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SPECIAL SOLUTIONS OF THE RICCATI EQUATION WITH APPLICATIONS TO THE GROSS-PITAEVSKII NONLINEAR PDE

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Abstract. A method for finding solutions of the Riccati differential equation $y' = P(x) + Q(x)y + R(x)y^2$ is introduced. Provided that certain relations exist between the coefficient $P(x)$, $Q(x)$ and $R(x)$, the above equation can be solved in closed form. We determine the required relations and find the general solutions to the aforementioned equation. The method is then applied to the Riccati equation arising in the solution of the multidimensional Gross-Pitaevskii equation of Bose-Einstein condensates by the F-expansion and the balance principle techniques.

1. Introduction

The Riccati equation (RE), named after the Italian mathematician Jacopo Francesco Riccati [10], is a basic first-order nonlinear ordinary differential equation (ODE) that arises in different fields of mathematics and physics [15]. It has the form

$$y' = P(x) + Q(x)y + R(x)y^2, \tag{1.1}$$

which can be considered as the lowest order *nonlinear* approximation to the derivative of a function in terms of the function itself. It is assumed that y, P, Q and R are real functions of the real argument x. It is well known that solutions to the general Riccati equation are not available, and only special cases can be treated [5, 3, 14, 7, 23, 12]. Even though the equation is nonlinear, similar to the second order inhomogeneous linear ODEs one needs only a particular solution to find the general solution.

In a standard manner Riccati equation can be reduced to a second-order linear ODE [10, 5] or to a Schrödinger equation (SE) of quantum mechanics [16]. In fact, Riccati equation naturally arises in many fields of quantum mechanics; in particular, in quantum chemistry [4], the Wentzel-Kramers-Brillouin approximation [17] and SUSY theories [8]. Recently, methods for solving the Gross-Pitaevskii equation (GPE) arising in Bose-Einstein condensates (BECs) [1, 20] based on Riccati equation were introduced. Our objective is to find new solutions of Riccati equation by

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utilizing relations between the coefficient functions $P(x)$, $Q(x)$ and $R(x)$ for which the above equation can be solved in closed form.

It is well known that any equation of the Riccati type can always be reduced to the second order linear ODE

$$u'' - \big[Q(x) + \frac{R'(x)}{R(x)}\big]u' + P(x)R(x)u = 0 \tag{1.2}$$

by a substitution $y = -u'/(uR)$. It is also known that if one can find a particular solution y_p to the original equation, then the general solution can be written as $y = y_p + 1/w$ [18], where w is the general solution of an associated linear ODE

$$w' + [Q(x) + 2R(x)y_p]\, w + R(x) = 0 \tag{1.3}$$

which does not contain $P(x)$. Solving this equation we get [13]: $w = w_0 e^{-\phi(x)} - e^{-\phi(x)} \int_{x_0}^{x} R(\xi)e^{\phi(\xi)}\, d\xi$, where $\phi(x) = \int_{x_0}^{x} [Q(\xi) + 2R(\xi)y_p]\, d\xi$. It is clearly seen from the relation above that $w_0 = \frac{1}{y_0 - y_{p0}}$. The general solution is therefore given by [13]:

$$y = y_p + e^{\phi(x)}\big[\frac{1}{y_0 - y_{p0}} - \int_{x_0}^{x} R(\xi)e^{\phi(\xi)}\, d\xi\big]^{-1}. \tag{1.4}$$

This article contains four sections. Section 2 introduces the solution method, Sec. 3 presents an application and Sec. 4 brings a conclusion.

2. Solution method

Equation (1.1) cannot be solved in closed form for arbitrary functions $P(x)$, $Q(x)$ and $R(x)$. However, if certain relations exist between these functions, then the above equation can be transformed into a second order linear ODE, which can be easily solved in two cases: If it contains constant coefficients, or if it contains certain coefficient functions.

For the sake of making our calculations clearer, we make the following two substitutions: $a(x) = -(Q + R'/R)$ and $b(x) = P(x)R(x)$. Now the above ODE for u becomes

$$\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u = 0. \tag{2.1}$$

Consider an arbitrary function of x, $z \equiv f(x)$, which we choose to be a new independent variable. The substitution looks arbitrary, but it will be made more specific in a moment. We compute the first and second derivatives of u with respect to x, but now in terms of the new independent variable z:

$$\frac{du}{dx} = \frac{du}{dz}\frac{dz}{dx} \tag{2.2}$$

$$\frac{d^2u}{dx^2} = \frac{d^2z}{dx^2}\frac{du}{dz} + \big(\frac{dz}{dx}\big)^2\frac{d^2u}{dz^2}. \tag{2.3}$$

We plug the last results into the differential equation (2.1), to get:

$$\Big(\frac{dz}{dx}\Big)^2\frac{d^2u}{dz^2} + \Big[\frac{d^2z}{dx^2} + a(x)\frac{dz}{dx}\Big]\frac{du}{dz} + b(x)u = 0. \tag{2.4}$$

Finally, dividing by $\left(\frac{dz}{dx}\right)^2$, we obtain [19]:

$$\frac{d^2u}{dz^2} + \Big[\frac{\frac{d^2z}{dx^2} + a(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\Big]\frac{du}{dz} + \Big[\frac{b(x)}{\left(\frac{dz}{dx}\right)^2}\Big]u = 0 \tag{2.5}$$

$$\equiv \frac{d^2u}{dz^2} + 2A\frac{du}{dz} + Bu = 0, \tag{2.6}$$

provided dz/dx is not equal to 0. The obtained equation can easily be solved in closed form if A and B are either constants [19] or if they are some special functions for which the closed-form solutions to (2.6) are known. In this paper we consider only the two special cases, namely when A and $B > 0$ are constants, or when $A = 0$ and B is an arbitrary function $B(x)$.

If $b(x)$ is positive, by considering the coefficient of u we define z to be the following function:

$$z \equiv z_0 + s\int_{x_0}^{x}\sqrt{\frac{b(\xi)}{B}}d\xi, \tag{2.7}$$

where $s = \pm 1$. The requirement that $b(x)$ is positive is equivalent to the condition that the product $P(x)R(x)$ is positive. To simplify bookkeeping, let $c = b/B$; then we have the following relations:

$$\frac{dz}{dx} = sc^{1/2}, \tag{2.8}$$

$$\frac{d^2z}{dx^2} = \frac{c'}{2sc^{-1/2}}. \tag{2.9}$$

From (2.8) it is clear that dz/dx cannot be equal to 0. Now we compare the coefficients of du/dz and use relations (2.8) and (2.9) to get:

$$\frac{c'}{2sc^{1/2}} + asc^{1/2} - 2Ac = 0, \tag{2.10}$$

or

$$\big(\frac{b}{B}\big)' + 2a\big(\frac{b}{B}\big) - 4As\big(\frac{b}{B}\big)^{3/2} = 0. \tag{2.11}$$

At this point it is more convenient to consider the two cases separately.

2.1. **Case 1: A and B are constants.** If (2.6) has constant coefficients $2A$ and B, then it is easily solvable in closed form. This means:

$$b' + 2ab - \frac{4sA}{\sqrt{B}}b^{3/2} = 0 \tag{2.12}$$

or:

$$\frac{b'(x) + 2a(x)b(x)}{[b(x)]^{3/2}} = \frac{4sA}{\sqrt{B}}. \tag{2.13}$$

Substituting back the original expressions for $a(x)$ and $b(x)$, we get the final result:

$$\frac{[P(x)R(x)]' - 2\left[Q(x) + R'(x)/R(x)\right]P(x)R(x)}{[P(x)R(x)]^{3/2}} = \frac{4sA}{\sqrt{B}}. \tag{2.14}$$

At this point a few comments are in order. First, note that we are stating that if the condition (2.14) is satisfied, then the general solution can be found. However, when the condition is *not* satisfied, this does not mean that the general solution cannot be found. In fact, most of the known special cases of Riccati equation (with known solutions) [12] do not satisfy the relation obtained.

Second, one may object that in place of the original nonlinear Riccati equation we obtained another *nonlinear* equation for $b(x)$, which might be equally difficult to solve. Luckily, this is not the case; (2.12) has a constant coefficient in front of the nonlinear term (which is also a *variable* parameter at our will) and hence is more manageable. It often allows easy solutions, as we display below, for which one can find nontrivial solutions of the original Riccati equation.

Now we proceed to solve (2.6). The general solution is given by:

$$u(x) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z}, \tag{2.15}$$

where z is the function defined in (2.7), c_1 and c_2 are some initial values, and λ_1 and λ_2 are the roots of the characteristic polynomial $\lambda^2 + 2A\lambda + B = 0$, given by:

$$\lambda_{1,2} = -A \pm \sqrt{A^2 - B}. \tag{2.16}$$

Hence, we assume that $A^2 \geq B > 0$, so that both lambdas are real and negative. This condition is not necessary for the solution procedure, but is convenient for the applications of solutions, which require real functions. We need only a particular solution of (2.6), so we consider only $u_p = e^{\lambda z}$, where λ is any of the roots to the polynomial.

From the substitution done in (1.2), namely $y = -u'/[uR(x)]$, we find the particular solution to be:

$$y_p = -\frac{s\lambda}{\sqrt{B}} \sqrt{\frac{P(x)}{R(x)}} \tag{2.17}$$

Finally, we plug y_p into the expression for the general solution of Riccati equation, to find:

$$y = -\frac{s\lambda}{\sqrt{B}} \sqrt{\frac{P(x)}{R(x)}} + e^{\phi(x)} \Big[\frac{1}{y_0 + \frac{s\lambda}{\sqrt{B}} \sqrt{\frac{P(0)}{R(0)}}} - \int_{x_0}^{x} R(\xi) e^{\phi(\xi)}\, d\xi \Big]^{-1} \tag{2.18}$$

Note that we have substituted y_{p0} by its value. To recapitulate, here A and B are two arbitrary constants satisfying $A^2 \geq B > 0$, λ is one of the roots of the characteristic polynomial, y_0 is the initial condition for y, and

$$\phi(x) = \int_{x_0}^{x} \Big[Q(\xi) - \frac{2s\lambda}{\sqrt{B}} \sqrt{P(\xi)R(\xi)} \Big]\, d\xi \tag{2.19}$$

is the integrating exponent. Below we apply this general result to some specific examples.

2.2. **Case 2: $A = 0$ and $B = B(x)$.** When $A = 0$, (2.10) reduces to the simple equation $c' = -2ac$. Solving for c, and remembering that $c = b/B$, we get the simple relation

$$\frac{b}{B} = \Big(\frac{b}{B}\Big)_0 \exp\Big(-2 \int_{x_0}^{x} a\, dx \Big), \tag{2.20}$$

where $a(x)$ and $b(x)$ are given by the original Riccati equation, and $B(x)$ is still an arbitrary function. Note that (2.6) now becomes

$$\frac{d^2u}{dz^2} + B(z)u = 0, \tag{2.21}$$

where z is given by (2.7). When $B(z)$ is chosen as $B(z) = B_0 + B_1(z)$, then the last equation becomes equivalent to the Schrödinger equation of quantum mechanics, which is a linear second order differential equation of the form:

$$\psi'' + \frac{2m}{\hbar^2}(E - V)\psi = 0. \tag{2.22}$$

This is an equation for the wave function $\psi = u(z)$ of a particle of mass m moving in a potential $V = -\hbar^2 B_1(z)/2m$ with an energy eigenvalue $E = \hbar^2 B_0/2m$ [9]. There are many specific potentials V for which the solutions ψ_n and the energies E_n in the above equation are known. Here n denotes some set of quantum numbers. Therefore, one can choose $B(z)$ such that the solutions $u_n(z)$ can be found. If $u_n(z)$ (and hence $u_n(x)$) are known, then the solutions y_n to Riccati equation can be easily written down from the substitution $y_n(x) = -u_n'/(u_n R)$ mentioned above. This in fact gives rise to various solutions of the various special cases of Riccati equation.

3. Application

In [11] we considered the generalized GPE in (3+1)D for the BEC wave function $u(x, y, z, t)$, with distributed time-dependent coefficients [1, 20, 2]:

$$i\partial_t u + \frac{\beta(t)}{2}\Delta u + \chi(t)|u|^2 u + \alpha(t) r^2 u = i\gamma(t)u. \tag{3.1}$$

Here t is time, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the 3D Laplacian, $r = \sqrt{x^2 + y^2 + z^2}$ is the position coordinate, and $\alpha(t)$ stands for the strength of the quadratic potential as a function of time. The functions β, χ and γ stand for the diffraction, nonlinearity and gain/loss coefficients, respectively.

According to the F-expansion and the balance principle techniques [22], in [11] we sought the solution in the form:

$$u(x, y, z, t) = \mathcal{M}(x, y, z, t) \exp[i\mathcal{P}(x, y, z, t)], \tag{3.2}$$

where the magnitude $\mathcal{M}(x, y, z, t)$ and the phase $\mathcal{P}(x, y, z, t)$ are given by

$$\mathcal{M}(x, y, z, t) = f(t)F(\theta) + g(t)F^{-1}(\theta), \tag{3.3}$$

$$\theta = k(t)x + l(t)y + m(t)z + \omega(t), \tag{3.4}$$

$$\mathcal{P}(x, y, z, t) = a(t)r^2 + b(t)(x + y + z) + e(t). \tag{3.5}$$

Here f, g, k, l, m, ω, a, b, e are parameter functions to be determined, and F is one of the Jacobi elliptic functions. The functions $a(t)$ and $b(t)$ should not be confused with the functions $a(x)$ and $b(x)$ used before. Of all the parameters, by far the most important is the chirp function $a(t)$, because all other parameters, as well as the general solution of GPE, can be expressed in terms of a. On the other hand, the equation for the determination of a is a Riccati equation of the following type [11]:

$$\frac{da}{dt} + 2\beta(t)a^2 - \alpha(t) = 0. \tag{3.6}$$

To this equation we apply the method developed in this paper. We take A and B to be constant here.

Put in the form of the original Riccati equation, the coefficients are:

$$P(t) = \alpha(t), \quad Q(t) = 0, \quad R(t) = -2\beta(t). \tag{3.7}$$

We write down relation (2.14) between α and β for which (3.6) is solvable in closed form:

$$\frac{\alpha\beta' - \alpha'\beta}{(-\alpha\beta)^{3/2}} = \frac{4\sqrt{2}sA}{\sqrt{B}}. \tag{3.8}$$

The prime is now the derivative with respect to t. Equation (3.8) can be manipulated to become a simple differential equation for $-\alpha/\beta$:

$$\frac{\left(-\frac{\alpha}{\beta}\right)'}{\left(-\frac{\alpha}{\beta}\right)^{3/2}\beta} = \frac{4\sqrt{2}sA}{\sqrt{B}}. \tag{3.9}$$

Solving this equation, one finds:

$$\sqrt{-\frac{\beta}{\alpha}} = \sqrt{-\frac{\beta_0}{\alpha_0}} - \frac{2\sqrt{2}sA}{\sqrt{B}}\int_0^t \beta\, dt. \tag{3.10}$$

Now one can write down the solution for $a(t)$ from (2.18), provided the above condition is satisfied:

$$a(t) = -\frac{s\lambda}{\sqrt{B}}\sqrt{-\frac{\alpha(t)}{2\beta(t)}} + e^{\phi(t)}\Big[\frac{1}{a_0 + \frac{s\lambda}{\sqrt{B}}\sqrt{-\frac{\alpha_0}{2\beta_0}}} + 2\int_0^t \beta(\tau)e^{\phi(\tau)}\, d\tau\Big]^{-1}, \tag{3.11}$$

where $\phi(t) = -2\sqrt{2}s\lambda\int_0^t \sqrt{-\alpha(\tau)\beta(\tau)}\, d\tau/\sqrt{B}$. Note that the $-$ sign in the square root indicates that α and β have to be of the opposite signs, which is consistent with the requirement that the original function b is positive. Hence, as long as the ratio of the diffraction coefficient to the strength of the parabolic potential can be made to satisfy (3.9), one can write down the exact solutions to GPE. It should be mentioned that these functions are the material parameters in BECs that are accessible to experimental manipulation.

Our solution method for the GPE requires that β be proportional to χ, and χ in turn be proportional to the s-wave scattering length [21]. To validate our proposed solution method, we present a couple of examples in which β, and hence the scattering length, are given by some representative functions of time. In all the examples we determine the corresponding chirp functions $a(t)$, from which one can write down the exact solutions of the GPEs in question [11]. To avoid singularities that are likely to appear in $\alpha(t)$ and $a(t)$ we are choosing s to be -1. Note that the appearance of singularities is not detrimental to our method or to the theory of BECs based on GPE, because that model is known to be valid only on a limited time interval.

3.1. **Example 1:** $\beta = \frac{1}{2}(e^{-\delta t} + 1)$**.** We consider first the case when β is an exponential function of time, $\beta(t) = \frac{1}{2}(e^{-\delta t} + 1)$, where δ is some arbitrary parameter. This function describes a smooth change in $\beta(t)$ from 1 to 1/2. First, (3.10) is solved for α, to obtain:

$$\alpha(t) = -\frac{1 + e^{-\delta t}}{2\left(1 + \frac{\sqrt{2}(1 - e^{-\delta t} + \delta t)}{\delta}\right)^2} \tag{3.12}$$

Then one finds ϕ:

$$\phi(t) = \delta t + \ln\Big|\frac{\delta}{-\sqrt{2} + e^{\delta t}\left(\sqrt{2} + \delta + \sqrt{2}\delta t\right)}\Big|. \tag{3.13}$$

Taking $\alpha_0 = -1$, $A = B = 1$, and performing the calculations, we obtain the following solution for a:

$$a(t) = \frac{-\delta}{2 - 2e^{-\delta t} + \sqrt{2}\delta + 2t\delta} - \frac{\delta\sqrt{2}e^{\delta t}}{\left[-\sqrt{2} + e^{\delta t}\left(\sqrt{2} + \delta + \sqrt{2}t\delta\right)\right]\zeta(t)} \tag{3.14}$$

where

$$\zeta(t) = \delta t + \ln\Big|\frac{\delta}{-\sqrt{2} + e^{t\delta}(\sqrt{2} + \delta + \sqrt{2}t\delta)}\Big| - \frac{2}{1 + \sqrt{2}a_0}.$$

Although this solution looks complicated, it allows simple expressions in the limit $\delta \to 0$, when β becomes constant. Figure 1 presents some representative cases of α and a functions for different values of δ.

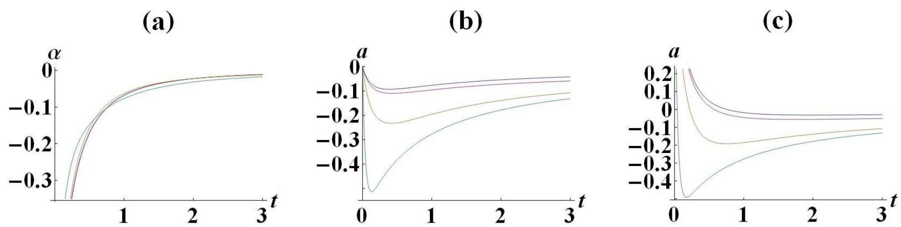

FIGURE 1. Graphs of (a): $\alpha(t)$, (b): $a(t)$ for $a_0 = 0$, and (c): $a(t)$ for $a_0 = 1$, for $\delta = 0.01, 0.1, 1, 10$ (top to bottom).

3.2. **Example 2:** $\beta = \sum_{n=0}^{N} \beta_n t^n$**.** Next, we consider the case when β is some power series of the form $\sum_{n=0}^{N} \beta_n t^n$, where $\beta_0 \neq 0$. We go through the same procedure and solve (3.10) for α, to get:

$$\alpha(t) = -\frac{\sum_{n=0}^{N} \beta_n t^n}{\left(1 + 2\sqrt{2}\sum_{n=0}^{N} \beta_n \frac{t^{n+1}}{n+1}\right)^2}. \tag{3.15}$$

Then we find ϕ to be:

$$\phi(t) = \ln\frac{1}{\left|1 + 2\sqrt{2}\sum_{n=0}^{N} \beta_n \frac{t^{n+1}}{n+1}\right|}. \tag{3.16}$$

Again, taking $\alpha_0 = -1$, $A = B = 1$, and performing the calculations, we arrive at the following closed-form solution:

$$a(t) = \frac{2\sqrt{2}a_0 - (a_0\sqrt{2} + 1)\ln\left|1 + 2\sqrt{2}\sum_{n=0}^{N} \beta_n \frac{t^{n+1}}{n+1}\right|}{\left(1 + 2\sqrt{2}\sum_{n=0}^{N} \beta_n \frac{t^{n+1}}{n+1}\right)\left[2\sqrt{2} + (2a_0 + \sqrt{2})\ln\left|1 + 2\sqrt{2}\sum_{n=0}^{N} \beta_n \frac{t^{n+1}}{n+1}\right|\right]}. \tag{3.17}$$

These solutions for α and a are plotted in Fig. 2. Note that by choosing different parameters β_n and letting $N \to \infty$ one can obtain closed-form expressions for different functions $\beta(t)$. Figure 3 presents the case with $\beta = \cos(\Omega t)$.

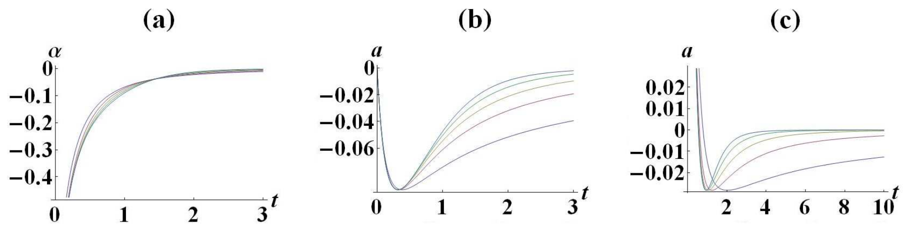

FIGURE 2. Same as Fig. 1. (a) $\alpha(t)$, (b) $a(t)$ for $a_0 = 0$, and (c) $a(t)$ for $a_0 = 1$. Parameters: $N = 0, 1, 2, 3, 4$ (top to bottom at $t = 0.5$ for α; bottom to top at $t = 3$ for a), $\beta_n = 1$.

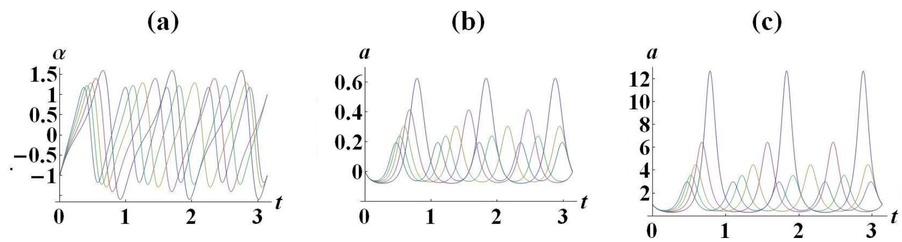

FIGURE 3. Same as Fig. 2, but for $\beta(t) = \cos(\Omega t)$. (a) $\alpha(t)$, (b) $a(t)$ for $a_0 = 0$, and (c) $a(t)$ for $a_0 = 1$. Here $\Omega = 6, 7, 8, 9, 10$; Curves with higher peaks correspond to lower values of Ω.

3.3. **Example 3:** $\beta = \tilde{\beta}\big(1 - \frac{D}{B_1 t - B_0}\big)$**.** Finally, we consider the case when β is of the form shown above. This form is dictated by the dependance of the scattering length on the magnetic field near the Feshbach resonance of cold BEC atoms [21]. The magnetic field $B(t) = B_1 t$ (again, not to be confused with the function $B(x)$ from the solution procedure) is assumed to be linearly ramped in time near the resonance field B_0. The parameter D stands for the width of the resonance. Such a dependence is found relevant not only on theoretical grounds [21] but most importantly experimentally [6].

The closed-form solution is again readily obtained; however, this time it includes integrals that cannot be evaluated in terms of elementary functions. The results for α, ϕ, and a are as follows:

$$\alpha(t) = -\tilde{\beta}\frac{1 - \frac{D}{B_1 t - B_0}}{\big[1 - 2\sqrt{2}\tilde{\beta}t + \frac{2\sqrt{2}\tilde{\beta}D}{B_1}\ln\big|\frac{B_1 t - B_0}{B_0}\big|\big]^2}, \tag{3.18}$$

$$\phi(t) = 2\sqrt{2}\,\tilde{\beta}\int_0^t \frac{1 - \frac{D}{B_1 \tau - B_0}}{1 - 2\sqrt{2}\tilde{\beta}\tau + \frac{2\sqrt{2}\tilde{\beta}D}{B_1}\ln\big|\frac{B_1 \tau - B_0}{B_0}\big|}\,d\tau, \tag{3.19}$$

$$a(t) = \frac{1}{\sqrt{2} - 2\tilde{\beta}t + \frac{4\tilde{\beta}D}{B_1}\ln\big|\frac{B_1 t - B_0}{B_0}\big|} + \frac{e^{\phi(t)}}{\frac{\sqrt{2}}{a_0\sqrt{2}-1} + 2\tilde{\beta}\int_0^t \big(1 - \frac{D}{B_1 \tau - B_0}\big)e^{\phi(\tau)}\,d\tau}. \tag{3.20}$$

4. Conclusion

We conclude the paper by restating our results. Provided for the case 1 that the following condition between the coefficient functions of Riccati equation $P(x)$, $Q(x)$, and $R(x)$ is met:

$$\frac{[P(x)R(x)]' - 2\left[Q(x) + R'(x)/R(x)\right]P(x)R(x)}{[P(x)R(x)]^{3/2}} = \frac{4sA}{\sqrt{B}}, \tag{4.1}$$

then the general solution of Riccati equation is given by

$$y = -\frac{s\lambda}{\sqrt{B}}\sqrt{\frac{P(x)}{R(x)}} + e^{\phi(x)}\Big[\frac{1}{y_0 + \frac{s\lambda}{\sqrt{B}}\sqrt{\frac{P(0)}{R(0)}}} - \int_{x_0}^{x} R(\xi)e^{\phi(\xi)}\,d\xi\Big]^{-1}. \tag{4.2}$$

Here A and B are two arbitrary constants satisfying $A^2 \geq B > 0$; $\lambda = -A \pm \sqrt{A^2 - B}$ is one of the two roots of the characteristic polynomial; y_0, $P(0)$, $R(0)$ are the given boundary conditions; and $\phi(x) = \int_{x_0}^{x}\left[Q(\xi) - \frac{2s\lambda}{\sqrt{B}}\sqrt{P(\xi)R(\xi)}\right]d\xi$.

In the other case when $A = 0$ and B is some arbitrary function of x, provided the following relation between the coefficient functions a and b is valid:

$$\frac{b}{B} = \big(\frac{b}{B}\big)_0 \exp\Big(-2\int_{x_0}^{x} a\,dx\Big), \tag{4.3}$$

then a simple relation between the second order ODE for u and the one-dimensional Schrödinger equation exists. Hence, many of the known exact solutions to the Schrödinger equation for different potentials can be utilized to arrive at the solutions to various types of new special cases of Riccati equation.

When applied to the multidimensional GPE of BECs, the case with constant A and B yields closed form solutions for the chirp function $a(t)$ of the matter wave:

$$a(t) = -\frac{s\lambda}{\sqrt{B}}\sqrt{-\frac{\alpha(t)}{2\beta(t)}} + e^{\phi(t)}\Big[\frac{1}{a_0 + \frac{s\lambda}{\sqrt{B}}\sqrt{-\frac{\alpha_0}{2\beta_0}}} + 2\int_0^t \beta(\tau)e^{\phi(\tau)}\,d\tau\Big]^{-1}, \tag{4.4}$$

given that the following relation holds between the diffraction coefficient β and the strength of the parabolic potential α:

$$\sqrt{-\frac{\beta}{\alpha}} = \sqrt{-\frac{\beta_0}{\alpha_0}} - \frac{2\sqrt{2}sA}{\sqrt{B}}\int_0^t \beta\,dt. \tag{4.5}$$

Here $\phi(t) = -2\sqrt{2}s\lambda\int_0^t \sqrt{-\alpha(\tau)\beta(\tau)}\,d\tau/\sqrt{B}$. The chirp function is an essential part of the exact solutions to GPE.

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