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EXISTENCE OF SOLUTIONS FOR NONLINEAR PARABOLIC SYSTEMS VIA WEAK CONVERGENCE OF TRUNCATIONS

ELHOUSSINE AZROUL, HICHAM REDWANE, MOHAMED RHOUDAF

Abstract. We prove an existence result for a class of nonlinear parabolic systems. Without assumptions on the growth of some nonlinear terms, we prove the existence of a renormalized solution.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N \geq 1)$, $T > 0$ and let $Q := (0,T) \times \Omega$. We prove the existence of a renormalized solution for the nonlinear parabolic systems

$$(b_i(u_i))_t - \operatorname{div}\Big(a(x, u_i, Du_i) + \Phi_i(u_i)\Big) + f_i(x, u_1, u_2) = 0 \quad \text{in } Q, \tag{1.1}$$

$$u_i = 0 \quad \text{on } \Gamma := (0,T) \times \partial\Omega, \tag{1.2}$$

$$b_i(u_i)(t = 0) = b_i(u_{i,0}) \quad \text{in } \Omega, \tag{1.3}$$

where $i = 1, 2$. Here, the vector field

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Carathéodory function such that} \tag{1.4}$$

- There exists $\alpha > 0$ with
$$a(x, s, \xi).\xi \geq \alpha|\xi|^p \tag{1.5}$$
for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \in \mathbb{R}^N$.
- For each $K > 0$, there exists $\beta_K > 0$ and a function C_K in $L^{p'}(\Omega)$ such that
$$|a(x, s, \xi)| \leq C_K(x) + \beta_K|\xi|^{p-1} \tag{1.6}$$
for almost every $x \in \Omega$, for every s such that $|s| \leq K$, and for every $\xi \in \mathbb{R}^N$.
- The vector field a is monotone in ξ; i.e.,
$$[a(x, s, \xi) - a(x, s, \xi')][\xi - \xi'] \geq 0, \tag{1.7}$$
for any $s \in \mathbb{R}$, for any $(\xi, \xi') \in \mathbb{R}^{2N}$ and for almost every $x \in \Omega$.

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Moreover, we suppose that for $i = 1, 2$,

$$\Phi_i : \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function,} \tag{1.8}$$

$$b_i : \mathbb{R} \to \mathbb{R} \text{ is a strictly increasing } C^1\text{-function with } b_i(0) = 0, \tag{1.9}$$

$f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with

$$f_1(x, 0, s) = f_2(x, s, 0) = 0 \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}. \tag{1.10}$$

and for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$,

$$sign(s_i) f_i(x, s_1, s_2) \geq 0. \tag{1.11}$$

The growth assumptions on f_i are as follows: For each $K > 0$, there exists $\sigma_K > 0$ and a function F_K in $L^1(\Omega)$ such that

$$|f_1(x, s_1, s_2)| \leq F_K(x) + \sigma_K \ |b_2(s_2)| \tag{1.12}$$

a.e. in Ω, for all s_1 such that $|s_1| \leq K$, for all $s_2 \in \mathbb{R}$.

For each $K > 0$, there exists $\lambda_K > 0$ and a function G_K in $L^1(\Omega)$ such that

$$|f_2(x, s_1, s_2)| \leq G_K(x) + \lambda_K \ |b_1(s_1)| \tag{1.13}$$

for almost every $x \in \Omega$, for every s_2 such that $|s_2| \leq K$, and for every $s_1 \in \mathbb{R}$. Finally, we assume the following condition on the initial data $u_{i,0}$:

$$u_{i,0} \text{ is a measurable function such that } b_i(u_{i,0}) \in L^1(\Omega), \text{ for } i = 1, 2. \tag{1.14}$$

The main difficulty when dealing with problem (1.1)-(1.3) is due to the fact that the functions $a(x, u_i, Du_i), \Phi_i(u_i)$ and $f_i(x, u_1, u_2)$ are not in $(L^1_{loc}(Q))^N$ in general, since the growth of $a(x, u_i, Du_i), \Phi_i(u_i)$ and $f_i(x, u_1, u_2)$ are not controlled with respect to u_i, so that proving existence of a weak solution (i.e. in the distribution meaning) seems to be an arduous task. To overcome this difficulty, we use in this paper the framework of renormalized solutions due to Lions and DiPerna [20] for the study of Boltzmann equations (see also Lions [21] for a few applications to fluid mechanics models). This notion was then adapted to the elliptic version of (1.1)-(1.3) in Boccardo, Diaz, Giachetti, Murat [11], in Lions and Murat [22] and Murat[22, 23]. At the same the equivalent notion of entropy solutions have been developed independently by Bénilan and al. [1] for the study of nonlinear elliptic problems.

The particular case where $b_i(u_i) = u_i$ and $\Phi_i = \Phi$, $i = 1, 2$ has been studied in Redwane [25] and for the parabolic version of (1.1)-(1.3), existence and uniqueness results are already proved in [4] (see also [30] and [24]) in the case where $f_i(x, u_1, u_2)$ is replaced by $f + \operatorname{div}(g)$ where $f \in L^1(Q)$ and $g \in L^{p'}(Q)^N$.

In the case where $a(t, x, s, \xi)$ is independent of s, $\Phi_i = 0$ and $g = 0$, existence and uniqueness are established in [2]; in [3], and in the case where $a(t, x, s, \xi)$ is independent of s and linear with respect to ξ, existence and uniqueness are established in [7].

In the case where $\Phi_i = 0$ and the operator $\Delta_p u = \operatorname{div} |\nabla u|^{p-2} \nabla u)$ p-Laplacian replaces a nonlinear term $\operatorname{div} a(x, s, \xi))$, existence of a solution for nonlinear parabolic systems (1.1)-(1.3) is investigated in [26, 27], in [28] and in [29], where an existence result of as (usual) weak solution is proved.

This article is organized as follows: in Section 2, we specify the notation and give the definition of a renormalized solution of (1.1)-(1.3). Then, in Section 3, we establish the existence of such a solution (see Theorem 3.1).

2. Notation

In this paper, for $K > 0$, we denote by $T_K : r \mapsto \min(K, max(r, -K))$ the truncation function at height K. For any measurable subset E of Q, we denote by $meas(E)$ the Lebesgue measure of E. For any measurable function v defined on Q and for any real number $s, \chi_{\{v<s\}}$ (respectively, $\chi_{\{v=s\}}, \chi_{\{v>s\}}$) denote the characteristic function of the set $\{(x,t) \in Q\ ;\ v(x,t) < s\}$ (respectively, $\{(x,t) \in Q; v(x,t) = s\}, \{(x,t) \in Q; v(x,t) > s\}$).

Definition 2.1. *A couple of functions (u_1, u_2) defined on Q is called a renormalized solution of (1.1)-(1.3) if for $i = 1, 2$ the function u_i satisfies*

$$T_K(u_i) \in L^p(0,T; W_0^{1,p}(\Omega)) \quad \text{and} \quad b_i(u_i) \in L^\infty(0,T; L^1(\Omega)), \tag{2.1}$$

for any $K \geq 0$.

$$\int_{\{(t,x)\in Q\ ;\ n\leq |u_i(x,t)|\leq n+1\}} a(x, u_i, Du_i) Du_i \,dx\,dt \to 0 \quad \text{as } n \to +\infty, \tag{2.2}$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have

$$\begin{aligned} &\frac{\partial b_{i,S}(u_i)}{\partial t} - \operatorname{div} S'(u_i) a(x, u_i, Du_i)) + S''(u_i) a(x, u_i, Du_i) Du_i \\ &- \operatorname{div} S'(u_i)\Phi_i(u_i)) + S''(u_i)\Phi_i(u_i) Du_i + f_i(x, u_1, u_2) S'(u_i) = 0 \quad \text{in } D'(Q), \end{aligned} \tag{2.3}$$

and

$$b_{i,S}(u_i)(t = 0) = b_{i,S}(u_{i,0}) \quad \text{in } \Omega, \tag{2.4}$$

where $b_{i,S}(r) = \int_0^r b_i'(s) S'(s)\,ds$.

Remark 2.2. Equation (2.3) is formally obtained through pointwise multiplication of equation (1.1) by $S'(u_i)$. Note that in Definition 2.1, the gradient Du_i is not defined even as a distribution, but that due to (2.1) each term in (2.3) has a meaning in $L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega))$.

Indeed if $K > 0$ is such that $\operatorname{supp} S' \subset [-K, K]$, the following identifications are made in (2.3):

- $b_{i,S}(u_i)$ belong to $L^\infty(Q) \cap L^p(0,T; W_0^{1,p}(\Omega))$. Indeed
$$Db_{i,S}(u) = S'(u_i) b_i'(T_K(u_i)) DT_K(u_i) \in (L^p(\Omega))^N$$
and
$$|b_{i,S}(u_i)| \leq \int_0^{|u_i|} |S'(s) b_i'(s)|\,ds \leq K \max_{|r|\leq K} |S'(r) b_i'(r)|.$$
- $S'(u_i) a(x, u_i, Du_i)$ can be identified with $S'(u_i) a(x, T_K(u_i), DT_K(u_i))$ a.e. in Q. Indeed, since $|T_K(u_i)| \leq K$ a.e. in Q, assumptions (1.4) and (1.6) imply that
$$\big|a(x, T_K(u_i), DT_K(u_i))\big| \leq C_K(t,x) + \beta_K |DT_K(u_i)|^{p-1} \quad \text{a.e. in } Q.$$
As a consequence of (2.1) and of $S'(u_i) \in L^\infty(Q)$, it follows that
$$S'(u_i) a(x, T_K(u_i), DT_K(u_i)) \in (L^{p'}(Q))^N.$$

- $S''(u_i)a(x, u_i, Du_i)Du_i$ can be identified with
$$S''(u_i)a(x, T_K(u_i), DT_K(u_i))DT_K(u_i)$$
and in view of (1.4), (1.6) and (2.1) one has
$$S''(u_i)a(x, T_K(u_i), DT_K(u_i))DT_K(u_i) \in L^1(Q).$$
- $S'(u_i)\Phi_i(u_i)$ and $S''(u_i)\Phi_i(u_i)Du_i$ respectively identify with $S'(u_i)\Phi_i(T_K(u_i))$ and $S''(u_i)\Phi(T_K(u_i))DT_K(u_i)$. Due to the properties of S and (1.8), the functions S', S'' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (2.1) implies that $S'(u_i)\Phi_i(T_K(u_i)) \in (L^\infty(Q))^N$ and $S''(u_i)\Phi_i(T_K(u_i))DT_K(u_i)$ belongs to $L^p(Q)$.
- $S'(u_i)f_i(x, u_1, u_2)$ identifies with $S'(u_i)f_1(x, T_K(u_1), u_2)$ a.e. in Q (or $S'(u_i)f_2(x, u_1, T_K(u_2))$ a.e. in Q). Indeed, since $|T_K(u_i)| \le K$ a.e. in Q, assumptions (1.12) and (1.13) imply that
$$\big|f_1(x, T_K(u_1), u_2)\big| \le F_K(x) + \sigma_K\ |b_2(u_2)| \quad \text{a.e. in } Q$$
and
$$\big|f_2(x, u_1, T_K(u_2))\big| \le G_K(x) + \sigma_K\ |b_1(u_1)| \quad \text{a.e. in } Q.$$
As a consequence of (2.1) and of $S'(u_i) \in L^\infty(Q)$, it follows that
$$S'(u_1)f_1(x, T_K(u_1), u_2) \in L^1(Q) \quad \text{and} \quad S'(u_2)f_2(x, u_1, T_K(u_2)) \in L^1(Q).$$

The above considerations show that (2.3) takes place in $D'(Q)$ and that $\frac{\partial b_{i,S}(u_i)}{\partial t}$ belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$, which in turn implies that $\frac{\partial b_{i,S}(u_i)}{\partial t}$ belongs to $L^1(0, T; W^{-1,s}(\Omega))$ for all $s < inf(p', \frac{N}{N-1})$. It follows that $b_{i,S}(u_i)$ belongs to $C^0([0, T]; W^{-1,s}(\Omega))$ so that the initial condition (2.4) makes sense.

3. Existence result

This section is devoted to the proof of the following existence theorem.

Theorem 3.1. *Under assumptions* (1.4)-(1.14)*, there exists at least a renormalized solution* (u_1, u_2) *of Problem* (1.1)-(1.3).

Proof. The proof is divided into 9 steps. In step1, we introduce an approximate problem and step 2 is devoted to establish a few *a priori* estimates. In step 3, we prove some properties of the limit u_i of the approximate solutions u_i^ε. In step 4, we define a time regularization of the field $T_K(u_i)$ and we establish Lemma 3.2 which allows to control the parabolic contribution that arises in the monotonicity method when passing to the limit. In step 5, we prove an energy estimate (see Lemma 3.3) which is a key point for the monotonicity arguments that are developed in Step 6 and Step 7. In Step 8, we prove that u_i satisfies (2.2) and finally, in step 9, we prove that u_i satisfies properties (2.3) and (2.4) of Definition 2.1. □

Step 1. Let us introduce the following regularization of the data: for $\varepsilon > 0$ and $i = 1, 2$
$$b_{i,\varepsilon}(s) = b_i(T_{\frac{1}{\varepsilon}}(s)) + \varepsilon\ s \quad \forall s \in \mathbb{R}, \tag{3.1}$$
$$a_\varepsilon(x, s, \xi) = a(x, T_{\frac{1}{\varepsilon}}(s), \xi) \text{ a.e. in } \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \tag{3.2}$$
$$\Phi_{i,\varepsilon} \text{ is a Lipschitz continuous bounded function from } \mathbb{R} \text{ into } \mathbb{R}^N \tag{3.3}$$

such that Φ_i^ε converges uniformly to Φ_i on any compact subset of $\mathbb{R}$ as ε tends to 0.

$$f_1^\varepsilon(x,s_1,s_2) = f_1(x,T_{\frac{1}{\varepsilon}}(s_1),T_{\frac{1}{\varepsilon}}(s_2)) \quad \text{a.e. in } \Omega, \forall s_1,s_2\in\mathbb{R}, \tag{3.4}$$

$$f_2^\varepsilon(x,s_1,s_2) = f_2(x,T_{\frac{1}{\varepsilon}}(s_1),T_{\frac{1}{\varepsilon}}(s_2)) \quad \text{a.e. in } \Omega, \forall s_1,s_2\in\mathbb{R}, \tag{3.5}$$

$$u_{i,0}^\varepsilon \in C_0^\infty(\Omega), b_{i,\varepsilon}(u_{i,0}^\varepsilon) \to b_i(u_{i,0}) \quad \text{in } L^1(\Omega) \text{ as } \varepsilon \text{ tends to } 0. \tag{3.6}$$

Let us now consider the regularized problem

$$\frac{\partial b_{i,\varepsilon}(u^\varepsilon)}{\partial t} - \operatorname{div}\big(a_\varepsilon(x,u^\varepsilon,Du^\varepsilon)+\Phi_{i,\varepsilon}(u^\varepsilon)\big) + f_i^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon) = 0 \quad \text{in } Q, \tag{3.7}$$

$$u_i^\varepsilon = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{3.8}$$

$$b_{i,\varepsilon}(u_i^\varepsilon)(t=0) = b_{i,\varepsilon}(u_{i,0}^\varepsilon) \quad \text{in } \Omega. \tag{3.9}$$

In view of (1.9) and (3.1), for $i=1,2$, we have

$$b'_{i,\varepsilon}(s)\ge\varepsilon, \quad |b_{i,\varepsilon}(s)| \le \max_{|s|\le\frac{1}{\varepsilon}} |b_i(s)|+1 \quad \forall s\in\mathbb{R},$$

In view of (1.6), (1.12) and (1.13), $a_\varepsilon, f_1^\varepsilon$ and f_2^ε satisfy: There exists $C_\varepsilon \in L^{p'}(\Omega), F_\varepsilon\in L^1(\Omega), G_\varepsilon\in L^1(\Omega)$ and $\beta_\varepsilon>0, \sigma_\varepsilon>0, \lambda_\varepsilon>0$, such that

$$|a_\varepsilon(x,s,\xi)| \le C_\varepsilon(x)+\beta_\varepsilon|\xi|^{p-1} \quad \text{a.e. in } x\in\Omega, \forall s\in\mathbb{R}, \forall\xi\in\mathbb{R}^N.$$

$$|f_1^\varepsilon(x,s_1,s_2)| \le F_\varepsilon(x)+\sigma_\varepsilon \max_{|s|\le\frac{1}{\varepsilon}}|b_i(s)| \quad \text{a.e. in } x\in\Omega, \forall s_1,s_2\in\mathbb{R},$$

$$|f_2^\varepsilon(x,s_1,s_2)| \le G_\varepsilon(x)+\lambda_\varepsilon \max_{|s|\le\frac{1}{\varepsilon}}|b_i(s)| \quad \text{a.e. in } x\in\Omega, \forall s_1,s_2\in\mathbb{R}.$$

As a consequence, proving the existence of a weak solution $u_i^\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ of (3.7)-(3.9) is an easy task (see e.g. [29, 26, 27]).

Step 2. The estimates derived in this step rely on usual techniques for problems of type (3.9)-(3.13) and we just sketch the proof of them (the reader is referred to [2, 3, 7, 10, 4, 5] or to [11, 22, 23] for elliptic versions of (3.9)-(3.13)).

Using $T_K(u_i^\varepsilon)$ as a test function in (3.7) leads to

$$\begin{aligned}&\int_\Omega b_{i,\varepsilon}^K(u_i^\varepsilon)(t)\,dx + \int_0^t\int_\Omega a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)DT_K(u_i^\varepsilon)\,dx\,ds\\ &+\int_0^t\int_\Omega \Phi_{i,\varepsilon}(u_i^\varepsilon)DT_K(u_i^\varepsilon)\,dx\,ds + \int_0^t\int_\Omega f_i^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)T_K(u_i^\varepsilon)\,dx\,ds\\ &=\int_\Omega b_{i,\varepsilon}^K(u_{i,0}^\varepsilon)\,dx\end{aligned} \tag{3.10}$$

for $i=1,2$, for almost every t in $(0,T)$, and where $b_{i,\varepsilon}^K(r)=\int_0^r T_K(s)b'_{i,\varepsilon}(s)\,ds$. The Lipschitz character of $\Phi_{i,\varepsilon}$, Stokes formula together with the boundary condition (3.8) allow to obtain obtain

$$\int_0^t\int_\Omega \Phi_{i,\varepsilon}(u_i^\varepsilon)DT_K(u_i^\varepsilon)\,dx\,ds = 0, \tag{3.11}$$

for almost any $t\in(0,T)$. Now, as $0\le b_{i,\varepsilon}^K(u_{i,0}^\varepsilon)\le K|b_{i,\varepsilon}(u_{i,0}^\varepsilon)|$ a.e. in Ω, it follows that $0\le\int_\Omega b_{i,\varepsilon}^K(u_{i,0}^\varepsilon)\,dx \le K\int_\Omega|b_{i,\varepsilon}(u_{i,0}^\varepsilon)|\,dx$. Since a_ε satisfies (3.2), f_i^ε satisfies (3.4), (3.5), we deduce from (3.14) (taking into account the properties of $b_{i,\varepsilon}^K$ and $u_{i,0}^\varepsilon$) that

$$T_K(u_i^\varepsilon) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)) \tag{3.12}$$

independently of ε for any $K \geq 0$.

Proceeding as in [3, 7, 4], we prove that for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compact ($\operatorname{supp} S' \subset [-K, K]$)

$$S(b_{i,\varepsilon}(u_i^\varepsilon)) \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.13}$$

and

$$\frac{\partial S(b_{i,\varepsilon}(u_i^\varepsilon))}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{3.14}$$

independently of ε, as soon as $\varepsilon < \frac{1}{K}$. Due to the definition (3.1) of b_ε, it is clear that

$$\{-K \leq b_{i,\varepsilon}(u_i^\varepsilon) \leq K\} \subset \{-K \leq b_i(u_i^\varepsilon) \leq K\} = \{b_i^{-1}(-K) \leq u_i^\varepsilon \leq b_i^{-1}(K)\}$$

as long as $\varepsilon < \frac{1}{K}$. As a first consequence we have

$$DS(b_{i,\varepsilon}(u_i^\varepsilon)) = S'(b_{i,\varepsilon}(u_i^\varepsilon))b'_{i,\varepsilon}(T_{K_i^*}(u_i^\varepsilon))DT_{K_i^*}(u_i^\varepsilon) \quad \text{a.e. in } Q. \tag{3.15}$$

as long as $\varepsilon < \frac{1}{K}$, and $K_i^* = \max(|b_i^{-1}(-K)|, b_i^{-1}(K))$. Secondly, the following estimate holds true

$$\|S'(b_{i,\varepsilon}(u_i^\varepsilon))b'_{i,\varepsilon}(T_{K_i^*}(u_i^\varepsilon))\|_{L^\infty(Q)} \leq \|S'\|_{L^\infty(\mathbb{R})}\big(\max_{|r|\leq K_i^*}(b_i'(r)) + 1\big),$$

as long as $\varepsilon < \frac{1}{K}$.

As a consequence of (3.12), (3.15) we obtain (3.13). To show that (3.14) holds, we multiply the equation for u^ε in (2.3) by $S'(b_{i,\varepsilon}(u_i^\varepsilon))$ to obtain

$$\begin{aligned} &\frac{\partial S(b_{i,\varepsilon}(u^\varepsilon))}{\partial t} \\ &= \operatorname{div}\Big(S'(b_\varepsilon(u_i^\varepsilon))a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon\Big) \\ &\quad - S''(b_{i,\varepsilon}(u_i^\varepsilon))b'_{i,\varepsilon}(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon Du_i^\varepsilon + \operatorname{div}\Phi_{i,\varepsilon}(u^\varepsilon)S'(b_{i,\varepsilon}(u^\varepsilon))) \\ &\quad - S''(b_{i,\varepsilon}(u_i^\varepsilon))b'_{i,\varepsilon}(u_i^\varepsilon)\Phi_{i,\varepsilon}(u_i^\varepsilon)Du_i^\varepsilon + f_i^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)S'(b_{i,\varepsilon}(u_i^\varepsilon)) = 0, \end{aligned} \tag{3.16}$$

in $D'(Q)$. Since $\operatorname{supp} S'$ and $\operatorname{supp} S''$ are both included in $[-K, K], u_i^\varepsilon$ may be replaced by $T_{K_i^*}(u_i^\varepsilon)$ in each of these terms, where $K_i^* = \max(|b_i^{-1}(-K)|, b_i^{-1}(K))$. As a consequence, each term in the right hand side of (3.16) is bounded either in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ or in $L^1(Q)$. (see [4, 7]). As a consequence of (3.12), (3.16) we then obtain (3.14).

Now for fixed $K > 0 : a_\varepsilon(x, T_K(u_i^\varepsilon), DT_K(u_i^\varepsilon)) = a(x, T_K(u_i^\varepsilon), DT_K(u_i^\varepsilon))$ a.e. in Q as long as $\varepsilon < \frac{1}{K}$, while assumption (1.6) gives

$$\big|a_\varepsilon(x,T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon))\big| \leq C_K(x) + \beta_K|DT_K(u_i^\varepsilon)|^{p-1}$$

where $\beta_K > 0$ and $C_K \in L^{p'}(Q)$. In view of (3.12), we deduce that

$$a\big(x,T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\big) \text{ is bounded in } (L^{p'}(Q))^N. \tag{3.17}$$

independently of ε for $\varepsilon < \frac{1}{K}$.

For any integer $n \geq 1$, consider the Lipschitz continuous function θ_n defined through

$$\theta_n(r) = T_{n+1}(r) - T_n(r)$$

We remark that $\|\theta_n\|_{L^\infty(\mathbb{R})} \leq 1$ for any $n \geq 1$ and that $\theta_n(r) \to 0$ for any r when n tends to infinity.

Using the admissible test function $\theta_n(u^\varepsilon)$ in (3.7) leads to

$$\begin{aligned} &\int_\Omega b^n_{i,\varepsilon}(u_i^\varepsilon)(t)\,dx + \int_0^t\int_\Omega a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)D\theta_n(u_i^\varepsilon)\,dx\,ds \\ &+ \int_0^t\int_\Omega \Phi_{i,\varepsilon}(u_i^\varepsilon)D\theta_n(u_i^\varepsilon)\,dx\,ds + \int_0^t\int_\Omega f_i^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)\theta_n(u_i^\varepsilon)\,dx\,ds \\ &= \int_\Omega b^n_{i,\varepsilon}(u^\varepsilon_{i,0})\,dx, \end{aligned} \tag{3.18}$$

for almost any t in $(0,T)$ and where $b^n_{i,\varepsilon}(r) = \int_0^r b'_{i,\varepsilon}(s)\theta_n(s)\,ds$.

The Lipschitz character of Φ_ε, Stokes formula together with the boundary condition (3.8) allow to obtain

$$\int_0^t\int_\Omega \Phi_{i,\varepsilon}(u^\varepsilon)D\theta_n(u_i^\varepsilon)\,dx\,ds = 0. \tag{3.19}$$

Since $b^n_{i,\varepsilon}(.) \ge 0$, f_i^ε satisfies (1.11), we have

$$\int_0^t\int_\Omega a(x,u_i^\varepsilon,Du_i^\varepsilon)D\theta_n(u_i^\varepsilon)\,dx\,ds \le \int_\Omega b^n_{i,\varepsilon}(u^\varepsilon_{i,0})\,dx, \tag{3.20}$$

for almost $t\in(0,T)$ and for $\varepsilon < \frac{1}{n+1}$.

Step 3. Arguing again as in [3, 7, 4, 5], estimates (3.13)and (3.14) imply that for a subsequence still indexed by ε,

$$u_i^\varepsilon \text{ converges almost every where to } u_i \text{ in } Q \tag{3.21}$$

and thanks to (3.12),

$$T_K(u_i^\varepsilon) \text{ converges weakly to } T_K(u_i) \text{ in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.22}$$

$$\theta_n(u_i^\varepsilon) \rightharpoonup \theta_n(u_i) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \tag{3.23}$$

$$a_\varepsilon\Big(x,T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\Big) \rightharpoonup X_{i,K} \text{ weakly in } (L^{p'}(Q))^N. \tag{3.24}$$

as ε tends to 0 for any $K>0$ and any $n\ge 1$. Here, for any $K>0$ and for $i=1,2$, $X_{i,K}$ belongs to $(L^{p'}(Q))^N$.

We now establish that $b_i(u_i)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Indeed using $\frac{1}{\sigma}T_\sigma(u_i^\varepsilon)$ as a test function in (3.7) leads to

$$\begin{aligned} &\frac{1}{\sigma}\int_\Omega b^\sigma_{i,\varepsilon}(u_i^\varepsilon)(t)\,dx + \frac{1}{\sigma}\int_0^t\int_\Omega a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)DT_\sigma(u_i^\varepsilon)\,dx\,ds \\ &+ \frac{1}{\sigma}\int_0^t\int_\Omega \Phi_{i,\varepsilon}(u_i^\varepsilon)DT_\sigma(u_i^\varepsilon)\,dx\,ds + \frac{1}{\sigma}\int_0^t\int_\Omega f_i^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)T_\sigma(u_i^\varepsilon)\,dx\,ds \\ &= \frac{1}{\sigma}\int_\Omega b^\sigma_{i,\varepsilon}(u^\varepsilon_{i,0})\,dx, \end{aligned} \tag{3.25}$$

for almost any t in $(0,T)$. Where, $b^n_{i,\varepsilon}(r) = \int_0^r b'_{i,\varepsilon}(s)T_\sigma(s)\,ds$.

The Lipschitz character of Φ_ε, Stokes formula together with the boundary condition (3.8) allow to obtain

$$\frac{1}{\sigma}\int_0^t\int_\Omega \Phi_{i,\varepsilon}(u_i^\varepsilon)DT_\sigma(u_i^\varepsilon)\,dx\,ds = 0. \tag{3.26}$$

Since a_ε satisfies (1.5) and f_i^ε satisfies (1.11), letting σ go to zero, it follows that

$$\int_\Omega |b_{i,\varepsilon}(u_i^\varepsilon)(t)|\,dx \le \|b_{i,\varepsilon}(u_{i,0}^\varepsilon)\|_{L^1(\Omega)} \tag{3.27}$$

for almost $t \in (0,T)$. Recalling (3.6), (3.21) and (3.27) makes it possible to pass to the limit-inf and we show that $b_i(u_i)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

We are now in a position to exploit (3.20). The pointwise convergence of u^ε to u and $b_{i,\varepsilon}(u_0^\varepsilon)$ to $b_i(u_0)$ imply that

$$\limsup_{\varepsilon\to 0}\int_0^t\int_\Omega a(x,u_i^\varepsilon,Du_i^\varepsilon)D\theta_n(u_i^\varepsilon)\,dx\,ds \le \int_\Omega b_i^n(u_{i,0})\,dx, \tag{3.28}$$

Since θ_n converge to zero everywhere as n goes to zero, the Lebesgue's convergence theorem permits to conclude that

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}\int_{\{n\le|u_i^\varepsilon|\le n+1\}} a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon\,dx\,dt = 0. \tag{3.29}$$

Step 4. This step is devoted to introduce for $K \ge 0$ fixed, a time regularization of the function $T_K(u_i)$ in order to perform the monotonicity method which will be developed in Step 5 and Step 6. This kind of regularization has been first introduced by Landes (see Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231 in [18]). More recently, it has been exploited in [9] and [16] to solve a few nonlinear evolution problems with L^1 or measure data.

This specific time regularization of $T_K(u_i)$ (for fixed $K \ge 0$) is defined as follows. let us consider the unique solution $T_K(u_i)_\mu \in L^\infty(Q)\cap L^p(0,T;W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial T_K(u_i)_\mu}{\partial t} + \mu\Big(T_K(u_i)_\mu - T_K(u_i)\Big) = 0 \quad \text{in } D'(Q). \tag{3.30}$$

$$T_K(u_i)_\mu(t=0) = 0 \text{ in } \Omega. \tag{3.31}$$

We remark that for $\mu > 0$ and $K \ge 0$,

$$\frac{\partial T_K(u_i)_\mu}{\partial t} \in L^p(0,T;W_0^{1,p}(\Omega)). \tag{3.32}$$

The behavior of $T_K(u_i)_\mu$ as $\mu \to +\infty$ is investigated in [18] (see also [16] and [17]) and we just recall here that (3.30)-(3.31) imply that

$$T_K(u_i)_\mu \to T_K(u_i) \quad \text{a.e. in } Q\,, \tag{3.33}$$

and in $L^\infty(Q)$ weak $\star$ and strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\mu \to +\infty$.

$$\|T_K(u_i)_\mu\|_{L^\infty(Q)} \le K \tag{3.34}$$

for any μ and any $K \ge 0$.

Let $v_{i,j} \in C_0^\infty(\Omega)$, such that $v_{i,j}$ converges almost everywhere to $u_{i,0}$ in Ω. And let us consider

$$T_K(u_i)_{\mu,j} = T_K(u_i)_\mu + e^{-\mu t}T_K(v_{i,j})$$

is a smooth approximation of $T_K(u_i)$. We remark that for $\mu > 0, j > 0$ and $K \geq 0$, we have $|T_K(u_i)_{\mu,j}| \leq K$ and

$$\frac{\partial T_K(u_i)_{\mu,j}}{\partial t} = \mu\Big(T_K(u_i) - T_K(u_i)_{\mu,j}\Big), \tag{3.35}$$

$$T_K(u_i)_{\mu,j}(0) = T_K(v_{i,j}), \tag{3.36}$$

$$T_K(u_i)_{\mu,j} \to T_K(u_i) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.37}$$

as μ tends to infinity.

We denote by $w(\varepsilon,\mu,j)$ the quantities such that

$$\lim_{j\to+\infty}\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0} w(\varepsilon,\mu,j) = 0.$$

The main estimate is as follows.

Lemma 3.2. *Let $K \geq 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r) = r$ for $|r| \leq K$ and $\operatorname{supp}(S')$ is compact. Then*

$$\liminf_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^s \Big\langle \frac{\partial b_{i,S}(u_i^\varepsilon)}{\partial t}, \big(T_K(u_i^\varepsilon) - (T_K(u_i))_\mu\big)\Big\rangle\,dt\,ds \geq 0$$

where $\langle,\rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. and where $b_{i,S}(r) = \int_0^r b_i'(s)S(s)\,ds$.

The proof of the above Lemma can be found in [24].

Step 5. In this step we prove the following Lemma which is the key point in the monotonocity arguments that will be developed in Step 6.

Lemma 3.3. *The subsequence of u^ε defined is Step 3 satisfies: For any $K \geq 0$,*

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega a(u_i^\varepsilon, DT_K(u_i^\varepsilon))DT_K(u_i^\varepsilon)\,dx\,ds\,dt\\ &\leq \int_0^T\int_0^t\int_\Omega X_{i,K}DT_K(u_i)\,dx\,ds\,dt\end{aligned} \tag{3.38}$$

Proof. We first introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_n such that, for any $n \geq 1$

$$S_n(r) = r \text{ for } |r| \leq n, \quad \operatorname{supp}(S_n') \subset [-(n+1),(n+1)], \quad \|S_n''\|_{L^\infty(\mathbb{R})} \leq 1. \tag{3.39}$$

Pointwise multiplication of (3.7) by $S_n'(u_i^\varepsilon)$ (which is licit) leads to

$$\begin{aligned}&\frac{\partial b_{i,S_n}(u_i^\varepsilon)}{\partial t} - \operatorname{div}\Big(S_n(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)\Big) + S_n''(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon\\ &-\operatorname{div}\Big(\Phi_{i,\varepsilon}(u_i^\varepsilon)S_n'(u_i^\varepsilon)\Big) + S_n''(u_i^\varepsilon)\Phi_{i,\varepsilon}(u_i^\varepsilon)Du_i^\varepsilon + f_i^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)S_n'(u_i^\varepsilon) = 0\end{aligned} \tag{3.40}$$

in $D'(Q)$. We use the sequence $T_K(u)_\mu$ of approximations of $T_K(u)$ defined by (3.30), (3.31) of Step 4 and plug the test function $T_K(u^\varepsilon) - T_K(u)_\mu$ (for $\varepsilon > 0$ and $\mu > 0$) in (3.40). Through setting, for fixed $K \geq 0$,

$$W_{i,\mu}^\varepsilon = T_K(u_i^\varepsilon) - T_K(u_i)_\mu \tag{3.41}$$

we obtain upon integration over $(0,t)$ and then over $(0,T)$,

$$\begin{aligned}
&\int_0^T \int_0^t \langle \frac{\partial b_{i,S_n}(u_i^\varepsilon)}{\partial t}, W_{i,\mu}^\varepsilon \rangle \, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega S_n'(u_i^\varepsilon) a_\varepsilon(x, u_i^\varepsilon, Du_i^\varepsilon) DW_{i,\mu}^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega S_n''(u_i^\varepsilon) W_{i,\mu}^\varepsilon a_\varepsilon(x, u_i^\varepsilon, Du_i^\varepsilon) Du_i^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega \Phi_{i,\varepsilon}(u_i^\varepsilon) S_n'(u_i^\varepsilon) DW_{i,\mu}^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega S_n''(u_i^\varepsilon) W_{i,\mu}^\varepsilon \Phi_{i,\varepsilon}(u_i^\varepsilon) Du_i^\varepsilon \, dx\, ds\, dt \\
&+ \int_0^T \int_0^t \int_\Omega f_i^\varepsilon(x, u_1^\varepsilon, u_2^\varepsilon) S_n'(u_i^\varepsilon) W_{i,\mu}^\varepsilon \, dx\, ds\, dt = 0
\end{aligned} \tag{3.42}$$

Next we pass to the limit as ε tends to 0, then μ tends to $+\infty$ and then n tends to $+\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $K \geq 0$:

$$\liminf_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \langle \frac{\partial b_{i,S_n}(u_i^\varepsilon)}{\partial t}, W_{i,\mu}^\varepsilon \rangle \, ds\, dt \geq 0 \quad \text{for any } n \geq K, \tag{3.43}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S_n'(u_i^\varepsilon) \Phi_{i,\varepsilon}(u_i^\varepsilon) DW_{i,\mu}^\varepsilon \, dx\, ds\, dt = 0 \quad \text{for any } n \geq 1, \tag{3.44}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega S_n''(u_i^\varepsilon) W_{i,\mu}^\varepsilon \Phi_{i,\varepsilon}(u_i^\varepsilon) Du_i^\varepsilon \, dx\, ds\, dt = 0 \quad \text{for any } n, \tag{3.45}$$

$$\lim_{n\to+\infty} \overline{\lim_{\mu\to+\infty}} \, \overline{\lim_{\varepsilon\to 0}} \Big| \int_0^T \int_0^t \int_\Omega S_n''(u_i^\varepsilon) W_{i,\mu}^\varepsilon a_\varepsilon(u_i^\varepsilon, Du_i^\varepsilon) Du_i^\varepsilon \, dx\, ds\, dt \Big| = 0, \tag{3.46}$$

$$\lim_{\mu\to+\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega f_i^\varepsilon(x, u_1^\varepsilon, u_2^\varepsilon) S_n'(u_i^\varepsilon) W_{i,\mu}^\varepsilon \, dx\, ds\, dt = 0 \quad \text{for any } n \geq 1. \tag{3.47}$$

□

Proof of (3.43)**.** In view of (3.41) of $W_{i,\mu}^\varepsilon$, Lemma 3.2 applies with $S = S_n$ for fixed $n \geq K$. As a consequence (3.43) holds.

Proof of (3.44)**.** For fixed $n \geq 1$, we have

$$S_n'(u_i^\varepsilon) \Phi_{i,\varepsilon}(u_i^\varepsilon) DW_{i,\mu}^\varepsilon = S_n'(u_i^\varepsilon) \Phi_{i,\varepsilon}(T_{n+1}(u_i^\varepsilon)) DW_{i,\mu}^\varepsilon \tag{3.48}$$

a.e. in Q, and for all $\varepsilon \leq \frac{1}{n+1}$, and where $\operatorname{supp} S_n' \subset [-(n+1), n+1]$.

Since S_n' is smooth and bounded, (1.8), (3.5) and (3.22) lead to

$$S_n'(u_i^\varepsilon) \Phi_{i,\varepsilon}(T_{n+1}(u_i^\varepsilon)) \to S_n'(u_i) \Phi_i(T_{n+1}(u_i)) \tag{3.49}$$

a.e. in Q and in $L^\infty(Q)$ weak $\star$, as ε tends to 0. For fixed $\mu > 0$, we have

$$W_{i,\mu}^\varepsilon \rightharpoonup T_K(u_i) - T_K(u_i)_\mu \quad \text{weakly in } L^p(0,T; W_0^{1,p}(\Omega)) \tag{3.50}$$

and a.e. in Q and in $L^\infty(Q)$ weak $\star$, as ε tends to 0. As a consequence of (3.48), (3.49) and (3.50) we deduce that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n'(u_i^\varepsilon)\Phi_{i,\varepsilon}(u_i^\varepsilon)DW_{i,\mu}^\varepsilon\,dx\,ds\,dt \\ &=\int_0^T\int_0^t\int_\Omega S_n'(u_i)\Phi_i(u_i)\big[DT_K(u_i)-DT_K(u_i)_\mu\big]\,dx\,ds\,dt \end{aligned}\tag{3.51}$$

for any $\mu>0$. Appealing now to (3.33) and passing to the limit as $\mu\to+\infty$ in (3.51) allows to conclude that (3.44) holds.

Proof of (3.45)**.** For fixed $n\geq 1$, and by the same arguments as those which lead to (3.48), we have

$$S_n''(u_i^\varepsilon)\Phi_{i,\varepsilon}(u_i^\varepsilon)Du_i^\varepsilon W_{i,\mu}^\varepsilon = S_n''(u_i^\varepsilon)\Phi_{i,\varepsilon}(T_{n+1}(u_i^\varepsilon))DT_{n+1}(u_i^\varepsilon)W_{i,\mu}^\varepsilon \quad \text{a.e. in } Q.$$

From (1.8), (3.3) and (3.22), it follows that for any $\mu>0$,

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega S_n''(u_i^\varepsilon)\Phi_{i,\varepsilon}(u_i^\varepsilon)Du_i^\varepsilon W_{i,\mu}^\varepsilon\,dx\,ds\,dt \\ &=\int_0^T\int_0^t\int_\Omega S_n''(u_i)\Phi_i(T_{n+1}(u_i))DT_{n+1}(u_i)W_{i,\mu}\big[DT_K(u_i)-DT_K(u_i)_\mu\big]\,dx\,ds\,dt \end{aligned}$$

with the help of (3.37) passing to the limit, as μ tends to $+\infty$, in the above equality, we find (3.45).

Proof of (3.46)**.** For any $n\geq 1$ fixed, we have $\operatorname{supp}(S_n'')\subset[-(n+1),-n]\cup[n,n+1]$. As a consequence

$$\begin{aligned} &\Big|\int_0^T\int_0^t\int_\Omega S_n''(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon W_{i,\mu}^\varepsilon\,dx\,ds\,dt\Big| \\ &\leq T\|S_n''\|_{L^\infty(\mathbb{R})}\|W_{i,\mu}^\varepsilon\|_{L^\infty(Q)}\int_{\{n\leq|u_i^\varepsilon|\leq n+1\}}a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon\,dx\,dt, \end{aligned}$$

for any $n\geq 1$, and any $\mu>0$. The above inequality together with (3.34) and (3.39) make it possible to obtain

$$\begin{aligned} &\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\Big|\int_0^T\int_0^t\int_\Omega S_n''(u_i^\varepsilon)a_\varepsilon(u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon W_{i,\mu}^\varepsilon\,dx\,ds\,dt\Big| \\ &\leq C\limsup_{\varepsilon\to 0}\int_{\{n\leq|u_i^\varepsilon|\leq n+1\}}a_\varepsilon(u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon\,dx\,dt, \end{aligned}\tag{3.52}$$

for any $n\geq 1$, where C is a constant independent of n. Using (3.29) we pass to the limit as n tends to $+\infty$ in (3.52) and establish (3.46).

Proof of (3.47)**.** For fixed $n\geq 1$, we have,

$$\begin{gathered} f_1^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)S_n'(u_1^\varepsilon)=f_1(x,T_{n+1}(u_1^\varepsilon),T_{\frac{1}{\varepsilon}}(u_2^\varepsilon))S_n'(u_1^\varepsilon), \\ f_2^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)S_n'(u_2^\varepsilon)=f_2(x,T_{\frac{1}{\varepsilon}}(u_1^\varepsilon),T_{n+1}(u_2^\varepsilon))S_n'(u_2^\varepsilon) \end{gathered}$$

a.e. in Q, and for all $\varepsilon \leq \frac{1}{n+1}$. In view of (1.10), (3.21) and (3.22), Lebesgue's convergence theorem implies that for any $\mu > 0$ and any $n \geq 1$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T \int_0^t \int_\Omega f_1^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)S_n'(u_i^\varepsilon)W_\mu^\varepsilon\,dx\,ds\,dt \\
&= \int_0^T \int_0^t \int_\Omega f_1(x,u_1,u_2)S_n'(u_i)\Big(T_K(u_i) - T_K(u_i)_\mu\Big)\,dx\,ds\,dt.
\end{aligned}$$

Now for fixed $n \geq 1$, using (3.33) permits to pass to the limit as μ tends to $+\infty$ in the above equality to obtain (3.47).

We now turn back to the proof of Lemma 3.3, due to (3.43), (3.44), (3.45), (3.46) and (3.47), we are in a position to pass to the lim-sup when ε tends to zero, then to the limit-sup when μ tends to $+\infty$ and then to the limit as n tends to $+\infty$ in (3.42). We obtain using the definition of W_μ^ε that for any $K \geq 0$,

$$\begin{aligned}
&\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_0^T \int_0^t \int_\Omega S_n'(u_i^\varepsilon)a_\varepsilon(u_i^\varepsilon,Du_i^\varepsilon)\big(DT_K(u_i^\varepsilon) \\
&- DT_K(u_i)_\mu\big)\,dx\,ds\,dt \leq 0.
\end{aligned}$$

Since $S_n'(u_i^\varepsilon)a_\varepsilon(u_i^\varepsilon,Du_i^\varepsilon)DT_K(u_i^\varepsilon) = a(u_i^\varepsilon,Du_i^\varepsilon)DT_K(u_i^\varepsilon)$ for $\varepsilon \leq \frac{1}{K}$ and $K \leq n$.

The above inequality implies that for $K \leq n$,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_0^T \int_0^t \int_\Omega a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)DT_K(u_i^\varepsilon)\,dx\,ds\,dt \\
&\leq \lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\varepsilon\to 0}\int_0^T \int_0^t \int_\Omega S_n'(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)DT_K(u_i)_\mu\,dx\,ds\,dt
\end{aligned} \tag{3.53}$$

The right hand side of (3.53) is computed as follows: In view of (3.2) and (3.40), we have for $\varepsilon \leq \frac{1}{n+1}$,

$$S_n'(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon) = S_n'(u_i^\varepsilon)a\Big(x,T_{n+1}(u_i^\varepsilon),DT_{n+1}(u_i^\varepsilon)\Big) \text{ a.e. in } Q.$$

Due to (3.24), it follows that for fixed $n \geq 1$,

$$S_n'(u_i^\varepsilon)a_\varepsilon(u_i^\varepsilon,Du_i^\varepsilon) \rightharpoonup S_n'(u_i)X_{i,n+1} \text{ weakly in } (L^{p'}(Q))^N,$$

when ε tends to 0.

The strong convergence of $T_K(u_i)_\mu$ to $T_K(u_i)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ as μ tends to $+\infty$, allows then to conclude that

$$\begin{aligned}
&\lim_{\mu\to+\infty}\lim_{\varepsilon\to 0}\int_0^T \int_0^t \int_\Omega S_n'(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)DT_K(u_i)_\mu\,dx\,ds\,dt \\
&= \int_0^T \int_0^t \int_\Omega S_n'(u_i)X_{i,n+1}DT_K(u_i)\,dx\,ds\,dt \\
&= \int_0^T \int_0^t \int_\Omega X_{i,n+1}DT_K(u_i)\,dx\,ds\,dt
\end{aligned} \tag{3.54}$$

as long as $K \leq n$, since $S_n'(r) = 1$ for $|r| \leq n$. Now for $K \leq n$, we have

$$a\big(x,T_{n+1}(u_i^\varepsilon),DT_{n+1}(u_i^\varepsilon)\big)\chi_{\{|u_i^\varepsilon|<K\}} = a\big(x,T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\big)\chi_{\{|u_i^\varepsilon|<K\}},$$

a.e. in Q. Passing to the limit as ε tends to 0, we obtain

$$X_{i,n+1}\chi_{\{|u_i|<K\}} = X_{i,K}\chi_{\{|u_i|<K\}} \quad \text{a.e. in } Q - \{|u_i| = K\} \text{ for } K \leq n. \tag{3.55}$$

As a consequence of (3.55), for $K \le n$, we have

$$X_{n+1} DT_K(u_i) = X_K DT_K(u_i) \quad \text{a.e. in } Q. \tag{3.56}$$

Taking into account (3.53), (3.54) and (3.56), we conclude that (3.38) holds true and the proof of Lemma 3.3 is complete.

Step 6. In this step, we prove the following monotonicity estimate.

Lemma 3.4. *The subsequence of u_i^ε defined in step 3 satisfies: For any $K \ge 0$,*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega \big[a(T_K(u_i^\varepsilon), DT_K(u_i^\varepsilon)) - a(T_K(u_i^\varepsilon), DT_K(u_i))\big] \\ &\times \big[DT_K(u_i^\varepsilon) - DT_K(u_i)\big]\, dx\, ds\, dt = 0\,. \end{aligned} \tag{3.57}$$

Proof. Let $K \ge 0$ be fixed. The monotone character (1.7) of $a(s,\xi)$ with respect to ξ implies that

$$\begin{aligned} &\int_0^T \int_0^t \int_\Omega \big[a(T_K(u_i^\varepsilon), DT_K(u_i^\varepsilon)) - a(T_K(u_i^\varepsilon), DT_K(u_i))\big] \\ &\times \big[DT_K(u_i^\varepsilon) - DT_K(u_i)\big]\, dx\, ds\, dt \ge 0, \end{aligned} \tag{3.58}$$

In order to pass to the limit-sup as ε tends to 0 in (3.58), let us recall first that (1.4), (1.6) and (3.21) imply

$$a(T_K(u_i^\varepsilon), DT_K(u_i)) \to a(T_K(u_i), DT_K(u_i)) \quad \text{a.e. in } Q,$$

as ε tends to 0, and that

$$\big|a(T_K(u_i^\varepsilon), DT_K(u_i))\big| \le C_K(t,x) + \beta_K |DT_K(u_i)|^{p-1}$$

a.e. in Q, uniformly with respect to ε. It follows that when ε tends to 0,

$$a\big(T_K(u_i^\varepsilon), DT_K(u_i)\big) \to a\big(T_K(u_i), DT_K(u_i)\big) \quad \text{strongly in } (L^{p'}(Q))^N. \tag{3.59}$$

Using (3.38) of Lemma 3.3, (3.22), (3.24) and (3.59), we can pass to the lim-sup as ε tends to zero in (3.58) to obtain (3.57) of Lemma 3.4. $\square$

Step 7. In this step we identify the weak limit $X_{i,K}$ and we prove the weak L^1 convergence of the "truncated" energy $a\big(T_K(x,u_i^\varepsilon), DT_K(u_i^\varepsilon)\big) DT_K(u_i^\varepsilon)$ as ε tends to 0.

Lemma 3.5. *For fixed $K \ge 0$, as ε tends to 0, we have*

$$X_{i,K} = a\big(x, T_K(u_i^\varepsilon), DT_K(u_i^\varepsilon)\big) \quad \text{a.e. in } Q. \tag{3.60}$$

Also, as ε tends to 0,

$$a\big(T_K(u_i^\varepsilon), DT_K(u_i^\varepsilon)\big) DT_K(u_i^\varepsilon) \rightharpoonup a\big(T_K(u_i), DT_K(u_i)\big) DT_K(u_i), \tag{3.61}$$

weakly in $L^1(Q)$.

Proof. The proof is standard once we remark that for any $K \ge 0$, any $0 < \varepsilon < \frac{1}{K}$ and any $\xi \in \mathbb{R}^N$

$$a_\varepsilon(x, T_K(u_i^\varepsilon), \xi) = a(x, T_K(u_i^\varepsilon), \xi) = a_{\frac{1}{K}}(x, T_K(u_i^\varepsilon), \xi) \quad \text{a.e. in } Q$$

which together with (3.24), (3.59) and (3.57) of Lemma 3.4 imply

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^T\int_0^t\int_\Omega a_{\frac{1}{K}}\Big(x,T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\Big)DT_K(u_i^\varepsilon)\,dx\,ds\,dt\\&=\int_0^T\int_0^t\int_\Omega \sigma_K DT_K(u_i)\,dx\,ds\,dt.\end{aligned}\tag{3.62}$$

Since, for fixed $K>0$, the function $a_{\frac{1}{K}}(x,s,\xi)$ is continuous and bounded with respect to s, the usual Minty's argument applies in view of (3.22), (3.24), and (3.62). It follows that (3.60) holds true (the case $K=0$ being trivial). In order to prove (3.61), we observe that thanks to the monotone character of a (with respect to ξ) and (3.57), for any $K\geq 0$ and any $T'<T$, we have

$$\big[a(T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon))-a(T_K(u_i^\varepsilon),DT_K(u))\big]\big[DT_K(u_i^\varepsilon)-DT_K(u_i)\big]\to 0 \tag{3.63}$$

strongly in $L^1((0,T')\times\Omega)$ as ε tends to 0. Moreover (3.22), (3.24), (3.59) and (3.60) imply that

$$a\Big(T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\Big)DT_K(u_i)\rightharpoonup a\Big(T_K(u_i),DT_K(u_i)\Big)DT_K(u_i)$$

weakly in $L^1(Q)$,

$$a\Big(T_K(u_i^\varepsilon),DT_K(u_i)\Big)DT_K(u_i^\varepsilon)\rightharpoonup a\Big(T_K(u_i),DT_K(u_i)\Big)DT_K(u_i)$$

weakly in $L^1(Q)$,

$$a\Big(T_K(u_i^\varepsilon),DT_K(u_i)\Big)DT_K(u_i)\to a\Big(T_K(u_i),DT_K(u_i)\Big)DT_K(u_i),$$

strongly in $L^1(Q)$, as ε tends to 0. Using the above convergence results in (3.63), we get for any $K\geq 0$ and any $T'<T$,

$$a\Big(T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\Big)DT_K(u_i^\varepsilon)\rightharpoonup a\Big(T_K(u_i),DT_K(u_i)\Big)DT_K(u) \tag{3.64}$$

weakly in $L^1((0,T')\times\Omega)$ as ε tends to 0. $\square$

We remark that for $\overline{T}>T$, (1.6)-(1.14) are satisfied with $\overline{T}$ in place of T and that the convergence result (3.64) is still true in $L^1(Q)$-weak which means that (3.61) holds.

Step 8. In this step we prove that u satisfies (2.2). To this end, we remark that for any fixed $n\geq 0$,

$$\begin{aligned}&\int_{\{(t,x)/\ n\leq|u_i^\varepsilon|\leq n+1\}} a(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon\,dx\,dt\\&=\int_Q a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)\Big[DT_{n+1}(u_i^\varepsilon)-DT_n(u_i^\varepsilon)\Big]\,dx\,dt\\&=\int_Q a_\varepsilon\Big(x,T_{n+1}(u_i^\varepsilon),DT_{n+1}(u_i^\varepsilon)\Big)DT_{n+1}(u_i^\varepsilon)\,dx\,dt\\&\quad-\int_Q a_\varepsilon\Big(x,T_n(u_i^\varepsilon),DT_n(u_i^\varepsilon)\Big)DT_n(u^\varepsilon)\,dx\,dt\end{aligned}$$

for $\varepsilon<\frac{1}{n+1}$.

According to (3.61), one can pass to the limit as ε tends to 0; for fixed $n \geq 0$ to obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\{(t,x)/\ n\leq|u_i^\varepsilon|\leq n+1\}} a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon\,dx\,dt \\ &= \int_Q a\Big(x,T_{n+1}(u_i),DT_{n+1}(u_i)\Big)DT_{n+1}(u_i)\,dx\,dt \\ &\quad - \int_Q a\Big(x,T_n(u_i),DT_n(u_i)\Big)DT_n(u_i)\,dx\,dt \\ &= \int_{\{(t,x)/\ n\leq|u_i|\leq n+1\}} a(x,u_i,Du_i)Du_i\,dx\,dt \end{aligned} \tag{3.65}$$

Taking the limit as n tends to $+\infty$ in (3.65) and using the estimate (3.29) show that u_i satisfies (2.2).

Step 9. In this step, u_i is shown to satisfy (2.3) and (2.4). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\operatorname{supp} S' \subset [-K,K]$. Pointwise multiplication of the approximate equation (3.7) by $S'(u_i^\varepsilon)$ leads to

$$\begin{aligned} &\frac{\partial b_{i,S}^\varepsilon(u_i^\varepsilon)}{\partial t} - \operatorname{div}\Big(S'(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)\Big) + S''(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon \\ &- \operatorname{div}\Big(S'(u_i^\varepsilon)\Phi_{i,\varepsilon}(u_i^\varepsilon)\Big) + S''(u_i^\varepsilon)\Phi_\varepsilon(u_i^\varepsilon)Du_i^\varepsilon + f_i^\varepsilon(x,u_1^\varepsilon,u_2^\varepsilon)S'(u_i^\varepsilon) = 0 \end{aligned} \tag{3.66}$$

in $D'(Q)$, for $i=1,2$. In what follows we pass to the limit as ε tends to 0 in each term of (3.66).

Limit of $\frac{\partial b_{i,S}^\varepsilon(u_i^\varepsilon)}{\partial t}$. Since S is bounded and continuous, and $b_{i,S}^\varepsilon(u_i^\varepsilon)$ converges to $S(u_i)$ a.e. in Q and in $L^\infty(Q)$ weak $\star$, $\frac{\partial b_{i,S}^\varepsilon(u_i^\varepsilon)}{\partial t}$ converges to $\frac{\partial b_{i,S}(u_i)}{\partial t}$ in $D'(Q)$ as ε tends to 0.

Limit of $-\operatorname{div}\Big(S'(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)\Big)$. Since $\operatorname{supp} S' \subset [-K,K]$, for $\varepsilon < \frac{1}{K}$, we have

$$S'(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon) = S'(u_i^\varepsilon)a_\varepsilon\Big(x,T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\Big) \quad \text{a.e. in } Q.$$

The pointwise convergence of u^ε to u as ε tends to 0, the bounded character of S, (3.24) and (3.60) of Lemma 3.5 imply that $S'(u_i^\varepsilon)a_\varepsilon\Big(x,T_K(u_i^\varepsilon),DT_K(u_i^\varepsilon)\Big)$ converges to $S'(u_i)a\Big(x,T_K(u_i),DT_K(u_i)\Big)$ weakly in $L^{p'}(Q)$, as ε tends to 0, because $S'(u_i) = 0$ for $|u_i| \geq K$ a.e. in Q. And $S'(u_i)a\Big(x,T_K(u_i),DT_K(u_i)\Big) = S'(u_i)a(x,u_i,Du_i)$ a.e. in Q.

Limit of $S''(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon$. Since $\operatorname{supp} S'' \subset [-K,K]$, for $\varepsilon \leq \frac{1}{K}$, we have

$$S''(u^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon = S''(u_i^\varepsilon)a_\varepsilon\Big(T_K(x,u_i^\varepsilon),DT_K(u_i^\varepsilon)\Big)DT_K(u_i^\varepsilon) \quad \text{a.e. in } Q.$$

The pointwise convergence of $S''(u^\varepsilon)$ to $S''(u_i)$ as ε tends to 0, the bounded character of S'', T_K and (3.61) of Lemma 3.5 allow to conclude that

$$S''(u_i^\varepsilon)a_\varepsilon(x,u_i^\varepsilon,Du_i^\varepsilon)Du_i^\varepsilon \rightharpoonup S''(u_i)a\Big(x,T_K(u_i),DT_K(u_i)\Big)DT_K(u_i)$$

weakly in $L^1(Q)$, as ε tends to 0. Also

$$S''(u_i)a\Big(T_K(u_i), DT_K(u_i)\Big)DT_K(u_i) = S''(u_i)a(u_i, Du_i)Du_i \quad \text{a.e. in } Q.$$

Limit of $S'(u_i^\varepsilon)\Phi_\varepsilon(u_i^\varepsilon)$. Since $\operatorname{supp} S' \subset [-K, K]$, for $\varepsilon \leq \frac{1}{K^\star}$ we have $S'(u_i^\varepsilon)\Phi_\varepsilon(u_i^\varepsilon) = S'(u_i^\varepsilon)\Phi_\varepsilon(T_K(u_i^\varepsilon))$ a.e. in Q. As a consequence of (1.8), (3.3) and (3.21), it follows that for any $1 \leq q < +\infty$: $S'(u_i^\varepsilon)\Phi_\varepsilon(u_i^\varepsilon) \to S'(u_i)\Phi(T_K(u_i))$ strongly in $L^q(Q)$, as ε tends to 0. The term $S'(u_i)\Phi(T_K(u_i))$ is denoted by $S'(u_i)\Phi(u_i)$.

Limit of $S''(u_i^\varepsilon)\Phi_\varepsilon(u_i^\varepsilon)Du_i^\varepsilon$. Since $S' \in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp} S' \subset [-K, K]$, we have $S''(u_i^\varepsilon)\Phi_\varepsilon(u_i^\varepsilon)Du_i^\varepsilon = \Phi_\varepsilon(T_K(u_i^\varepsilon))DS'(u_i^\varepsilon$ a.e. in Q. Then, $DS'(u_i^\varepsilon)$ converges to $DS'(u)$ weakly in $L^p(Q)^N$ as ε tends to 0, while $\Phi_\varepsilon(T_K(u_i^\varepsilon))$ is uniformly bounded with respect to ε and converges a.e. in Q to $\Phi(T_K(u_i))$ as ε tends to 0. Therefore

$$S''(u^\varepsilon)\Phi_\varepsilon(u_i^\varepsilon)Du_i^\varepsilon \rightharpoonup \Phi_\varepsilon(T_K(u_i^\varepsilon))DS'(u_i^\varepsilon) \text{ weakly in } L^p(Q).$$

Limit of $f_i^\varepsilon(x, u_1^\varepsilon, u_2^\varepsilon)S'(u_i^\varepsilon)$. Due to (1.10), (1.12), (1.13), (3.4) and (3.5), we have $f_i^\varepsilon(x, u_1^\varepsilon, u_2^\varepsilon)S'(u_i^\varepsilon)$ converges to $f_i(x, u_1, u_2)S'(u_i)$ strongly in $L^1(Q)$, as ε tends to 0.

As a consequence of the above convergence result, we are in a position to pass to the limit as ε tends to 0 in equation (3.66) and to conclude that u satisfies (2.3).

It remains to show that $b_{i,S}(u_i)$ satisfies the initial condition (2.4). To this end, firstly remark that, S being bounded, $b_{i,S}^\varepsilon(u_i^\varepsilon)$ is bounded in $L^\infty(Q)$. Secondly, (3.66) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial b_{i,S}^\varepsilon(u_i^\varepsilon)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type lemma (see, e.g, [31], Corollary 4), $b_{i,S}^\varepsilon(u_i^\varepsilon)$ lies in a compact set of $C^0([0, T]; W^{-1,s}(\Omega))$ for any $s < inf\Big(p', \frac{N}{N-1}\Big)$. It follows that $b_{i,S}^\varepsilon(u_i^\varepsilon)(t = 0) = b_{i,S}^\varepsilon(u_{i,0}^\varepsilon)$ converges to $b_{i,S}(u_i)(t = 0)$ strongly in $W^{-1,s}(\Omega)$. On the order hand, (3.9) and the smoothness of S imply that $b_{i,S}^\varepsilon(u_{i,0}^\varepsilon)$ converges to $b_{i,S}(u_{i,0})(t = 0)$ strongly in $L^q(\Omega)$ for all $q < +\infty$ and this in turn implies (2.4). As a conclusion of step 3, step 8 and step 9, we prove theorem 3.1.

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Elhoussine Azroul
Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz. B.P. 1796 Atlas Fès, Morocco
E-mail address: `azroul_elhoussine@yahoo.fr`

Hicham Redwane
Faculté des Sciences Juridiques, Economiques et Sociales, Université Hassan 1, B.P. 784, Settat 26 000, Morocco
E-mail address: `redwane_hicham@yahoo.fr`

Mohamed Rhoudaf
Département des Mathématiques, Faculté des Sciences et Techniques de Tanger. B.P. 416, Tanger, Morocco
E-mail address: rhoudaf_mohamed@yahoo.fr