B_s(z))\frac{f^{(s)}}{f}+\dots+(A_1(z)e^{P_1(z)}+B_1(z))\frac{f'}{f}.\end{aligned} \tag{3.21}$$

Hence by (3.16) and (3.19)-(3.21), we obtain for all z with $\arg z=\theta$, $|z|=r\notin[0,1]\cup E_1\cup E_3$ and a sufficiently large r,

$$\begin{aligned}&(1-o(1))\exp\{(1-\epsilon)\delta(P_0,\theta)r^n\}\\&\leq(1+(k-1)\exp\{r^{\beta+\epsilon}\})Br[T(2r,f)]^{k+1}\\&\leq kBr\exp\{r^{\beta+\epsilon}\}[T(2r,f)]^{k+1}.\end{aligned} \tag{3.22}$$

Thus $0<2\epsilon<\min\{1,n-\beta\}$ implies $\sigma(f)=+\infty$ and $\sigma_2(f)\geq n$. By Lemma 2.4, we have $\sigma_2(f)=n$.

Case (b): $0\leq c<1$. Using the same reasoning as above, there exists a ray $\arg z=\theta\in[0,2\pi)\setminus H$, where H is defined as above, such that $\delta(P_0,\theta)>0$, and $\delta(P_s,\theta)<0$. Hence

$$\delta(-cP_0,\theta)=-c\delta(P_0,\theta)<0,\delta((1-c)P_0,\theta)=(1-c)\delta(P_0,\theta)>0, \tag{3.23}$$
$$\delta(P_j,\theta)=|a_{n,j}|\cos(\theta_s+n\theta)<0, \tag{3.24}$$

where $j\in\{1,\dots,s-1,s+1,\dots,k-1\}\setminus\{j_1,\dots,j_m\}$,

$$\delta(P_j-cP_0,\theta)<0,j\in\{1,\dots,k-1\}\setminus\{j_1,\dots,j_m\}, \tag{3.25}$$
$$\delta(P_{j_\alpha}-cP_0,\theta)=(c_{j_\alpha}-c)\delta(P_0,\theta)<0(\alpha=1,\dots,m). \tag{3.26}$$

By Lemma 2.3, for any given ϵ $(0<2\epsilon<1)$, there exists a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z=\theta$, $|z|=r\notin[0,1]\cup E_3$

and a sufficiently large r, we have

$$|A_0(z)e^{(1-c)P_0(z)}| \geq \exp\{(1-\epsilon)(1-c)\delta(P_0,\theta)r^n\}, \tag{3.27}$$

$$|e^{-cP_0(z)}| \leq \exp\{-(1-\epsilon)c\delta(P_0,\theta)r^n\} < M, \tag{3.28}$$

$$|B_j(z)e^{-cP_0(z)}| \leq \exp\{-(1-\epsilon)c\delta(P_0,\theta)r^n\} < M, \tag{3.29}$$

$$|A_j(z)e^{P_j(z)-cP_0(z)}| \leq \exp\{(1-\epsilon)\delta(P_j-cP_0,\theta)r^n\} < M, \tag{3.30}$$

where $j=1,\dots,k-1$, and M is a positive constant. We can rewrite (1.6) as

$$\begin{aligned} A_0(z)e^{(1-c)P_0(z)} &= -B_0(z)e^{-cP_0(z)} + e^{-cP_0(z)}\frac{f^{(k)}}{f} \\ &\quad + (A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)} + B_{k-1}(z)e^{-cP_0(z)})\frac{f^{(k-1)}}{f} + \dots \\ &\quad + (A_s(z)e^{P_s(z)-cP_0(z)} + B_s(z)e^{-cP_0(z)})\frac{f^{(s)}}{f} + \dots \\ &\quad + (A_1(z)e^{P_1(z)-cP_0(z)} + B_1(z)e^{-cP_0(z)})\frac{f'}{f}. \end{aligned} \tag{3.31}$$

By (3.16), (3.27)-(3.31), for all z with $|z|=r\notin[0,1]\cup E_1\cup E_3$ and a sufficiently large r, we have

$$\exp\{(1-\epsilon)(1-c)\delta(P_0,\theta)r^n\} \leq M'r[T(2r,f)]^{k+1}, \tag{3.32}$$

where M' is a positive constant. Thus $0<2\epsilon<1$ and (3.32) implie $\sigma(f)=+\infty$ and $\sigma_2(f)\geq n$. By Lemma 2.4, we have $\sigma_2(f)=n$.

Now we prove that equation (1.6) cannot have a nonzero polynomial solution. Let $f(z)$ be a nonzero polynomial solution of (1.6) with $\deg f(z)=q$. Suppose first that $\max\{c_{j_1},\dots,c_{j_m}\}=c<0$. Using the same reasoning as above, there is a ray $\arg z=\theta\in[0,2\pi)\setminus H$, where H is defined as above, such that $\delta(P_0,\theta)>0$, and $\delta(P_s,\theta)<0$. By Lemma 2.3, for any given ϵ $(0<2\epsilon<\min\{1,n-\beta\})$, there exists a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z=\theta$, $|z|=r\notin[0,1]\cup E_3$ and a sufficiently large r, we have (3.19) and (3.20).

By (1.6), (3.19) and (3.20), for all z with $\arg z=\theta$, $|z|=r\notin[0,1]\cup E_3$ and a sufficiently large r, we have

$$\begin{aligned} \gamma_1 r^q(1-o(1))\exp\{(1-\epsilon)\delta(P_0,\theta)r^n\} &\leq |A_0(z)e^{P_0(z)}+B_0(z)||f(z)| \\ &\leq k\gamma_2 r^{q-1}\exp\{r^{\beta+\epsilon}\}. \end{aligned} \tag{3.33}$$

where γ_1 and γ_2 are positive constants. From (3.33),

$$\exp\{(1-\epsilon)\delta(P_0,\theta)r^n\} \leq \frac{\gamma_3}{r}, \tag{3.34}$$

where γ_3 is a positive constant. This is a contradiction. Suppose now that $0\leq c<1$. Using the same reasoning as above, there is a ray $\arg z=\theta\in[0,2\pi)\setminus H$, where H is defined as above, such that $\delta(P_0,\theta)>0$, and $\delta(P_s,\theta)<0$. By Lemma 2.3, for any ϵ $(0<2\epsilon<1)$, there exists a set $E_3\subset[1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z=\theta$, $|z|=r\notin[0,1]\cup E_3$ and a sufficiently large r, we have (3.27)-(3.30).

By (1.6), (3.27)-(3.30), for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r, we have

$$\begin{aligned} &\gamma_4 r^q \exp\{(1-\epsilon)(1-c)\delta(P_0,\theta)r^n\} \\ &\le |A_0(z)e^{(1-c)P_0(z)}||f(z)| \\ &\le |B_0(z)e^{-cP_0(z)}||f(z)| + |e^{-cP_0(z)}||f^{(k)}(z)| \\ &\quad + |A_{k-1}(z)e^{P_{k-1}(z)-cP_0(z)} + B_{k-1}(z)e^{-cP_0(z)}||f^{(k-1)}(z)| \\ &\quad + \dots + |A_1(z)e^{P_1(z)-cP_0(z)} + B_1(z)e^{-cP_0(z)}||f'(z)| \\ &\le \gamma_5 r^q, \end{aligned} \tag{3.35}$$

where γ_4 and γ_5 are positive constants. From (3.35), we obtain for $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r,

$$\exp\{(1-\epsilon)(1-c)\delta(P_0,\theta)r^n\} \le \frac{\gamma_5}{\gamma_4}\,. \tag{3.36}$$

This is a contradiction; hence (1.6) cannot have a nonzero polynomial solution.

If $\arg a_{n,j} = \theta_s$ ($j = 1,\dots,s-1,s+1,\dots,k-1$), then $\arg a_{n,j} \ne \arg a_{n,0}$ ($j = 1,\dots,k-1$) and by Theorem 1.3, it follows that every solution f ($\not\equiv 0$) of (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.

Proof of Theorem 1.9. Assume f is a transcendental solution of (1.6). By Lemma 2.1, there exist a constant $B > 0$ and a set $E_1 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$|\frac{f^{(j)}(z)}{f^{(d)}(z)}| \le Br[T(2r,f)]^{j-d+1} \quad (j = d+1,\dots,k) \tag{3.37}$$

$$|\frac{f^{(j)}(z)}{f(z)}| \le Br[T(2r,f)]^{j+1} \quad (j = 1,2,\dots,d-1). \tag{3.38}$$

By Lemma 2.2, there exists a set $E_2 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r,f)$, we have

$$|\frac{f(z)}{f^{(d)}(z)}| \le 2r^d. \tag{3.39}$$

Set $\beta = \max\{\sigma(B_j)\ (j = 0,\dots,k-1)\}$. Suppose that $a_{n,j_1},\dots,a_{n,j_m}$ satisfy $a_{n,j_\alpha} = c_{j_\alpha} a_{n,d}$, $j_\alpha \in \{0,\dots,k-1\} \setminus \{d,s\}$, $\alpha \in \{1,\dots,m\}$, $1 \le m \le k-2$ and $\arg a_{n,j} = \theta_s$ for $j \in \{0,\dots,k-1\} \setminus \{d,s,j_1,\dots,j_m\}$. Choose a constant c satisfying $\max\{c_{j_1},\dots,c_{j_m}\} = c < 1$. We divide the proof into two cases: $c < 0$ and $0 \le c < 1$.

Case (a): $c < 0$. Since $\theta_d \ne \theta_s$, there is a ray $\arg z = \theta \in [0,2\pi) \setminus H$, where $H = \{\theta \in [0,2\pi) : \delta(P_d,\theta) = 0 \text{ or } \delta(P_s,\theta) = 0\}$ such that $\delta(P_d,\theta) > 0$ and $\delta(P_s,\theta) < 0$. Hence

$$\delta(P_{j_\alpha},\theta) = c_{j_\alpha}\delta(P_d,\theta) < 0 \quad (\alpha = 1,\dots,m), \tag{3.40}$$

$$\delta(P_j,\theta) = |a_{n,j}|\cos(\theta_s + n\theta) < 0, \quad j \in \{0,\dots,k-1\} \setminus \{d,s,j_1,\dots,j_m\}. \tag{3.41}$$

By Lemma 2.3, for any ϵ ($0 < 2\epsilon < \min\{1, n-\beta\}$), there exists a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r, we have

$$|A_d(z)e^{P_d(z)} + B_d(z)| \ge (1-o(1))\exp\{(1-\epsilon)\delta(P_d,\theta)r^n\} \tag{3.42}$$

and

$$\begin{aligned} |A_j(z)e^{P_j(z)} + B_j(z)| &\le \exp\{(1-\epsilon)\delta(P_j,\theta)r^n\} + \exp\{r^{\sigma(B_j)+\frac{\epsilon}{2}}\} \\ &\le \exp\{r^{\sigma(B_j)+\epsilon}\} \\ &\le \exp\{r^{\beta+\epsilon}\}(j \ne d). \end{aligned} \tag{3.43}$$

By (1.6), we have

$$\begin{aligned} &A_d(z)e^{P_d(z)} + B_d(z) \\ &= \frac{f^{(k)}}{f^{(d)}} + \big(A_{k-1}(z)e^{P_{k-1}(z)} + B_{k-1}(z)\big)\frac{f^{(k-1)}}{f^{(d)}} + \dots \\ &\quad + \big(A_{d+1}(z)e^{P_{d+1}(z)} + B_{d+1}(z)\big)\frac{f^{(d+1)}}{f^{(d)}} \\ &\quad + \big(A_{d-1}(z)e^{P_{d-1}(z)} + B_{d-1}(z)\big)\frac{f^{(d-1)}}{f}\frac{f}{f^{(d)}} + \dots \\ &\quad + (A_1(z)e^{P_1(z)} + B_1(z))\frac{f'}{f}\frac{f}{f^{(d)}} + (A_0(z)e^{P_0(z)} + B_0(z))\frac{f}{f^{(d)}}\,. \end{aligned} \tag{3.44}$$

Hence by (3.37)-(3.39) and (3.42)-(3.44), we get for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_3$, $|f(z)| = M(r,f)$ and a sufficiently large r,

$$(1-o(1))\exp\{(1-\epsilon)\delta(P_d,\theta)r^n\} \le M_1 r^{d+1}\exp\{r^{\beta+\epsilon}\}[T(2r,f)]^{k+1}, \tag{3.45}$$

where M_1 is a positive constant. Thus $0 < 2\epsilon < \min\{1, n-\beta\}$ implies $\sigma(f) = +\infty$ and $\sigma_2(f) \ge n$. By Lemma 2.4, we have $\sigma_2(f) = n$.

Case (b): $0 \le c < 1$. Using the same reasoning as above, there exists a ray $\arg z = \theta \in [0, 2\pi) \setminus H$, where H is defined as above, such that $\delta(P_d,\theta) > 0$, and $\delta(P_s,\theta) < 0$. Hence

$$\delta(-cP_d,\theta) = -c\delta(P_d,\theta) < 0, \delta((1-c)P_d,\theta) = (1-c)\delta(P_d,\theta) > 0, \tag{3.46}$$

$$\delta(P_j,\theta) = |a_{n,j}|\cos(\theta_s + n\theta) < 0, \quad j \in \{0,\dots,k-1\} \setminus \{d,s,j_1,\dots,j_m\}, \tag{3.47}$$

$$\delta(P_j - cP_d,\theta) < 0 \quad j \in \{0,\dots,k-1\} \setminus \{d,j_1,\dots,j_m\}, \tag{3.48}$$

$$\delta(P_{j_\alpha} - cP_d,\theta) = (c_{j_\alpha} - c)\delta(P_d,\theta) < 0 \quad (\alpha = 1,\dots,m). \tag{3.49}$$

By Lemma 2.3, for any given ϵ $(0 < 2\epsilon < 1)$, there exists a set $E_3 \subset [1,+\infty)$ having finite logarithmic measure such that for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_3$ and a sufficiently large r, we have

$$|A_d(z)e^{(1-c)P_d(z)}| \ge \exp\{(1-\epsilon)(1-c)\delta(P_d,\theta)r^n\}, \tag{3.50}$$

$$|e^{-cP_d(z)}| \le \exp\{-(1-\epsilon)c\delta(P_d,\theta)r^n\} < M_2, \tag{3.51}$$

$$|B_j(z)e^{-cP_d(z)}| \le \exp\{-(1-\epsilon)c\delta(P_d,\theta)r^n\} < M_2 \quad (j = 0,\dots,k-1). \tag{3.52}$$

$$|A_j(z)e^{P_j(z)-cP_d(z)}| \le \exp\{(1-\epsilon)\delta(P_j - cP_d,\theta)r^n\} < M_2 \quad (j \ne d), \tag{3.53}$$

where M_2 is a positive constant. We can rewrite (1.6) as

$$
\begin{aligned}
&A_d(z)e^{(1-c)P_d(z)}\\
&= -B_d(z)e^{-cP_d(z)} + e^{-cP_d(z)}\frac{f^{(k)}}{f^{(d)}}\\
&\quad + (A_{k-1}(z)e^{P_{k-1}(z)-cP_d(z)} + B_{k-1}(z)e^{-cP_d(z)})\frac{f^{(k-1)}}{f^{(d)}} + \dots\\
&\quad + (A_{d+1}(z)e^{P_{d+1}(z)-cP_d(z)} + B_{d+1}(z)e^{-cP_d(z)})\frac{f^{(d+1)}}{f^{(d)}}\\
&\quad + (A_{d-1}(z)e^{P_{d-1}(z)-cP_d(z)} + B_{d-1}(z)e^{-cP_d(z)})\frac{f^{(d-1)}}{f}\frac{f}{f^{(d)}}\\
&\quad + \dots + (A_1(z)e^{P_1(z)-cP_d(z)} + B_1(z)e^{-cP_d(z)})\frac{f'}{f}\frac{f}{f^{(d)}}\\
&\quad + (A_0(z)e^{P_0(z)-cP_d(z)} + B_0(z)e^{-cP_d(z)})\frac{f}{f^{(d)}}.
\end{aligned}
\tag{3.54}
$$

By (3.37)-(3.39) and (3.50)-(3.54), for all z with $\arg z = \theta$, $|z| = r \notin [0,1] \cup E_1 \cup E_2 \cup E_3$, $|f(z)| = M(r,f)$ and a sufficiently large r, we have

$$
\exp\{(1-\epsilon)(1-c)\delta(P_d,\theta)r^n\} \le M_3 r^{d+1}[T(2r,f)]^{k+1}, \tag{3.55}
$$

where M_3 is a positive constant. Thus $0 < 2\epsilon < 1$ implies $\sigma(f) = +\infty$ and $\sigma_2(f) \ge n$. By Lemma 2.4, we have $\sigma_2(f) = n$.

If $\arg a_{n,j} = \theta_s$ ($j \ne d, s$), then $\arg a_{n,j} \ne \arg a_{n,d}$ ($j \ne d$) and by Theorem 1.6, it follows that every transcendental solution f of equation (1.6) is of infinite order and satisfies $\sigma_2(f) = n$.

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Karima Hamani
Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem, B. P. 227 Mostaganem, Algeria
E-mail address: hamanikarima@yahoo.fr